axioms

Article

Some Hadamard-Type Integral Inequalities Involving Modified Harmonic Exponential Type Convexity

Asif Ali Shaikh [1,2], Evren Hincal [2], Sotiris K. Ntouyas [3], Jessada Tariboon [4,*] and Muhammad Tariq [1]

[1] Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan; asif.shaikh@faculty.muet.edu.pk (A.A.S.); captaintariq2187@gmail.com (M.T.)
[2] Department of Mathematics, Faculty of Arts and Sciences Near East University, Mersin 99138, Turkey; evren.hincal@neu.edu.tr
[3] Department of Mathematics, School of Sciences, University of Ioannina, 45110 Ioannina, Greece; sntouyas@uoi.gr
[4] Intelligent and Nonlinear Dynamic Innovations Research Center, Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
* Correspondence: jessada.t@sci.kmutnb.ac.th

Abstract: The term convexity and theory of inequalities is an enormous and intriguing domain of research in the realm of mathematical comprehension. Due to its applications in multiple areas of science, the theory of convexity and inequalities have recently attracted a lot of attention from historians and modern researchers. This article explores the concept of a new group of modified harmonic exponential s-convex functions. Some of its significant algebraic properties are elegantly elaborated to maintain the newly described idea. A new sort of Hermite–Hadamard-type integral inequality using this new concept of the function is investigated. In addition, several new estimates of Hermite–Hadamard inequality are presented to improve the study. These new results illustrate some generalizations of prior findings in the literature.

Keywords: convex function; *m*-convexity; Holder's inequality; Hermite–Hadamard inequality

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

Citation: Shaikh, A.A.; Hincal, E.; Ntouyas, S.K.; Tariboon, J.; Tariq, M. Some Hadamard-Type Integral Inequalities Involving Modified Harmonic Exponential Type Convexity. *Axioms* **2023**, *12*, 454. https://doi.org/10.3390/axioms12050454

Academic Editor: Mircea Merca

Received: 22 March 2023
Revised: 26 April 2023
Accepted: 3 May 2023
Published: 5 May 2023

1. Introduction

In recent decades, the theory of convexity and inequalities has become an amazing and deep source of attention and inspiration in different areas of science. The combined study of these terminologies has had not only interesting and deep results in numerous subjects of applied and engineering sciences but also contributed equally towards numerical optimization. The concept of convexity is based and depends on the theory of inequalities and also plays a prominent and meaningful role in this field. The novel literature on inequalities always provides an excellent glimpse of the beauty and fascination of science. Integral inequalities have many applications in probability theory, information technology, statistics, numerical integration, stochastic processes, optimization theory, and integral operator theory. For detailed concepts on inequalities, see [1–19]. In [20], İşcan explores an extended form of convex function, namely the n-polynomial convex function. The harmonic convex set in 2003 was first defined by Shi in [21]. On this harmonic convex set, the harmonic convex function was introduced by Anderson et al. [22]. Noor [23] continued his work on estimations and extensions and investigated the harmonic convex function in a polynomial version and also made some improvements in the frame variational inequality (see [24,25]).

Dragomir [26] was the first to define and research the term "exponential convex function" in the literature. After Dragomir, Awan [27] conducted the study and refined this function.

Kadakal [28] presented a revised definition of exponential convexity. The remarkable significance and applications of exponential convexity are exploited in information sciences, stochastic optimization, data mining, sequential prediction, and statistical learning.

The construction of this manuscript is as follows. In Section 2, we give some basic definitions and concepts which will be required throughout the manuscript's following sections. In Section 3, we introduce the modified harmonic exp s-convex functions and discuss some properties of it. In Section 4, using a newly introduced concept, a new sort of Hadamard-type inequality is achieved. Next, we prove and examine some extensions of the Hadamard-type inequality regarding the new definition with the help of Holder's inequality in Section 5. Finally, in Section 6, future scopes of the present study and a brief conclusion are provided.

2. Preliminaries

For the reader's interest and the quality of the manuscript, it will be best to study and explain some ideas, concepts, definitions, corollaries, theorems, and remarks in this part. The main aim of this part is to mention and discuss some already published definitions and ideas, which we require in our study in the following sections. We start by introducing the convex function and its generalizations in different versions and the Hermite–Hadamard-type inequality. In addition, some theorems regarding harmonic convex functions are added. We sum up this part by stating Holder's and the power mean inequality, which will be needed in our further investigation.

Definition 1 ([1]). *Assume that $\mathbb{X}$ is a convex subset of a real vector space $\mathbb{R}$. A function $\mathrm{Q} : \mathbb{X} \to \mathbb{R}$ is convex if*

$$\mathrm{Q}(\lambda \mathrm{v}_1 + (1-\lambda)\mathrm{v}_2) \leq \lambda \mathrm{Q}(\mathrm{v}_1) + (1-\lambda)\mathrm{Q}(\mathrm{v}_2), \tag{1}$$

holds $\forall$ $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, and $\lambda \in [0,1]$.

The Hermite–Hadamard-type inequality performs a good role in the literature due to its importance and popularity. A lot of scientists have worked on numerous ideas and definitions on the subject of inequalities. In the field of analysis, this inequality has great interest due to its applications. This inequality states that, if function $\mathrm{Q} : \mathbb{X} \to \mathbb{R}$ is convex for $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$ with the condition $\mathrm{v}_1 < \mathrm{v}_2$, then

$$\mathrm{Q}\left(\frac{\mathrm{v}_1 + \mathrm{v}_2}{2}\right) \leq \frac{1}{\mathrm{v}_2 - \mathrm{v}_1} \int_{\mathrm{v}_1}^{\mathrm{v}_2} \mathrm{Q}(\chi) d\chi \leq \frac{\mathrm{Q}(\mathrm{v}_1) + \mathrm{Q}(\mathrm{v}_2)}{2}. \tag{2}$$

We recommend that readers refer to [29–32].

Definition 2 ([33]). *Let $s \in (0,1]$. A function $\mathrm{Q} : [0, +\infty) \to \mathbb{R}$ is s-convex in the second sense if*

$$\mathrm{Q}(\lambda \mathrm{v}_1 + (1-\lambda)\mathrm{v}_2) \leq \lambda^s \mathrm{Q}(\mathrm{v}_1) + (1-\lambda)^s \mathrm{Q}(\mathrm{v}_2) \tag{3}$$

holds $\forall$ $\mathrm{v}_1, \mathrm{v}_2 \in [0, +\infty)$, and $\lambda \in [0,1]$.

Definition 3 ([28]). *Let $\mathbb{X}$ be a non-negative real interval. A function $\mathrm{Q} : \mathbb{X} \to \mathbb{R}$ is exponentially convex if*

$$\mathrm{Q}(\lambda \mathrm{v}_1 + (1-\lambda)\mathrm{v}_2) \leq \left(e^{\lambda} - 1\right)\mathrm{Q}(\mathrm{v}_1) + \left(e^{(1-\lambda)} - 1\right)\mathrm{Q}(\mathrm{v}_2), \tag{4}$$

for all $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, and $\lambda \in [0,1]$.

The notation $EXPC(I)$ represents the family of all exponentially convex functions on the interval $\mathbb{X}$.

Definition 4 ([34]). *Let $\mathbb{X} \subset \mathbb{R} \setminus \{0\}$ be a real interval. A function $\mathrm{Q} : \mathbb{X} \subseteq (0, +\infty) \to \mathbb{R}$ is harmonically convex if*

$$\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1}\right) \leq \lambda\mathrm{Q}(\mathrm{v}_1) + (1-\lambda)\mathrm{Q}(\mathrm{v}_2), \tag{5}$$

holds for all $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, and $\lambda \in [0,1]$.

Theorem 1 ([34]). *Assume that a real-valued function Q on $\mathbb{X} \subseteq (0, +\infty) \to \mathbb{R}$ is harmonically convex. If Q is defined on integrable space, i.e., $L[\mathrm{v}_1, \mathrm{v}_2]$, for all $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$ with $\mathrm{v}_1 < \mathrm{v}_2$, then*

$$\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1 + \mathrm{v}_2}\right) \leq \frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2 - \mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx \leq \frac{\mathrm{Q}(\mathrm{v}_1) + \mathrm{Q}(\mathrm{v}_2)}{2}. \tag{6}$$

Definition 5 ([20]). *Let $\mathbb{X}$ be a non-negative real interval. A function $\mathrm{Q} : \mathbb{X} \to [0, \infty)$. Then Q is m-polynomial convex if*

$$\mathrm{Q}(\lambda\mu_1 + (1-\lambda)\mathrm{v}_2) \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1 - (1-\lambda)^\eta]\mathrm{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1 - \lambda^\eta]\mathrm{Q}(\mathrm{v}_2), \tag{7}$$

holds for every $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, $\mathrm{m} \in \mathbb{N}$, and $\lambda \in [0,1]$.

Definition 6 ([35]). *Assume that $\mathrm{Q} : \mathbb{X} = (0, +\infty) \to [0, \infty)$. Then Q is m-polynomial exponential s-convex if*

$$\mathrm{Q}\Big(\lambda\mathrm{v}_1 + (1-\lambda)\mathrm{v}_2\Big) \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\Big(e^{s\lambda} - 1\Big)^\eta\mathrm{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\Big(e^{s(1-\lambda)} - 1\Big)^\eta\mathrm{Q}(\mathrm{v}_2), \tag{8}$$

holds $\forall$ $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, $\mathrm{m} \in \mathbb{N}$, $s \in [\ln 2.5, 1]$, and $\lambda \in [0,1]$.

Definition 7 ([23]). *Let us assume that $\mathrm{Q} : \mathbb{X} \to [0, \infty)$. Then Q is m-polynomial harmonically convex if*

$$\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1}\right) \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1 - (1-\lambda)^\eta]\mathrm{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1 - \lambda^\eta]\mathrm{Q}(\mathrm{v}_2), \tag{9}$$

holds for every $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, $\mathrm{m} \in \mathbb{N}$, and $\lambda \in [0,1]$.

Remark 1. *Assume that $\mathrm{m} = 1$; then Definition 7 is referred to Definition 4.*

Remark 2. *If the following inequalities $\lambda \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1 - (1-\lambda)^\eta]$ and $1 - \lambda \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1 - \lambda^\eta]$ hold, then every harmonic convex function is an m-polynomial harmonic convex function.*

Definition 8 ([36]). *Let us assume that $\mathrm{Q} : \mathbb{X} \to [0, \infty)$. Then Q is m-polynomial harmonic exponential convex if*

$$\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1}\right) \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\Big(e^{\lambda} - 1\Big)^\eta\mathrm{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\Big(e^{1-\lambda} - 1\Big)^\eta\mathrm{Q}(\mathrm{v}_2), \tag{10}$$

holds for every $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, $\mathrm{m} \in \mathbb{N}$, and $\lambda \in [0,1]$.

Remark 3 ([36]). *Every nonnegative m-polynomial harmonic convex function is also an m-polynomial harmonic exponential-type convex function. Indeed, for all $\lambda \in [0,1]$ this case is clear from the following inequalities:*

$$\frac{1}{n}\sum_{\eta=1}^{\mathrm{m}}[1-(1-\lambda)^{\eta}] \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{\lambda}-1\right)^{\eta} \quad \text{and} \quad \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}[1-\lambda^{\eta}] \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{1-\lambda}-1\right)^{\eta}.$$

Theorem 2 ([37]). *Assume that $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$. If Q_1 and Q_2 are real functions defined on Lebesgue measurable space of a and b, i.e., $L[a,b]$, and if $|\mathrm{Q}_1|^p$ and $|\mathrm{Q}_2|^q$ are integrable functions on $[a,b]$, then*

$$\int_a^b |\mathrm{Q}_1(\nu)\mathrm{Q}_2(\nu)|d\nu \leq \left(\int_a^b |\mathrm{Q}_1(x)|^p dx\right)^{\frac{1}{p}}\left(\int_a^b |\mathrm{Q}_2(x)|^q dx\right)^{\frac{1}{q}}. \tag{11}$$

The equality holds if and only if $A|\mathrm{Q}_1|^p = B|\mathrm{Q}_2|^q$, almost everywhere, where A and B are constants.

3. Modified Harmonic Exponential s-Convex Function and Its Algebraic Properties

The term convexity has gained an amazing image due to many applications in the realms of engineering, optimizations, and applied mathematics. Although many outcomes have been deduced under convexity, the majority of the problems regarding real life are nonconvex in nature. In the 20th century, many researchers gave attention to the term convexity, such as Jensen, Hermite, Holder, and Stolz. Throughout this century, an unprecedented amount of research was carried out, and important results were obtained in the field of convex analysis.

We will provide our basic definition of the modified harmonic exp s-convex function and its corresponding features as the main topic of this section.

Definition 9. *Assume that $\mathrm{Q} : \mathbb{X} = (0,+\infty) \to [0,\infty)$. Then Q is modified harmonic exponential s-convex if*

$$\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) \leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathrm{Q}(\mathrm{v}_1)+\frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathrm{Q}(\mathrm{v}_2), \tag{12}$$

holds $\forall\ \mathrm{v}_1,\mathrm{v}_2 \in \mathbb{X}$, $\mathrm{m}\in\mathbb{N}$, $s\in[\ln 2.5, 1]$, and $\lambda\in[0,1]$.

Remark 4. *Assume that $\mathrm{m}=1$ in the above inequality (12); then*

$$\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) \leq \left(e^{s\lambda}-1\right)\mathrm{Q}(\mathrm{v}_1)+\left(e^{s(1-\lambda)}-1\right)\mathrm{Q}(\mathrm{v}_2). \tag{13}$$

Remark 5. *Assume that $\mathrm{m}=2$ in the above inequality (12); then*

$$\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) \leq \left(\frac{e^{2s\lambda}-e^{s\lambda}}{2}\right)\mathrm{Q}(\mathrm{v}_1)+\left(\frac{e^{2s(1-\lambda)}-e^{s(1-\lambda)}}{2}\right)\mathrm{Q}(\mathrm{v}_2). \tag{14}$$

Remark 6. *Assume that $s=1$ in the above inequality (12); we obtain Definition 8.*

Remark 7. *Assume that $\mathrm{m}=1$ and $s=1$ in the above inequality (12); we obtain Remark 3 in [36].*

Remark 8. *Assume that $\mathrm{m}=2$ and $s=1$ in the above inequality (12); we obtain Remark 4 in [36].*

That is the best advantage of the novel concept. If we take m and s at their given values, then we obtain the new inequalities and discover their connections with previous results.

Lemma 1. *Let us assume that $\lambda\in[0,1]$ and $s\in[\ln 2.5,1]$; then $\frac{1}{\mathrm{m}}\sum\limits_{\eta=1}^{\mathrm{m}}(e^{s\lambda}-1)^{\eta} \geq \lambda$ and $\frac{1}{\mathrm{m}}\sum\limits_{\eta=1}^{\mathrm{m}}(e^{s(1-\lambda)}-1)^{\eta} \geq (1-\lambda)$ hold.*

Lemma 2. *The following inequalities* $\frac{1}{m}\sum_{\eta=1}^{m}(e^{s\lambda}-1)^{\eta} \geq \frac{1}{n}\sum_{\eta=1}^{m}[1-(1-\lambda)^{\eta}]$ *and* $\frac{1}{m}\sum_{\eta=1}^{m}(e^{s(1-\lambda)}-1)^{\eta} \geq \frac{1}{m}\sum_{\eta=1}^{m}[1-\lambda^{\eta}]$ *hold, for all* $\lambda \in [0,1]$ *and* $s \in [\ln 2.5, 1]$.

Proposition 1. *Every harmonic convex function* $Q : I \subset (0,+\infty) \to [0,\infty)$ *is a modified harmonic exp s-convex function.*

Proof. Since the given function is a harmonic convex, by definition, we have

$$Q\left(\frac{v_1 v_2}{\lambda v_2 + (1-\lambda)v_1}\right) \leq \lambda Q(v_1) + (1-\lambda)Q(v_2).$$

Employing Lemma 1, we have

$$Q\left(\frac{v_1 v_2}{\lambda v_2 + (1-\lambda)v_1}\right) \leq \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s\lambda}-1\right)^{\eta}Q(v_1) + \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s(1-\lambda)}-1\right)^{\eta}Q(v_2).$$

□

Proposition 2. *Every* m*-polynomial harmonically convex function is a modified harmonically exp s-convex function.*

Proof. Since the given function is m-polynomial harmonic convex, by definition, we have

$$Q\left(\frac{v_1 v_2}{\lambda v_2 + (1-\lambda)v_1}\right) \leq \frac{1}{m}\sum_{\eta=1}^{m}[1-(1-\lambda)^{\eta}]Q(v_1) + \frac{1}{m}\sum_{\eta=1}^{m}[1-\lambda^{\eta}]Q(v_2).$$

Employing Lemma 2, we have

$$Q\left(\frac{v_1 v_2}{\lambda v_2 + (1-\lambda)v_1}\right) \leq \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s\lambda}-1\right)^{\eta}Q(v_1) + \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s(1-\lambda)}-1\right)^{\eta}Q(v_2).$$

□

Next, regarding this new definition, we add some examples.

Example 1. *Let* $Q(x) = x^2 e^{x^2}$ *be a non-decreasing convex function on* $(0,1)$*; then it is harmonic convex (see [38]). Employing Proposition 1, we claim that it is a modified harmonic exp s-convex function.*

Example 2. *Let* $Q(x) = e^{x}$ *be a non-decreasing convex function; then it is harmonic convex (see [38]). Employing Proposition 1, we claim that it is a modified harmonic exp s-convex function.*

Example 3. *Let* $Q(x) = \sin(-x)$ *be a non-decreasing convex function on* $(0,\frac{\pi}{2})$*; then it is harmonically convex* $\forall x \in (0,\frac{\pi}{2})$ *(see [38]). Employing Proposition 1, it is a modified harmonic exp s-convex function.*

Example 4. *Let* $Q(x) = x$ *be a non-decreasing convex function on* $(0,\infty)$*; then it is harmonically convex for all* $x \in (0,\infty)$ *(see [38]). Employing Remark 2, we claim that it is* m*-polynomial harmonic convex. Employing Proposition 2, we claim that it is a modified harmonic exp s-convex function.*

Example 5. *Let* $Q(x) = \ln x$ *be a harmonic convex on the interval* $(0,\infty)$ *(see [38]). Employing Remark 2 and Proposition 2, we obtain that* $Q(x)$ *is a modified harmonic exp s-convex function.*

In addition, we add some properties regarding the newly introduced idea, namely the modified harmonic exp s-convex function.

Theorem 3. *The sum of two modified harmonic exp s-convex functions is a modified harmonic exp s-convex function.*

Proof. Let us assume that the functions Q and H are modified harmonic exp s-convex and $\lambda \in [0,1]$; then

$$\begin{aligned}
&(\mathsf{Q}+\mathsf{H})\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) \\
&= \mathsf{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) + \mathsf{H}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) \\
&\leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathsf{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathsf{Q}(\mathrm{v}_2) \\
&+ \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathsf{H}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathsf{H}(\mathrm{v}_2) \\
&= \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}[\mathsf{Q}(\mathrm{v}_1)+\mathsf{H}(\mathrm{v}_1)] + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}[\mathsf{Q}(\mathrm{v}_2)+\mathsf{H}(\mathrm{v}_2)] \\
&= \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}(\mathsf{Q}+\mathsf{H})(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}(\mathsf{Q}+\mathsf{H})(\mathrm{v}_2).
\end{aligned}$$

This completes the proof. □

Remark 9. *If we assume that* $\mathrm{m} = 1$*, then we obtain* $\mathsf{Q}+\mathsf{H}$ *as the harmonic exp s-convex function.*

Remark 10. *If we assume that* $s = 1$*, then we obtain* $\mathsf{Q}+\mathsf{H}$ *as a modified harmonic exp convex function.*

Remark 11. *If we assume that* $\mathrm{m} = 1$ *and* $s = 1$*, then we obtain* $\mathsf{Q}+\mathsf{H}$ *as a harmonic exp convex function.*

Theorem 4. *Scalar multiplication of a modified harmonic exp s-convex function is a modified harmonic exp s-convex function.*

Proof. Let assume that the function Q is modified harmonic exp s-convex, $\lambda \in [0,1]$; then

$$\begin{aligned}
&(c\mathsf{Q})\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right) \\
&\leq c\left[\frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathsf{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathsf{Q}(\mathrm{v}_2)\right] \\
&= \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}c\mathsf{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}c\mathsf{Q}(\mathrm{v}_2) \\
&= \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}(c\mathsf{Q})(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}(c\mathsf{Q})(\mathrm{v}_2).
\end{aligned}$$

This completes the proof. □

Remark 12. *If we assume that* $\mathrm{m} = 1$*, then the scalar multiplication of a harmonic exp s-convex function is a harmonic exp s-convex function.*

Remark 13. *If we assume that s = 1, then the scalar multiplication of the modified harmonic exp convex function is a modified harmonic exp convex function.*

Remark 14. *If we assume that* $\mathrm{m} = 1$ *and* $s = 1$*, then scalar multiplication of a harmonically exp convex function is a harmonic exp convex function.*

Theorem 5. *Assume that the function* $\mathrm{Q}_1 : \mathbb{X} \to [0, +\infty)$ *is harmonic convex and the function* $\mathrm{Q}_2 : [0, +\infty) \to [0, +\infty)$ *is increasing and m-polynomial exp s-convex. Then* $\mathrm{Q}_2 \circ \mathrm{Q}_1 : \mathbb{X} \to [0, +\infty)$ *is a modified harmonic exp s-convex function.*

Proof. For all $\mathrm{v}_1, \mathrm{v}_2 \in \mathbb{X}$, and $\lambda \in [0, 1]$, we have

$$\begin{aligned}
&(\mathrm{Q}_2 \circ \mathrm{Q}_1)\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1}\right) \\
&= \mathrm{Q}_2\left(\mathrm{Q}_1\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1}\right)\right) \\
&\leq \mathrm{Q}_2(\lambda\mathrm{Q}_1(\mathrm{v}_1) + (1-\lambda)\mathrm{Q}_1(\mathrm{v}_2)) \\
&\leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathrm{Q}_2(\mathrm{Q}_1(\mathrm{v}_1)) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathrm{Q}_2(\mathrm{Q}_1(\mathrm{v}_2)) \\
&= \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}(\mathrm{Q}_2 \circ \mathrm{Q}_1)(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}(\mathrm{Q}_2 \circ \mathrm{Q}_1)(\mathrm{v}_2).
\end{aligned}$$

□

Theorem 6. *Let* $0 < \mathrm{v}_1 < \mathrm{v}_2$ *and assume that non-negative real-valued function* Q_j *is a class of modified harmonic exp s-convex and* $\mathrm{Q}(u) = \sup_j \mathrm{Q}_j(u)$*. Then the function* Q *is a modified harmonic exp s-convex and* $U = \{u \in [\mathrm{v}_1, \mathrm{v}_2] : \mathrm{Q}(u) < +\infty\}$ *is an interval.*

Proof. Let $\mathrm{v}_1, \mathrm{v}_2 \in U$ and $\lambda \in [0, 1]$; then

$$\begin{aligned}
&\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1}\right) \\
&= \sup_j \mathrm{Q}_j\left(\frac{\mathrm{v}_1\mathrm{v}_2}{(\lambda\mathrm{v}_2 + (1-\lambda)\mathrm{v}_1)}\right) \\
&\leq \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\sup_j \mathrm{Q}_j(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\sup_j \mathrm{Q}_j(\mathrm{v}_2) \\
&= \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathrm{Q}(\mathrm{v}_1) + \frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathrm{Q}(\mathrm{v}_2) < +\infty.
\end{aligned}$$

This shows simultaneously that U is an interval, since it contains every point between any two of its points, and that Q is a modified harmonic exp s-convex function on U. This is the required proof. □

Theorem 7. *If* $\mathrm{Q} : \mathbb{X} \to [0, +\infty)$ *is modified harmonic exp s-convex, then the function* Q *is bounded on* $[\mathrm{v}_1, \mathrm{v}_2]$*.*

Proof. Let us assume that $x \in [v_1, v_2]$ and $L = \max\{Q(v_1), Q(v_2)\}$. Then $\exists\, \lambda \in [0,1]$ such that $x = \frac{v_1 v_2}{\lambda v_2 + (1-\lambda)v_1}$. Here, we clearly know about the obvious following inequalities, i.e., $e^{s\lambda} \leq e$ and $e^{s(1-\lambda)} \leq e$; then

$$\begin{aligned}
Q(x) &= Q\left(\frac{v_1 v_2}{\lambda v_2 + (1-\lambda)v_1}\right) \\
&\leq \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s\lambda}-1\right)^{\eta} Q(v_1) + \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s(1-\lambda)}-1\right)^{\eta} Q(v_2) \\
&\leq \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s\lambda}-1\right)^{\eta} L + \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s(1-\lambda)}-1\right)^{\eta} L \\
&\leq \frac{2L}{m}\sum_{\eta=1}^{m}(e-1)^{\eta} = M.
\end{aligned}$$

□

4. Generalized Form of Hadamard Inequality via Modified Harmonic Exponential *s*-Convex Function

Convexity is important and crucial in many branches of the pure and applied sciences. Massive generalizations of mathematical inequalities for multiple convex functions have significantly influenced traditional research. Numerous fields, including linear programming, combinatorics, theory of relativity, optimization theory, quantum theory, number theory, dynamics, and orthogonal polynomials are affected by and use integral inequalities. This issue has received much attention from researchers. The Hadamard inequality is the most widely used and popular inequality in the history and literature pertaining to convex theory.

This purpose of this section is to establish a new kind of the Hadamard inequality pertaining to modified harmonic exp *s*-convexity.

Theorem 8. *Let non-negative real-valued* Q *be modified harmonic exp s-convex. If* $Q \in L[v_1, v_2]$, *then*

$$\frac{m}{2\sum_{\eta=1}^{m}\left(\sqrt{e^s}-1\right)^{\eta}} Q\left(\frac{2v_1v_2}{v_1+v_2}\right) \leq \frac{v_1v_2}{v_2-v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx \leq \left[\mathcal{A}_1 Q(v_1) + \mathcal{A}_2 Q(v_2)\right], \tag{15}$$

where
$\mathcal{A}_1 = \frac{1}{m}\sum_{\eta=1}^{m}\int_0^1 (e^{s\lambda}-1)^{\eta} d\lambda$ *and* $\mathcal{A}_2 = \frac{1}{m}\sum_{\eta=1}^{m}\int_0^1 \left(e^{s(1-\lambda)}-1\right)^{\eta} d\lambda.$

Proof. Since Q is modified harmonic exp *s*-convex, then we have

$$Q\left(\frac{xy}{\lambda y + (1-\lambda)x}\right) \leq \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s\lambda}-1\right)^{\eta} Q(x) + \frac{1}{m}\sum_{\eta=1}^{m}\left(e^{s(1-\lambda)}-1\right)^{\eta} Q(y),$$

which leads to

$$Q\left(\frac{2xy}{x+y}\right) \leq \frac{1}{m}\sum_{\eta=1}^{m}\left(\sqrt{e^s}-1\right)^{\eta} Q(x) + \frac{1}{m}\sum_{\eta=1}^{m}\left(\sqrt{e^s}-1\right)^{\eta} Q(y).$$

Employing the change in variables, we have

$$Q\left(\frac{2v_1v_2}{v_1+v_2}\right) \leq \frac{1}{m}\sum_{\eta=1}^{m}\left[\left(\sqrt{e^s}-1\right)^{\eta}\right]\left[Q\left(\frac{v_1v_2}{(\lambda v_2 + (1-\lambda)v_1)}\right) + Q\left(\frac{v_1v_2}{(\lambda v_1 + (1-\lambda)v_2)}\right)\right]. \tag{16}$$

Integrating inequality (16) w.r.t. λ on $[0,1]$ yields

$$\frac{\mathrm{m}}{2\sum_{\eta=1}^{\mathrm{m}}\left(\sqrt{e^s}-1\right)^{\eta}}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\leq\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx.$$

This is the required inequality.

For the other inequality, first we suppose $x=\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}$ and employ Definition 9 for the function Q; we have

$$\begin{aligned}
&\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\\
&=\int_0^1\mathrm{Q}\left(\frac{\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_2+(1-\lambda)\mathrm{v}_1}\right)d\lambda\\
&\leq\int_0^1\left[\frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}\mathrm{Q}(\mathrm{v}_1)+\frac{1}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}\mathrm{Q}(\mathrm{v}_2)\right]d\lambda\\
&=\frac{\mathrm{Q}(\mathrm{v}_1)}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1\left(e^{s\lambda}-1\right)^{\eta}d\lambda+\frac{\mathrm{Q}(\mathrm{v}_2)}{\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1\left(e^{s(1-\lambda)}-1\right)^{\eta}d\lambda\\
&=\left[\mathcal{A}_1\mathrm{Q}(\mathrm{v}_1)+\mathcal{A}_2\mathrm{Q}(\mathrm{v}_2)\right].
\end{aligned}$$

This completes the proof. □

Corollary 1. *Assume that $m=1$ in the above inequality (15); then*

$$\frac{1}{2\left(\sqrt{e^s}-1\right)}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\leq\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\leq\left(\frac{e^s-s-1}{s}\right)\left[\mathrm{Q}(\mathrm{v}_1)+\mathrm{Q}(\mathrm{v}_2)\right].$$

Remark 15. *Assume that $s=1$ in the above inequality (15); then we obtain Theorem 4.1 in [36].*

5. Refinements of Hadamard Inequality Involving Modified Harmonic Exponential *s*-Convex Function

In recognition of the importance of convexity, various researchers have created numerous generalizations of convexity and validated a lot of features in these new generalized cases. Convex sequences, their characteristics, and the accompanying inequalities with applications have received increased attention from researchers. The most viewed and discussed inequality in history connected with the field of convex analysis is the Hermite–Hadamard inequality.

Given the following lemma, with the aid of Holder's inequality and involving the newly introduced concept, we obtained some extensions of the Hermite–Hadamard inequality.

Lemma 3 ([23])**.** *Let us assume that $\rho,\sigma\in[0,1]$ and a non-negative real-valued function Q is a differentiable mapping. If $\mathrm{Q}'\in L[\mathrm{v}_1,\mathrm{v}_2]$, then the following identity holds:*

$$\begin{aligned}
&\frac{\rho\mathrm{Q}(\mathrm{v}_1)+\sigma\mathrm{Q}(\mathrm{v}_2)}{2}+\frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\\
&=\frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4}\int_0^1\left[\frac{4(1-\rho-\lambda)}{((1-\lambda)\mathrm{v}_2+(1+\lambda)\mathrm{v}_1)^2}\mathrm{Q}'\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{(1-\lambda)\mathrm{v}_2+(1+\lambda)\mathrm{v}_1}\right)\right.\\
&\left.+\frac{4(\sigma-\lambda)}{(\lambda\mathrm{v}_1+(2-\lambda)\mathrm{v}_2)^2}\mathrm{Q}'\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_1+(2-\lambda)\mathrm{v}_2}\right)\right]d\lambda.
\end{aligned}\tag{17}$$

For simplicity, we denote

$$A_{\mathrm{v}_1,\mathrm{v}_2}=(1-\lambda)\mathrm{v}_2+(1+\lambda)\mathrm{v}_1\quad\text{and}\quad B_{\mathrm{v}_1,\mathrm{v}_2}=\lambda\mathrm{v}_1+(2-\lambda)\mathrm{v}_2.\tag{18}$$

The following notations will be used in this way:

$$\Gamma(\mathrm{v}) = \int_0^{+\infty} e^{-\lambda}\lambda^{\mathrm{v}-1}d\lambda, \quad \mathrm{v} > 0;$$

$$\beta(\mathrm{v}_1, \mathrm{v}_2) = \int_0^1 \lambda^{\mathrm{v}_1-1}(1-\lambda)^{\mathrm{v}_2-1}d\lambda, \quad \mathrm{v}_1, \mathrm{v}_2 > 0;$$

This is a hypergeometric function in integral form first introduced by Euler [39]. This function states that

$$\beta(\mathrm{v}_1, \mathrm{v}_2) = \frac{\Gamma(\mathrm{v}_1)\Gamma(\mathrm{v}_2)}{\Gamma(\mathrm{v}_1 + \mathrm{v}_2)}, \quad \mathrm{v}_1, \mathrm{v}_2 > 0;$$

$${}_2F_1(\mathrm{v}_1, \mathrm{v}_2; \mathrm{v}_3; \mathrm{v}) = \frac{1}{\beta(\mathrm{v}_2, \mathrm{v}_3 - \mathrm{v}_2)} \int_0^1 \lambda^{\mathrm{v}_2-1}(1-\lambda)^{\mathrm{v}_3-\mathrm{v}_2-1}(1-\mathrm{v}\lambda)^{-\mathrm{v}_1}d\lambda,$$

where $\mathrm{v}_3 > \mathrm{v}_2 > 0$ and $|\mathrm{v}| < 1$.

Theorem 9. *Let us assume that $\rho, \sigma \in [0,1]$ and $\mathrm{Q} : [\mathrm{v}_1, \mathrm{v}_2] \subseteq (0, +\infty) \to \mathbb{R}$ is a differentiable mapping such that $\mathrm{Q}' \in L[\mathrm{v}_1, \mathrm{v}_2]$. Suppose $|\mathrm{Q}'|^q$ is modified harmonic exp s-convex; then for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned} &\left| \frac{\rho \mathrm{Q}(\mathrm{v}_1) + \sigma \mathrm{Q}(\mathrm{v}_2)}{2} + \frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1 + \mathrm{v}_2}\right) - \frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2 - \mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2} \frac{\mathrm{Q}(x)}{x^2}dx \right| \\ &\leq \mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2 - \mathrm{v}_1) \\ &\times \left[\varphi_1^{\frac{1}{p}} \left(\mathfrak{T}_1|\mathrm{Q}'(\mathrm{v}_1)|^q + \mathfrak{T}_2|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}} + \varphi_2^{\frac{1}{p}} \left(\mathfrak{T}_3|\mathrm{Q}'(\mathrm{v}_1)|^q + \mathfrak{T}_4|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}} \right], \end{aligned} \tag{19}$$

where

$$\varphi_1 = \int_0^1 |1-\rho-\lambda|^p d\lambda = \frac{(1-\rho)^{p+1} + \rho^{p+1}}{p+1},$$

$$\varphi_2 = \int_0^1 |\sigma-\lambda|^p d\lambda = \frac{(1-\sigma)^{p+1} + \sigma^{p+1}}{p+1},$$

$$\mathfrak{T}_1 = \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{n}\int_0^1 \frac{1}{A_{\mathrm{v}_1,\mathrm{v}_2}^{2q}}(e^{s(1-\lambda)}-1)^{\eta}d\lambda, \quad \mathfrak{T}_2 = \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 \frac{1}{A_{\mathrm{v}_1,\mathrm{v}_2}^{2q}}(e^{s(1+\lambda)}-1)^{\eta}d\lambda,$$

$$\mathfrak{T}_3 = \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 \frac{1}{B_{\mathrm{v}_1,\mathrm{v}_2}^{2q}}(e^{s(2-\lambda)}-1)^{\eta}d\lambda, \quad \mathfrak{T}_4 = \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 \frac{1}{B_{\mathrm{v}_1,\mathrm{v}_2}^{2q}}(e^{s\lambda}-1)^{\eta}d\lambda,$$

and $A_{\mathrm{v}_1,\mathrm{v}_2}, B_{\mathrm{v}_1,\mathrm{v}_2}$ are defined from (18).

Proof. From Lemma 3, we have

$$\begin{aligned} &\left| \frac{\rho \mathrm{Q}(\mathrm{v}_1) + \sigma \mathrm{Q}(\mathrm{v}_2)}{2} + \frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1 + \mathrm{v}_2}\right) - \frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2 - \mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2} \frac{\mathrm{Q}(x)}{x^2}dx \right| \\ &\leq \frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2 - \mathrm{v}_1)}{4}\left[\int_0^1 \left| \frac{4(1-\rho-\lambda)}{((1-\lambda)\mathrm{v}_2 + (1+\lambda)\mathrm{v}_1)^2} \right| \left| \mathrm{Q}'\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{(1-\lambda)\mathrm{v}_2 + (1+\lambda)\mathrm{v}_1}\right) \right| d\lambda \right. \\ &\left. + \int_0^1 \left| \frac{4(\sigma-\lambda)}{(\lambda\mu_1 + (2-\lambda)\mathrm{v}_2)^2} \right| \left| \mathrm{Q}'\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_1 + (2-\lambda)\mathrm{v}_2}\right) \right| d\lambda \right]. \end{aligned}$$

Employing the property of Hölder's inequality and modified harmonic exp s-convex function, we have

$$\begin{aligned}
&\left|\frac{\rho\mathrm{Q}(\mathrm{v}_1)+\sigma\mathrm{Q}(\mathrm{v}_2)}{2}+\frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right| \\
&\leq \mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)\left\{\left(\int_0^1|1-\rho-\lambda|^p d\lambda\right)^{\frac{1}{p}}\right. \\
&\times\left[\int_0^1\frac{1}{A^{2q}_{\mathrm{v}_1,\mathrm{v}_2}}\left(\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^{\eta}|\mathrm{Q}'(\mathrm{v}_1)|^q+\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1+\lambda)}-1\right)^{\eta}|\mathrm{Q}'(\mathrm{v}_2)|^q\right)d\lambda\right]^{\frac{1}{q}} \\
&+\left(\int_0^1|\sigma-\lambda|^p d\lambda\right)^{\frac{1}{p}} \\
&\left.\times\left[\int_0^1\frac{1}{B^{2q}_{\mathrm{v}_1,\mathrm{v}_2}}\left(\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(2-\lambda)}-1\right)^{\eta}|\mathrm{Q}'(\mathrm{v}_1)|^q+\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^{\eta}|\mathrm{Q}'(\mathrm{v}_2)|^q\right)d\lambda\right]^{\frac{1}{q}}\right\} \\
&=\frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4} \\
&\times\left[\varphi_1^{\frac{1}{p}}\left(\mathfrak{T}_1|\mathrm{Q}'(\mathrm{v}_1)|^q+\mathfrak{T}_2|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}+\varphi_2^{\frac{1}{p}}\left(\mathfrak{T}_3|\mathrm{Q}'(\mathrm{v}_1)|^q+\mathfrak{T}_4|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2. *Assume that* $\mathrm{m}=1$ *in inequality (19); then*

$$\begin{aligned}
&\left|\frac{\rho\mathrm{Q}(\mathrm{v}_1)+\sigma\mathrm{Q}(\mathrm{v}_2)}{2}+\frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right| \\
&\leq \mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1) \\
&\times\left[\varphi_1^{\frac{1}{p}}\left(D_1|\mathrm{Q}'(\mathrm{v}_1)|^q+D_2|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}+\varphi_2^{\frac{1}{p}}\left(D_3|\mathrm{Q}'(\mathrm{v}_1)|^q+D_4|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$D_1=\frac{1}{2}\int_0^1\frac{1}{A^{2q}_{\mathrm{v}_1,\mathrm{v}_2}}(e^{s(1-\lambda)}-1)d\lambda,\quad D_2=\frac{1}{2}\int_0^1\frac{1}{A^{2q}_{\mathrm{v}_1,\mathrm{v}_2}}(e^{s(1+\lambda)}-1)d\lambda,$$

$$D_3=\frac{1}{2}\int_0^1\frac{1}{B^{2q}_{\mathrm{v}_1,\mathrm{v}_2}}(e^{s(2-\lambda)}-1)d\lambda,\quad D_4=\frac{1}{2}\int_0^1\frac{1}{B^{2q}_{\mathrm{v}_1,\mathrm{v}_2}}(e^{s\lambda}-1)d\lambda.$$

Corollary 3. *Assume that* $\rho=\sigma$ *in inequality (19); then*

$$\begin{aligned}
&\left|\rho\frac{\mathrm{Q}(\mathrm{v}_1)+\mathrm{Q}(\mathrm{v}_2)}{2}+(1-\rho)\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right| \\
&\leq \mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)\varphi^{\frac{1}{p}} \\
&\times\left[\left(\mathfrak{T}_1|\mathrm{Q}'(\mathrm{v}_1)|^q+\mathfrak{T}_2|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}+\left(\mathfrak{T}_3|\mathrm{Q}'(\mathrm{v}_1)|^q+\mathfrak{T}_4|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\varphi_1=\varphi_2=\varphi$.

Corollary 4. *Assume that $\rho=\sigma=0$ in inequality (19); then*

$$\begin{aligned}&\left|Q\left(\frac{2v_1v_2}{v_1+v_2}\right)-\frac{2v_1v_2}{v_2-v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx\right|\leq\frac{v_1v_2(v_2-v_1)}{\sqrt[p]{p+1}}\\&\times\left[\left(\mathfrak{T}_1|Q'(v_1)|^q+\mathfrak{T}_2|Q'(v_2)|^q\right)^{\frac{1}{q}}+\left(\mathfrak{T}_3|Q'(v_1)|^q+\mathfrak{T}_4|Q'(v_2)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 5. *Assume that $\rho=\sigma=\frac{1}{2}$ in inequality (19); then*

$$\begin{aligned}&\left|\frac{Q(v_1)+Q(v_2)}{2}+Q\left(\frac{2v_1v_2}{v_1+v_2}\right)-\frac{2v_1v_2}{v_2-v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx\right|\\&\leq v_1v_2(v_2-v_1)\sqrt[p]{\frac{4}{p+1}}\\&\times\left[\left(\mathfrak{T}_1|Q'(v_1)|^q+\mathfrak{T}_2|Q'(v_2)|^q\right)^{\frac{1}{q}}+\left(\mathfrak{T}_3|Q'(v_1)|^q+\mathfrak{T}_4|Q'(v_2)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 6. *Assume that $\rho=\sigma=\frac{1}{3}$ in inequality (19); then*

$$\begin{aligned}&\left|\frac{Q(v_1)+Q(v_2)}{2}+2\psi\left(\frac{2v_1v_2}{v_1+v_2}\right)-\frac{3v_1v_2}{v_2-v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx\right|\\&\leq 3v_1v_2(v_2-v_1)\sqrt[p]{4\left(\frac{\left(\frac{2}{3}\right)^{p+1}+\left(\frac{1}{3}\right)^{p+1}}{p+1}\right)}\\&\times\left[\left(\mathfrak{T}_1|Q'(v_1)|^q+\mathfrak{T}_2|Q'(v_2)|^q\right)^{\frac{1}{q}}+\left(\mathfrak{T}_3|Q'(v_1)|^q+\mathfrak{T}_4|Q'(v_2)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Corollary 7. *Assume that $\rho=\sigma=1$ in inequality (19); then*

$$\begin{aligned}&\left|\frac{Q(v_1)+Q(v_2)}{2}-\frac{v_1v_2}{v_2-v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx\right|\leq\frac{v_1v_2(v_2-v_1)}{\sqrt[p]{p+1}}\\&\times\left[\left(\mathfrak{T}_1|Q'(v_1)|^q+\mathfrak{T}_2|Q'(v_2)|^q\right)^{\frac{1}{q}}+\left(\mathfrak{T}_3|Q'(v_1)|^q+\mathfrak{T}_4|Q'(v_2)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 10. *Assume that $\rho,\sigma\in[0,1]$ and $Q:[v_1,v_2]\subseteq(0,+\infty)\to\mathbb{R}$ is a differentiable mapping such that $Q'\in L[v_1,v_2]$. Suppose $|Q'|^q$ is modified harmonic exp s-convex; then for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have*

$$\begin{aligned}&\left|\frac{\rho Q(v_1)+\sigma Q(v_2)}{2}+\frac{2-\rho-\sigma}{2}Q\left(\frac{2v_1v_2}{v_1+v_2}\right)-\frac{v_1v_2}{v_2-v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx\right|\\&\leq\frac{v_1v_2(v_2-v_1)}{4}\\&\times\left[\frac{4}{(v_1+v_2)^2}\left({}_2F_1\left(2p,1;2;\frac{v_1-v_2}{v_1+v_2}\right)\right)^{\frac{1}{p}}\left(C_5|Q'(v_1)|^q+C_6|Q'(v_2)|^q\right)^{\frac{1}{q}}\right.\\&\left.+\frac{1}{v_1^2}\left({}_2F_1\left(2p,1;2;\frac{v_1-v_2}{2v_1}\right)\right)^{\frac{1}{p}}\left(C_7|Q'(v_1)|^q+C_8|Q'(v_2)|^q\right)^{\frac{1}{q}}\right],\end{aligned}\tag{20}$$

where

$$C_5=\frac{1}{2m}\sum_{\eta=1}^{m}\int_0^1|1-\rho-\sigma|^q(e^{s(1-\lambda)}-1)^\eta d\lambda,$$

$$C_6=\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|1-\rho-\sigma|^q(e^{s(1+\lambda)}-1)^\eta d\lambda,$$

$$C_7=\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|\sigma-\lambda|^q(e^{s(2-\lambda)}-1)^\eta d\lambda,$$

$$C_8=\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|\sigma-\lambda|^q(e^{s\lambda}-1)^\eta d\lambda.$$

Proof. According to the Lemma 3, we have

$$\begin{aligned}&\left|\frac{\rho \mathrm{Q}(\mathrm{v}_1)+\sigma \mathrm{Q}(\mathrm{v}_2)}{2}+\frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right|\\&\leq\frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4}\left[\int_0^1\left|\frac{4(1-\rho-\lambda)}{((1-\lambda)\mathrm{v}_2+(1+\lambda)\mathrm{v}_1)^2}\right|\left|\mathrm{Q}'\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{(1-\lambda)\mathrm{v}_2+(1+\lambda)\mathrm{v}_1}\right)\right|d\lambda\right.\\&\left.+\int_0^1\left|\frac{4(\sigma-\lambda)}{(\lambda\mathrm{v}_1+(2-\lambda)\mathrm{v}_2)^2}\right|\left|\mathrm{Q}'\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\lambda\mathrm{v}_1+(2-\lambda)\mathrm{v}_2}\right)\right|d\lambda\right].\end{aligned}$$

Employing the property of Hölder's inequality and modified harmonic exp s-convex function, we have

$$\begin{aligned}&\left|\frac{\rho \mathrm{Q}(\mathrm{v}_1)+\sigma \mathrm{Q}(\mathrm{v}_2)}{2}+\frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right|\\&\leq\frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4}\left\{4\left(\int_0^1\frac{1}{A_{\mathrm{v}_1,\mathrm{v}_2}^{2p}}d\lambda\right)^{\frac{1}{p}}\right.\\&\times\left[\int_0^1|1-\rho-\sigma|^q\left(\tfrac{1}{2\mathrm{m}}\textstyle\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1-\lambda)}-1\right)^\eta|\mathrm{Q}'(\mathrm{v}_1)|^q+\tfrac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(1+\lambda)}-1\right)^\eta|\mathrm{Q}'(\mathrm{v}_2)|^q\right)d\lambda\right]^{\frac{1}{q}}\\&+4\left(\int_0^1\frac{1}{B_{\mathrm{v}_1,\mathrm{v}_2}^{2p}}d\lambda\right)^{\frac{1}{p}}\\&\left.\times\left[\int_0^1|\sigma-\lambda|^q\left(\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s(2-\lambda)}-1\right)^\eta|\mathrm{Q}'(\mathrm{v}_1)|^q+\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\left(e^{s\lambda}-1\right)^\eta|\mathrm{Q}'(\mathrm{v}_2)|^q\right)d\lambda\right]^{\frac{1}{q}}\right\}\\&=\frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4}\\&\times\left[\frac{4}{(\mathrm{v}_1+\mathrm{v}_2)^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\right)^{\frac{1}{p}}\left(C_5|\mathrm{Q}'(\mathrm{v}_1)|^q+C_6|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right.\\&\left.+\frac{1}{\mathrm{v}_1^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{2\mathrm{v}_1}\right)\right)^{\frac{1}{p}}\left(C_7|\mathrm{Q}'(\mathrm{v}_1)|^q+C_8|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right].\end{aligned}$$

This completes the proof. □

Corollary 8. *Assume that* $\mathrm{m} = 1$ *in inequality (20); then*

$$
\begin{aligned}
&\left|\frac{\rho \mathrm{Q}(\mathrm{v}_1) + \sigma \mathrm{Q}(\mathrm{v}_2)}{2} + \frac{2-\rho-\sigma}{2}\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right) - \frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right| \\
&\leq \frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4} \\
&\times \left[\frac{4}{(\mathrm{v}_1+\mathrm{v}_2)^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\right)^{\frac{1}{p}}\left(D_5|\mathrm{Q}'(\mathrm{v}_1)|^q + D_6|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \frac{1}{\mathrm{v}_1^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{2\mathrm{v}_1}\right)\right)^{\frac{1}{p}}\left(D_7|\mathrm{Q}'(\mathrm{v}_1)|^q + D_8|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
D_5 &= \frac{1}{2}\int_0^1 |1-\rho-\sigma|^q\left(e^{s(1-\lambda)}-1\right)d\lambda, \\
D_6 &= \frac{1}{2}\int_0^1 |1-\rho-\sigma|^q\left(e^{s(1+\lambda)}-1\right)d\lambda,
\end{aligned}
$$

$$
D_7 = \frac{1}{2}\int_0^1 |\sigma-\lambda|^q\left(e^{s(2-\lambda)}-1\right)d\lambda, \quad D_8 = \frac{1}{2}\int_0^1 |\sigma-\lambda|^q\left(e^{s\lambda}-1\right)d\lambda.
$$

Corollary 9. *Assume that* $\rho = \sigma$ *in inequality (20); then*

$$
\begin{aligned}
&\left|\rho\frac{\mathrm{Q}(\mathrm{v}_1) + \mathrm{Q}(\mathrm{v}_2)}{2} + (1-\rho)\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right) - \frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right| \\
&\leq \frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4} \\
&\times \left[\frac{4}{(\mathrm{v}_1+\mathrm{v}_2)^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\right)^{\frac{1}{p}}\left(E_1|\mathrm{Q}'(\mathrm{v}_1)|^q + E_2|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \frac{1}{\mathrm{v}_1^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{2\mathrm{v}_1}\right)\right)^{\frac{1}{p}}\left(E_3|\mathrm{Q}'(\mathrm{v}_1)|^q + E_4|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where

$$
\begin{aligned}
E_1 &= \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 |1-2\rho|^q\left(e^{s(1-\lambda)}-1\right)^\eta d\lambda, \\
E_2 &= \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 |1-2\rho|^q\left(e^{s(1+\lambda)}-1\right)^\eta d\lambda, \\
E_3 &= \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 |\rho-\lambda|^q\left(e^{s(2-\lambda)}-1\right)^\eta d\lambda, \\
E_4 &= \frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1 |\rho-\lambda|^q\left(e^{s\lambda}-1\right)^\eta d\lambda.
\end{aligned}
$$

Corollary 10. *Assume that* $\rho = \sigma = 0$ *in inequality (20); then*

$$
\begin{aligned}
&\left|\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right) - \frac{\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right| \leq \frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4} \\
&\times \left[\frac{4}{(\mathrm{v}_1+\mathrm{v}_2)^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\right)^{\frac{1}{p}}\left(E_5|\mathrm{Q}'(\mathrm{v}_1)|^q + E_6|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \frac{1}{\mathrm{v}_1^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{2\mathrm{v}_1}\right)\right)^{\frac{1}{p}}\left(E_7|\mathrm{Q}'(\mathrm{v}_1)|^q + E_8|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
$$

where

$$E_5=\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1(e^{s(1-\lambda)}-1)^{\eta}d\lambda,$$

$$E_6=\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1(e^{s(1+\lambda)}-1)^{\eta}d\lambda,$$

$$E_7=\frac{1}{2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1\lambda^q(e^{s(2-\lambda)}-1)^{\eta}d\lambda,$$

$$E_8=\frac{1}{2n}\sum_{\eta=1}^{\mathrm{m}}\int_0^1\lambda^q(e^{s\lambda}-1)^{\eta}d\lambda.$$

Corollary 11. *Assume that $\rho=\sigma=\frac{1}{2}$ in inequality (20); then*

$$\begin{aligned}&\left|\frac{\mathrm{Q}(\mathrm{v}_1)+\mathrm{Q}(\mathrm{v}_2)}{2}+\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right|\leq\frac{\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{2}\\&\times\left[\frac{1}{\mathrm{v}_1^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{2\mathrm{v}_1}\right)\right)^{\frac{1}{p}}\left(E_9|\mathrm{Q}'(\mathrm{v}_1)|^q+E_{10}|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$E_9=\frac{1}{2^{q+1}\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|1-2\lambda|^q(e^{s(2-\lambda)}-1)^{\eta}d\lambda,$$

$$E_{10}=\frac{1}{2^{q+1}\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|1-2\lambda|^q(e^{s\lambda}-1)^{\eta}d\lambda.$$

Corollary 12. *Assume that $\rho=\sigma=\frac{1}{3}$ in inequality (20); then*

$$\begin{aligned}&\left|\frac{\mathrm{Q}(\mathrm{v}_1)+\mathrm{Q}(\mathrm{v}_2)}{2}+2\mathrm{Q}\left(\frac{2\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)-\frac{3\mathrm{v}_1\mathrm{v}_2}{\mathrm{v}_2-\mathrm{v}_1}\int_{\mathrm{v}_1}^{\mathrm{v}_2}\frac{\mathrm{Q}(x)}{x^2}dx\right|\leq\frac{3\mathrm{v}_1\mathrm{v}_2(\mathrm{v}_2-\mathrm{v}_1)}{4}\\&\times\left[\frac{4}{(\mathrm{v}_1+\mathrm{v}_2)^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{\mathrm{v}_1+\mathrm{v}_2}\right)\right)^{\frac{1}{p}}\left(G_1|\mathrm{Q}'(\mathrm{v}_1)|^q+G_2|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right.\\&\left.+\frac{1}{\mathrm{v}_1^2}\left({}_2F_1\left(2p,1;2;\frac{\mathrm{v}_1-\mathrm{v}_2}{2\mathrm{v}_1}\right)\right)^{\frac{1}{p}}\left(G_3|\mathrm{Q}'(\mathrm{v}_1)|^q+G_4|\mathrm{Q}'(\mathrm{v}_2)|^q\right)^{\frac{1}{q}}\right],\end{aligned}$$

where

$$G_1=\frac{1}{3^q2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1(e^{s(1-\lambda)}-1)^{\eta}d\lambda,$$

$$G_2=\frac{1}{3^q2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1(e^{s(1+\lambda)}-1)^{\eta}d\lambda,$$

$$G_3=\frac{1}{3^q2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|1-3\lambda|^q(e^{s(2-\lambda)}-1)^{\eta}d\lambda,$$

$$G_4=\frac{1}{3^q2\mathrm{m}}\sum_{\eta=1}^{\mathrm{m}}\int_0^1|1-3\lambda|^q(e^{s\lambda}-1)^{\eta}d\lambda.$$

Corollary 13. *Assume that $\rho = \sigma = 1$ in inequality (20); then*

$$\left|\frac{Q(v_1)+Q(v_2)}{2} - \frac{v_1 v_2}{v_2 - v_1}\int_{v_1}^{v_2}\frac{Q(x)}{x^2}dx\right| \le \frac{v_1 v_2(v_2 - v_1)}{4}$$
$$\times\left[\frac{4}{(v_1+v_2)^2}\left({}_2F_1\left(2p,1;2;\frac{v_1-v_2}{v_1+v_2}\right)\right)^{\frac{1}{p}}\left(G_5|Q'(v_1)|^q + G_6|Q'(v_2)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\frac{1}{v_1^2}\left({}_2F_1\left(2p,1;2;\frac{v_1-v_2}{2v_1}\right)\right)^{\frac{1}{p}}\left(G_7|Q'(v_1)|^q + G_8|Q'(v_2)|^q\right)^{\frac{1}{q}}\right],$$

where

$$G_5 = \frac{1}{2m}\sum_{\eta=1}^{m}\int_0^1 (e^{s(1-\lambda)} - 1)^\eta d\lambda,$$

$$G_6 = \frac{1}{2m}\sum_{\eta=1}^{m}\int_0^1 (e^{s(1+\lambda)} - 1)^\eta d\lambda,$$

$$G_7 = \frac{1}{2m}\sum_{\eta=1}^{m}\int_0^1 |1-\lambda|^q (e^{s(2-\lambda)} - 1)^\eta d\lambda,$$

$$G_8 = \frac{1}{2m}\sum_{\eta=1}^{m}\int_0^1 |1-\lambda|^q (e^{s\lambda} - 1)^\eta d\lambda.$$

6. Conclusions

The study of integral inequalities in association with convex analysis presents an intriguing and stimulating area of study in the domain of mathematical interpretation. Due to their pivotal role and beneficial importance in many disciplines of science, the subject of inequalities has been described as an attractive field for mathematicians. Many mathematicians try to use and employ new ideas in order to advance the theory of inequalities. A great framework for starting and creating numerical tools for solving and researching challenging mathematical problems is provided by the word inequalities. This work has shown a new variant of Hadamard inequalities involving a new family of convex functions, namely the modified harmonic exp s-convex function. A new class of these functions has been investigated by introducing some algebraic properties. The new family of modified harmonic exp s-convex functions is an extended and generalized class of functions, including convex and harmonically convex functions, which have been proved. Furthermore, the new type of Hadamard-type inequality and its estimations have been achieved. Many researchers add efforts to the term inequality hypotheses to reveal a new dimension of applied analysis because working on this hypothesis has its own importance and wide scope. It is a fascinating and engrossing field of research for researchers. Now is the time to explore the significance of convex analysis and inequalities along with innovative numerical techniques.

Author Contributions: Conceptualization, A.A.S., E.H., S.K.N., J.T. and M.T.; methodology, A.A.S., E.H., S.K.N., J.T. and M.T.; validation, A.A.S., E.H., S.K.N., J.T. and M.T.; formal analysis, A.A.S., E.H., S.K.N., J.T. and M.T.; writing—original draft preparation, A.A.S., E.H., S.K.N., J.T. and M.T. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by National Science, Research and Innovation Fund (NSRF) and King Mongkut's University of Technology North Bangkok with Contract no. KMUTNB-FF-66-11.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Niculescu, C.P.; Persson, L.E. *Convex Functions and Their Applications*; Springer: New York, NY, USA, 2006.
2. Özdemir, M.E.; Yildiz, C.; Akdemir, A.O.; Set, E. On some inequalities for s–convex functions and applications. *J. Inequal. Appl.* **2013**, *333*, 2–11. [CrossRef]
3. Butt, S.I.; Rashid, S.; Tariq, M.; Wang, X.H. Novel refinements via n-polynomial harmonically s-type convex functions and Applications in special functions. *J. Funct. Spaces* **2021**, *2021*, 6615948. [CrossRef]
4. Ali, S.; Mubeen, S.; Ali, R.S.; Rahman, G.; Morsy, A.; Nisar, K.S.; Purohit, S.D.; Zakarya, M. Dynamical significance of generalized fractional integral inequalities via convexity. *AIMS Math.* **2021**, *6*, 9705–9730. [CrossRef]
5. Xi, B.Y.; Qi, F. Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means. *J. Funct. Spaces***2012**, *2012*, 980438. [CrossRef]
6. Zhang, X.M.; Chu, Y.M.; Zhang, X.Y. The Hermite–Hadamard type inequality of GA–convex functions and its applications. *J. Inequal. Appl.* **2010**, *2010*, 507560. [CrossRef]
7. Tariq, M. New Hermite–Hadamard type inequalities via p–harmonic exponential type convexity and applications. *Univ. J. Math. Appl.* **2021**, *4*, 59–69.
8. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Nasir, J.; Aydi, H.; Mukheimer, A. New Ostrowski-type fractional integral inequalities via generalized exponential-type convex functions and applications. *Symmetry* **2021**, *13*, 1429. [CrossRef]
9. Butt, S.I.; Pecaric, J.; Rehman, A.U. Exponential convexity of Petrovic and related functional. *J. Inequal. Appl.* **2011**, *89*, 2011. [CrossRef]
10. Zhou, S.S.; Rashid, S.; Noor, M.A.; Noor, K.I.; Safdar, F.; Chu, Y.M. New Hermite-Hadamard type inequalities for exponentially convex functions and applications. *AIMS Math.* **2020**, *5*, 6874–6901. [CrossRef]
11. Omotoyinbo, O.; Mogbodemu, A. Some new Hermite–Hadamard integral inequalities for convex functions. *Int. J. Sci. Innov. Tech.* **2014**, *1*, 1–12.
12. Tariq, M.; Nasir, J.; Sahoo, S.K.; Mallah, A.A. A note on some Ostrowski-type inequalities via generalized exponentially convex function. *J. Math. Anal. Model.* **2021**, *2*, 2021. [CrossRef]
13. Butt, S.I.; Tariq, M.; Aslam, A.; Ahmad, H.; Nofel, T.A. Hermite–Hadamard type inequalities via generalized harmonic exponential convexity. *J. Funct. Spaces* **2021**, *2021*, 5533491. [CrossRef]
14. Rafiq, A.; Mir, N.A.; Ahmad, F. Weighted Chebysev–Ostrowski type inequalities. *Appl. Math. Mech.* **2007**, *28*, 901–906. [CrossRef]
15. Khan, M.A.; Chu, Y.M.; Khan, T.U. Some new inequalities of Hermite–Hadamard type for s–convex functions with applications. *Open Math.* **2017**, *15*, 1414–1430. [CrossRef]
16. Nasir, J.; Qaisar, S.; Butt, S.I.; Aydi, H.; Sen, M.D.L. Hermite-Hadamard like inequalities for fractional integral operator via convexity and quasi-convexity with their applications. *AIMS Math.* **2022**, *7*, 3418–3439. [CrossRef]
17. Aljaaidia, T.A.; Pachpatte, D. New generalization of reverse Minkowski's inequality for fractional integral. *Adv. Theory. Nonlinear Anal. Appl.* **2021**, *1*, 72–81.
18. Aljaaidia, T.A.; Pachpatte, D. Reverse Hermite-Hadamard's inequalities using Q-fractional integral. *Eng. Appl. Sci. Lett.* **2020**, *3*, 75–84. [CrossRef]
19. Aljaaidi, T.A.; Pachpatte, D.B.; Shatanawi, W.; Abdo, M.S.; Abodayeh, K. Generalized proportional fractional integral functional bounds in Minkowski's inequalities. *Adv. Differ. Equ.* **2021**, *2021*, 419.
20. Toplu, T.; Kadakal, M.; İşcan, Í. On n–polynomial convexity and some related inequalities. *AIMS Math.* **2020**, *5*, 1304–1318. [CrossRef]
21. Shi, H.N.; Zhang, J. Some new judgement theorems of Schur geometric and schur harmonic convexities for a class of symmetric function. *J. Inequal. Appl.* **2013**, *2013*, 527. [CrossRef]
22. Anderson, G.D.; Vamanamurthy, M.K.; Vuorinen, M. Generalized convexity and inequalities. *J. Math. Anal. Appl.* **2007**, *335*, 1294–1308. [CrossRef]
23. Awan, M.U.; Akhtar, N.; Iftikhar, S.; Noor, M.A.; Chu, Y.-M. New Hermite–Hadamard type inequalities for n–polynomial harmonically convex functions. *J. Inequal. Appl.* **2020**, *2020*, 125. [CrossRef]
24. Noor, M.A.; Noor, K.I. Harmonic variational inequalities. *Appl. Math. Inf. Sci.* **2016**, *10*, 1811–1814. [CrossRef]
25. Noor, M.A.; Noor, K.I. Some implicit methods for solving harmonic variational inequalities. *Inter. J. Anal. Appl.* **2016**, *12*, 10–14.
26. Dragomir, S.S.; Gomm, I. Some Hermite–Hadamard's inequality functions whose exponentials are convex. *Babes Bolyai Math.* **2015**, *60*, 527–534.
27. Awan, U.; Akhtar, N.; Iftikhar, S.; Noor, M.A.; Chu, Y.M. Hermite–Hadamard type inequalities for exponentially convex functions. *Appl. Math. Inf. Sci.* **2018**, *12*, 405–409. [CrossRef]
28. Kadakal, M.; İşcan, İ. Exponential type convexity and some related inequalities. *J. Inequal. Appl.* **2020**, *2020*, 82. [CrossRef]
29. Alomari, M.; Darus, M.; Kirmaci, U.S. Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means. *Comput. Math. Appl.* **2010**, *59*, 225–232. [CrossRef]
30. Chen, F.X.; Wu, S.H. Several complementary inequalities to inequalities of Hermite–Hadamard type for s–convex functions. *J. Nonlinear Sci. Appl.* **2016**, *9*, 705–716. [CrossRef]
31. Samraiz, M.; Saeed, K.; Naheed, S.; Rahman, G.; Nonlaopon, K. On inequalities of Hermite-Hadamard type via n-polynomial exponential type s-convex functions. *AIMS Math.* **2022**, *7*, 14282–14298. [CrossRef]

32. Kashuri, A.; Liko, R. Some new Hermite–Hadamard type inequalities and their applications. *Stud. Sci. Math. Hung.* **2019**, *56*, 103–142. [CrossRef]
33. Set, E.; Ozdemir, M.E.; Sarikaya, M.Z. New inequalities of Ostrowski's type for s-convex functions in the second sense with applications. *Facta Univ. (Nis) Ser. Math.* **2012**, *27*, 67–82.
34. İşcan, İ. Hermite–Hadamard type inequalities for harmonically convex functions. *Hacet. J. Math. Stat.* **2014**, *43*, 935–942. [CrossRef]
35. Tariq, M.; Sahoo, S.K.; Nasir, J.; Aydi, H.; Alsamir, H. Some Ostrowski type inequalities via npolynomial exponentially s-convex functions and their applications. *AIMS Math.* **2021**, *6*, 13272–13290. [CrossRef]
36. Geo, W.; Kashuri, A.; Butt, S.I.; Tariq, M.; Aslam, A.; Nadeem, M. New inequalities via *n*-polynomial harmonically exponential type functions. *AIMS Math.* **2020**, *5*, 6856–6873. [CrossRef]
37. Mitrinovic, D.S.; Pećaric, J.E.; Fink, A.M. *Classical and New Inequalities in Analysis*; Kluwer Academic: Dordrecht, The Netherlands, 1993.
38. Baloch, I.A.; Sen, M.D.L.; İşcan, İ. Characterizations of classes of harmonic convex functions and applications. *Int. J. Anal. Appl.* **2019**, *17*, 722–733.
39. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions with Formulas. Graphs, and Mathematical Tables*; Dover: New York, NY, USA, 1965.