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Existence and Qualitative Properties of Solution for a Class of Nonlinear Wave Equations with Delay Term and Variable-Exponents Nonlinearities

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Abstract: This article is devoted to a study of the question of existence (in time) of weak solutions and the derivation of qualitative properties of such solutions for the nonlinear viscoelastic wave equation with variable exponents and minor damping terms. By using the energy method combined with the Faedo–Galerkin method, the local and global existence of solutions are established. Then, the stability estimate of the solution is obtained by introducing a suitable Lyapunov function.

Keywords: Viscoelastic wave equation; Faedo–Galerkin method; local solution; global solution; exponential decay; Lyapunov function

MSC: 35B40; 35L70

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n, n \in \mathbb{N}$, with a smooth boundary $\partial\Omega = \Gamma$. For $x \in \Omega, t \in (0, \infty)$, we consider the following BVP:

$$\begin{cases} \partial_{tt}(u + \Delta_x u) - \Delta_x \left(u - \int\limits_0^t h(t-\tau) u(\tau) d\tau \right) & \\ +\mu_1 \partial_t u + \int\limits_{\tau_1}^{\tau_2} \mu_2(s) \partial_t u(t-s) ds = b|u|^{p(x)-2} u, & \text{in } \Omega \times (0, \infty) \\ u(x,t) = 0, & \text{in } \Gamma \times (0, \infty) \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), & \text{in } \Omega \\ \partial_t u(x,-t) = f_0(x,t), & \text{in } \Omega \times (0, \tau_2), \end{cases} \quad (1)$$

where $\rho > 0, \mu_1, b$ is a positive real number and h is a positive non-increasing function defined on $\mathbb{R}_+$. The values (u_0, u_1, f_0) are initial data belonging to a suitable function space.

Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real numbers that satisfy $0 \leq \tau_1 \tau_2$. The exponent $p(.)$ is given a measurable function on Ω satisfying

$$2 \leq p_1 \leq p(x) \leq p_2 < \infty,$$

with

$$p_1 = ess \inf_{x\in\Omega} p(x), \qquad p_2 = ess \sup_{x\in\Omega} p(x),$$

we also assume that $p(.)$ is log-continuous in Ω such that

$$\forall(a,b) \in \Omega^2, \ |p(a) - p(b)| \leq -\frac{C}{log|a-b|}, \quad \text{with } |a-b| < \delta, \tag{2}$$

where $C > 0, \ 0 < \delta < \frac{1}{2}$.

We can consider the Equation (1) as a generalization of a viscoelastic equation

$$\partial_{tt}u - \Delta_x\Big(u - \int_0^t h(t-\tau)u(\tau)d\tau\Big) + \mu_1 g_1(\partial_t u) + \mu_2 g_2(\partial_t u(t-\tau)) = 0, \tag{3}$$

for $x \in \Omega$ and $t > 0$, when h is of a general decay rate and g_1, g_2 are non-linear functions. The existence of global solutions and decay estimates has been discussed by Benaissa et al. in [1].

Mustafa and Kafini [2] have discussed the following problem:

$$\begin{cases} \partial_{tt}u - \Delta_x^2 u - \int_0^t h(t-\tau)\Delta_x^2 u(\tau)d\tau + \mu_1\partial_t u & \\ + \int_{\tau_1}^{\tau_2} \mu_2(s)\partial_t u(t-s)ds = u|u|^\gamma, & \text{in } \Omega \times (0,\infty) \\ u = \frac{\partial u}{\partial \nu} = 0 & \text{in } \partial\Omega \times]0,+\infty[, \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), & \text{in } \Omega \\ \partial_t u(x,-t) = f_0(x,t), & \text{in } \Omega \times (0,\tau_2), \end{cases} \tag{4}$$

here μ_1, μ_2 and f_0, as stated above under suitable conditions on the delay and source terms, established an explicit and general decay rate result without imposing restrictive assumptions on the behavior of the relaxation function at infinity.

Recently, many authors studied the existence and nonexistence of solutions for problems with variable exponents.

Messaoudi et al. in [3] used the Faedo–Galerkin method to find the existence of a weak local solution of the following equation:

$$\partial_{tt}u - \Delta_x u + a|\partial_t u|^{m(x)-2}\partial_t u = b|u|^{p(x)-2}u.$$

Alaoui et al. [4] proposed the related system

$$\partial_t u - div\Big(|\nabla_x u|^{m(x)-2}\nabla_x u\Big) = |u|^{p(x)-2}u + f, \qquad \text{in } \Omega \times (0,T),$$

where ω is a bounded domain in $\mathbb{R}^n$, $n > 1$, with a smooth boundary $\partial\Omega$. Under suitable conditions on m and p and for $f = 0$, they showed that any solution with a nontrivial initial datum blows up in finite time.

Our article is structured as follows. In Section 2, we describe our system and review several pertinent features and definitions pertaining to fractional Sobolev spaces. In Section 3, we discuss the local and global existence of solutions for Problem (1). As we will see, Section 4 will concentrate on decay estimates for solutions to the issue.

2. Preliminaries

Here, we state the results related with Lebesgue $L^{p(.)}(\Omega)$ and Soblev $W^{1,p(.)}(\Omega)$ spaces with variable exponents (see [5–10]). Let $p : \Omega \longrightarrow [1, \infty)$ be a measurable function. The variable exponent Lebesgue space with $p(.)$ is defined by

$$L^{p(x)}(\Omega) = \left\{ w : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_\Omega |w|^{p(x)} dx < \infty \right\},$$

equipped with a Luxemburg-type norm

$$\|u\|_{L^{p(x)}(\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u}{\lambda} \right|^{p(x)} dx \leq 1 \right\},$$

the space $L^{p(.)}(\Omega)$ is a Banach space (see [9]).

Next, we define the variable-exponent Sobolev space $W^{1,p(.)}(\Omega)$ as the following:

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega), \nabla_x u \in L^{p(x)}(\Omega) \right\},$$

equipped with the norm

$$\|u\|_{1,p(x)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla_x u\|_{L^{p(x)}(\Omega)},$$

is a Banach space. $W_0^{1,p(B)}(\Omega)$ is the space, which is defined as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(.)}(\Omega)$. For $u \in W_0^{1,p(.)}$ we can define an equivalent norm

$$\|u\|_{1,p(.)} = \|\nabla_x u\|_{p(.)},$$

the dual of $W_0^{1,p(x)}(\Omega)$ is defined as $W^{-1,p(x)}(\Omega)$, similar to Sobolev spaces, where

$$\frac{1}{p(.)} + \frac{1}{p'(.)} = 1.$$

We also assume that

$$|p(a) - p(b)| \leq \frac{A}{|\log(|a-b|)|}, \forall a, b \in \bar{\Omega}, \text{ such that } |a-b| < \frac{1}{2}, \tag{5}$$

for all $a, b \in \Omega$, $A > 0$ and $0 < \delta < 1$ with $|a - b| < \delta$ (log-Hölder condition).

Lemma 1 (Poincaré's inequality [5])**.** *Let Ω be a bounded domain of $\mathbb{R}^n$ and suppose that $p(.)$ satisfies (5). Then,*

$$\|u\|_{p(.)} \leq c(\Omega)\|\nabla_x u\|_{p(.)}, \qquad \forall u \in W_0^{1,p(.)}(\Omega), \tag{6}$$

where $c = c(p_1, p_2, |\Omega|) > 0$.

Next, we have a Sobolev–Poincaré's inequality.

Lemma 2 (Sobolev–Poincaré's inequality)**.** *Let q be a number with*

$$2 \leq q < \infty (n = 1, 2), 2 \leq q \leq \frac{2n}{n-2} (n \geq 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \leq c\|\nabla_x u\|_{L^2}, \ for \ u \in H_0^1(\Omega). \tag{7}$$

Lemma 3 ([5]). *If $p : \overline{\Omega} \longrightarrow [1, \infty)$ is continuous,*

$$2 \leq p_1 \leq p(x) \leq p_2 \leq \frac{2n}{n-2}, \qquad n \geq 3, \tag{8}$$

satisfies, then the embedding $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is continuous.

Lemma 4 ([5]). *If $p_2 < \infty$ and $p : \overline{\Omega} \longrightarrow [1, \infty)$ is a measurable function, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.*

Lemma 5 (Hölder's inequality [5]). *Let p, q, s be measurable functions defined on Ω and*

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)}, \qquad \text{for a.e } y \in \Omega.$$

If $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{q(\cdot)}(\Omega)$, then

$$\|f.g\|_{s(.)} \leq \|f\|_{p(.)} \|g\|_{q(.)}.$$

Lemma 6 ([5]). *If $p \geq 1$ is a measurable function on Ω, then*

$$\min\left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\} \leq \rho_{p(.)}(u) \leq \max\left\{ \|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2} \right\},$$

for any $u \in L^{p(\cdot)}(\Omega)$ and for a.e. $x \in \Omega$.

Lemma 7 (Gronwall's inequality). *Let $C > 0, u(t)$ and $y(t)$ be continuous non-negative functions defined for $0 \leq t < \infty$ satisfying the inequality*

$$u(t) \leq C + \int_0^t u(s) y(r) dr,$$

then

$$u(t) \leq C \, \exp\left(\int_0^t y(r) dr \right).$$

Lemma 8 (Modified Gronwall's inequality). *Let u and h be continuous non-negative functions defined for $0 \leq t < \infty$ satisfying the inequality*

$$0 \leq u(t) \leq C + \int_0^t u(s) h(r) dr,$$

with $C > 0$

$$u(t) \leq \left(C^{-r} - r \int_0^t h(r) dr \right)^{\frac{-1}{r}}.$$

We have the following assumptions:

(A**1**) The relaxation function $h : R_+ \to R_+$ is a bounded function of C^1 so that

$$\int_0^\infty h(\tau) d\tau = \beta < 1 \text{ and } 1 - \int_0^\infty h(\tau) d\tau = l, \; h(0) > 0, \tag{9}$$

and we suppose that there exists a positive constant ς to satisfy

$$h'(t) \leq -\varsigma h(t). \tag{10}$$

(A**2**) We assume

$$\int_{\tau_1}^{\tau_2} |\mu_2| ds < \mu_1.$$

Let ζ be a positive constant that satisfies

$$\int_{\tau_1}^{\tau_2} |\mu_2| ds + \frac{\zeta(\tau_2 - \tau_1)}{2} < \mu_1. \tag{11}$$

Lemma 9. *For $h, \Psi \in C^1([0, +\infty[, R)$ we have*

$$\begin{aligned} \int_\Omega \int_0^t h(t-s)\Psi(r) dr \partial_t \Psi dx = \frac{1}{2} h(t) \|\Psi(t)\|_{L^2}^2 - \frac{1}{2}(h' \circ \Psi) \\ + \frac{1}{2} \frac{d}{dt}\left[(h \circ \Psi) - \left(\int_0^t h(r) dr\right) \|\Psi\|_{L^2}^2\right], \end{aligned}$$

where

$$(h \circ u) = \int_\Omega \left(\int_0^t h(t-s)(u(t) - u(s))^2 ds\right) dx.$$

Lemma 10. *Suppose that h satisfies (A1). Then, for $u \in H_0^1(\Omega)$, we obtain*

$$\int_\Omega \left(\int_0^t h(t-s)(u(t) - u(s)) ds\right)^2 dx \le c(h \circ \nabla_x u), \tag{12}$$

and

$$\int_\Omega \left(\int_0^t h'(t-s)(u(t) - u(s)) ds\right)^2 dx \le -c(h' \circ \nabla_x u). \tag{13}$$

3. Statement of the Existence Results with Their Proofs

3.1. Reformulate the Problem

Firstly, we introduce, similar to [11], the new variable

$$z(x, \rho, s, t) = \partial_t u(x, t - \rho s), (x, \rho, s, t) \in \Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty),$$

which implies that

$$s \partial_t z(x, \rho, s, t) + z_\rho(x, \rho, s, t) = 0 \; in \; \Omega \times (0,1) \times (\tau_1, \tau_2) \times (0, \infty).$$

Hence, Problem (1) can be transformed as follows:

$$\begin{cases} \partial_{tt} u - \Delta_x u - \Delta_x \partial_{tt} u - \int\limits_0^t h(t-\tau) \Delta_x u(x,\tau) ds & \\ +\mu_1 \partial_t u + \int\limits_{\tau_1}^{\tau_2} \mu_2(s) z(x,t,1,s,t) dt = b|u|^{p(x)-2} u, & \text{in } \Omega \times (0,\infty) \\ s \partial_t z(x,\rho,s,t) + z_\rho(x,\rho,s,t) = 0, & \text{in } \Omega \times (0,1) \times (\tau_1,\tau_2) \times (0,\infty) \\ u = 0, & \text{on } \Gamma \times]0,\infty[, \\ z(x,0,s,t) = \partial_t u(x,t), & \text{in } \Omega \times (0,\infty) \\ u(x,0) = u_0(x), \partial_t u(x,0) = u_1(x), & \text{in } \Omega \\ z(x,\rho,s,0) = f_0(x,\rho s), & \text{in } \Omega \times (0,1) \times (0,\tau_2). \end{cases} \tag{14}$$

The Lyapunov functional of solution for (14) is defined by

$$\mathcal{E}(t) = \frac{1}{2}\|\partial_t u\|_{L^2}^2 + \frac{1}{2}\|\nabla_x \partial_t u\|_{L^2}^2 + \frac{1}{2}\left(1 - \int_0^t h(\tau)d\tau\right)\|\nabla_x u\|_{L^2}^2 + \frac{1}{2}(h \circ \nabla_x u)$$
$$- \int_\Omega \frac{b}{p(x)}|u|^{p(x)} + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\mu_2(s) + \zeta)z^2(x,\rho,s,t)dsd\rho dx. \tag{15}$$

Lemma 11. *Assume that (u,z) is a solution of Problem (14) and suppose that $(A1) - (A2)$ are verified. Then, $\mathcal{E}(t)$, defined by (15), satisfies.*

$$\mathcal{E}'(t) \leq -\left(\mu_1 - \int_{\tau_1}^{\tau_2}|\mu_2(s)|ds - \frac{\zeta(\tau_2 - \tau_1)}{2}\right)\|\partial_t u\|_{L^2}^2$$
$$- \frac{\zeta}{2}\int_\Omega \int_{\tau_2}^{\tau_2} z^2(x,1,s,t)dsdx - \frac{1}{2}h(t)\|\nabla_x u\|_{L^2}^2 + \frac{1}{2}(h' \circ \nabla_x u)$$
$$\leq 0, \quad \forall t \geq 0. \tag{16}$$

Proof. We multiply $(14)_1$ by $\partial_t u$ and integrate over Ω and use the integration by parts, and we obtain

$$\frac{d}{dt}\left[\frac{1}{2}\|\partial_t u\|_{L^2}^2 + \frac{1}{2}\|\nabla_x \partial_t u\|_{L^2}^2 + \frac{1}{2}\|\nabla_x u\|_{L^2}^2 - \int_\Omega \frac{b}{p(x)}|u|^{p(x)}\right]$$
$$+ \mu_1\|\partial_t u\|_{L^2}^2 + \int_{\tau_1}^{\tau_2}\mu_2(s)\int_\Omega z(x,1,s,t)\partial_t u dsdx$$
$$= \int_\Omega \int_0^t h(t-\tau)\nabla_x u(\tau)\nabla_x \partial_t u d\sigma dx. \tag{17}$$

Owing to Lemma 9, the RHS of (17) can be rewritten as

$$\int_\Omega \int_0^t h(t-\tau)\nabla_x u(\tau)\nabla_x \partial_t u d\sigma dx$$
$$= \frac{1}{2}\frac{d}{dt}\left[\int_0^t h(\tau)\|\nabla_x u(\tau)\|_{L^2}^2 - (h \circ \nabla_x u)\right]$$
$$- \frac{1}{2}h(t)\|\nabla_x u\|_{L^2}^2 + \frac{1}{2}(h' \circ \nabla_x u). \tag{18}$$

Utilizing Young's inequality, we have

$$\int_{\tau_1}^{\tau_2}\mu_2(s)\int_\Omega z(x,1,s,t)\partial_t u dsdx$$
$$\leq \frac{1}{2}\left(\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right)\int_\Omega \partial_t u^2 dx + \frac{1}{2}\int_\Omega \int_{\tau_1}^{\tau_2}|\mu_2(s)|\left|z^2(x,1,s,t)\right|dsdx. \tag{19}$$

Multiplying $(14)_2$ by $(|\mu_2(s)|+\zeta)z$ and integrating over $\Omega\times(0,1)\times(\tau_1,\tau_2)$ with respect to ρ, x and s, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(|\mu_2(s)|+\zeta)z^2(x,\rho,s,t)dsd\rho dx\\
&=-\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\zeta)\frac{\partial}{\partial\rho}z^2(x,\rho,s,t)dsd\rho dx\\
&=\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\zeta)\int_\Omega \partial_t u^2(x,t)dsdx\\
&-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\zeta)z^2(x,1,s,t)dsdx\\
&=\frac{1}{2}\left[\zeta(\tau_2-\tau_1)+\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\right]\int_\Omega \partial_t u^2(x,t)dx\\
&-\frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\zeta)z^2(x,1,s,t)dsdx. \qquad (20)
\end{aligned}$$

By combining (17)–(20) and using (9)–(11) give (16), which concludes the proof. □

3.2. Local Existence

We prove the existence of the local solution to the Problem (14).

Theorem 1. *Let $u_0\in H_0^1(\Omega), u_1\in L_2(\Omega)$ and $f_0\in H_0^1(\Omega, H^1(0,1))$ satisfies the compatibility condition*

$$f_0(.,0)=u_{1.}$$

Suppose that $(A1)-(A2)$ hold, hence the Problem (14) has a weak solution

$$\begin{aligned}
&u\in L^\infty\left(\mathbb{R}_+; H_0^1(\Omega)\right),\\
&\partial_t u\in L^\infty\left(\mathbb{R}_+; H_0^1(\Omega)\right), \qquad (21)\\
&\partial_{tt} u\in L^2\left(\mathbb{R}_+; H_0^1(\Omega)\right).
\end{aligned}$$

Proof. To prove Theorem 1, we need the local existence of the solution of the following related hyperbolic equation:

$$\begin{aligned}
&(\partial_{tt}u,\varphi)+(\nabla_x u,\nabla_x\varphi)+(\nabla_x\partial_{tt}u,\nabla_x\varphi)+\mu_1(\partial_t u,\varphi)\\
&-\int_0^t h(t-s)(\nabla_x u(s),\nabla_x\varphi)+\int_0^t \mu_2(s)(z(x,1,s,t),\varphi)ds\\
&=b\int_0^t\left(u|u|^{p(x)-1},\varphi\right), \qquad (22)
\end{aligned}$$

and

$$z(x,0,s,t)=\partial_t u(x,t). \qquad (23)$$

So, we start to prove the local solution of (14).

We shall use the standard of Faedo–Galerkin method to assured the existence of the local solution.

Introducing the sequence functions (φ_j) having the following properties:

- $\forall j\in\{1,\ldots,m\}, \varphi_j\in\mathcal{V}^{p(x)}$,
- The family $\{\varphi_1,\varphi_2,\ldots,\varphi_k\}$ is linearly independent,
- The space $\mathcal{V}_k=[\varphi_j]_{1\le j\le m}$ generated by the family, $\{\varphi_1,\varphi_2,\ldots,\varphi_k\}$, is dense in $\mathcal{V}^{p(x)}$.

Let $u_k = u_k(t)$ be an approached solution of the Problem (14) such that for all $1 \leq j \leq k$, the sequence $\phi^j(x,\rho)$ as follows:
$$\phi^j(x,0) = w^j.$$

We extend $\phi^j(x,0)$ by $\phi^j(x,\rho)$ over $L^2(\Omega \times (0,1))$ such that $(\phi^j)_{1\leq j\leq k}$ forms a basis of $L^2(\Omega) \times H^1(0,1)$ and show Z_k the sequence generated by $\left\{\phi^k\right\}$. We may be construct approximate solutions $(u^k, z^k), k = 1,2,\ldots$ in the form

$$u^k(t) = \sum_{i=1}^{k} \eta^{jk}(t)\varphi^j, z^k(t) = \sum_{i=1}^{k} c^{jk}(t)\phi^j,\, k = 1,2,\ldots,$$

satisfy the system of equations

$$\begin{aligned}
&\left(\partial_{tt}u^k, \varphi^j\right) + \left(\nabla_x u^k, \nabla_x u\varphi^j\right) + \left(\nabla_x \partial_{tt}u^k, \nabla_x u\varphi^j\right) + \mu_1\left(\partial_t u^k, \varphi^j\right) \\
&- \int_0^t h(t-s)\left(\nabla_x u^k(s), \nabla_x u\varphi^j\right) + \int_0^t \mu_2(s)\left(z^k(x,1,s,t), \varphi^j\right)ds \\
&= b\int_0^t \left(u^k|u^k|^{p(x)-1}, \varphi^j\right),
\end{aligned} \tag{24}$$

and
$$z^k(x,0,s,t) = \partial_t u^k(x,t), \tag{25}$$

which is a nonlinear system of ordinary differential equations and will be completed by the following initial conditions:

$$u^k(x,0) = u_0^k = \sum_{i=1}^{k} \omega^{jk}(t)\varphi^j \to u_0 \;\text{ when }\; k \to \infty \text{ in } H_0^1(\Omega), \tag{26}$$

and
$$\partial_t u^k(x,0) = u_1^k = \sum_{i=1}^{k} \chi^{jk}(t)\varphi_j \to u_1 \;\text{ when } k \to \infty \text{ in } L^2(\Omega), \tag{27}$$

$$\left(s\partial_t z^k + z_\rho^k, \phi\right) = 0, \quad 0 \leq j \leq k, \tag{28}$$

$$z^k(0,\rho,s,0) = z_0^k = \sum_{j=1}^{k} (f_0, \phi_j)\phi_j \to f_0 \text{ in } H_0^1(\Omega, H^1(0,1)) \;\text{ when }\; k \to +\infty. \tag{29}$$

Then, for any given $\varphi \in span\{\varphi_1, \varphi_2, \varphi_3, \ldots\}$, we have
$$\begin{aligned}
&\left(\partial_{tt}u^k, \varphi^k\right) + \left(\nabla_x u^k, \nabla_x \varphi^k\right) + \left(\nabla_x \partial_{tt}u^k, \nabla_x u\varphi^k\right) + \mu_1\left(u_t^k, \varphi^k\right) \\
&- \int_0^t h(t-s)\left(\nabla_x u^k(s), \nabla_x u\varphi^k\right) + \int_0^t \mu_2(s)\left(z^k(x,1,s,t), \varphi^k\right)ds \\
&= b\int_0^t \left(u^k|u^k|^{p(x)-1}, \varphi^k\right).
\end{aligned} \tag{30}$$

From the general results on systems of differential equations, we assured the existence of the solution of (14) (note that $\det(\varphi_i, \varphi_j) \neq 0$ and $\det(\phi_i, \phi_j) \neq 0$) thanks to the linear independence of $\varphi_1, \varphi_2, \ldots, \varphi_m$ and $\phi_1, \phi_2, \ldots, \phi_m$ in an interval $[0, t_m]$. Owing to the Galerkin method, we prove the result related to the existence of the local solution of (14).

3.2.1. First Estimate

By Lemma 11, since the sequences u_0^k, u_1^k converge, we find $C_1 > 0$ independent of k, satisfying

$$\begin{aligned}\mathcal{E}^k(t) - \mathcal{E}^k(0) \le &-\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds - \frac{\xi(\tau_2 - \tau_1)}{2}\right) \int_0^t \|\partial_t u^k(s)\|_{L^2}^2 ds \\ &- \frac{\xi}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^t |z^k(x,1,s,\rho)|^2 dsdxd\rho \\ &- \frac{1}{2} \int_0^t h(s) \|\nabla_x u^k\|_{L^2}^2 ds + \frac{1}{2} \int_0^t (h'k)(r) dr \\ &\le -\left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds - \frac{\xi(\tau_2 - \tau_1)}{2}\right) \int_0^1 \|\partial_t u^k(s)\|_{L^2}^2 ds \\ &- \frac{\xi}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 |z^k(x,1,s,\rho)|^2 dsdxd\rho. \end{aligned} \tag{31}$$

Since h is a positive non-increasing function, we have

$$\begin{aligned}&\mathcal{E}^k(t) + \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds - \frac{\xi(\tau_2 - \tau_1)}{2}\right) \int_0^t \|u_t^k(s)\|_{L^2}^2 ds \\ &+ \frac{\xi}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^t |z^k(x,1,s,\rho)|^2 dsdxd\rho \\ &\le \mathcal{E}^k(0) \le C_1, \end{aligned} \tag{32}$$

which

$$\begin{aligned}E^k(t) = &\frac{1}{2}\|\partial_t u^k\|_{L^2}^2 + \frac{1}{2}\left(1 - \int_0^t h(r)dr\right)\|\nabla_x u^k\|_{L^2}^2 + \frac{1}{2}(h \circ \nabla_x u^k) + \frac{1}{2}\|\nabla_x \partial_t u^k\|_{L^2}^2 \\ &- b \int_0^t \int_\Omega |u^k|^{p(x)-1} u^k \partial_t u^k + \frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(|\mu_2| + \xi)|z^k(x,k,s,t)|^2 dsdkdx.\end{aligned}$$

So, since (32), we obtain

$$\begin{aligned}&\frac{1}{2}\|\partial_t u^k\|_{L^2}^2 + \frac{1}{2}\left(1 - \int_0^t h(\tau)d\tau\right)\|\nabla_x u^k\|_{L^2}^2 \\ &+ \left(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds - \frac{\xi(\tau_2 - \tau_1)}{2}\right) \int_0^t \|\partial_t u^k(s)\|_{L^2}^2 ds \\ &+ \frac{1}{2}(h \circ \nabla_x u^k) + \frac{1}{2}\|\nabla_x \partial_t u^k\|_{L^2}^2 + \frac{1}{2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(|\mu_2| + \xi)|z^k(x,k,s,t)|^2 dsdkdx \\ &+ \frac{\xi}{2} \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^t |z^k(x,1,s,\rho)|^2 dsdxd\rho \\ &\le C_1 + b \int_0^t \int_\Omega |u^k|^{p(x)-1} u^k \partial_t u^k.\end{aligned}$$

Then, Young's inequality gives and Sobolev embedding

$$
\begin{aligned}
&|\int_\Omega |u^k|^{p(x)-1}u^k\partial_t u^k dx| \\
&\leq \int_\Omega |u^k|^{p(x)-1}|u^k||\partial_t u^k|dx \\
&\leq \frac{1}{2}C_\varepsilon \max\left(\int_\Omega |u^k|^{2p^+}dx, \int_\Omega |u^k|^{2p^-}dx\right) + \frac{1}{2}\varepsilon\int_\Omega |\partial_t u^k|^2 dx \\
&\leq \frac{1}{2}C_\varepsilon\left(\|\nabla_x u^k\|^{2p^+} + \|\nabla_x u^k\|^{2p^-}\right) + \frac{1}{2}\varepsilon\|\partial_t u^k\|^2_{L^2}.
\end{aligned}
$$

Thus, there exist $B_0 > 0, \beta_0 > 0$ and $r_0 > 0$ such that

$$
\left\|\nabla_x u^k\right\|^2_{L^2} + \left\|\partial_t u^k\right\|^2_2 \leq B_0 + \beta_0\int_0^t\left[1 + \left(\left\|\nabla_x u^k(s)\right\|^2_{L^2} + \left\|\partial_t u^k(s)\right\|^2_{L^2}\right)^{r_0+1}\right]\mathrm{d}s,
$$

where we note that B_0 and β_0 are independent of k and r_0. Since $r_0 > 0$, there exists enough small time $T_0 := T_0(u_0, u_1, \mu_1) \in (0, T)$ satisfying

$$
(B_0 + \beta_0 T_0)^{-r_0} - r_0\beta_0 T_0 > 0.
$$

Thus, we have by the modified Gronwall lemma (Lemma 8)

$$
\left\|\nabla_x u^k\right\|^2_{L^2} + \left\|\partial_t u^k\right\|^2_2 \leq \left((B_0 + \beta_0 T_0)^{-r_0} - r_0\beta_0 T_0\right)^{\frac{-1}{\sigma_0}}.
$$

Therefore, there exist constants $c_i = c_i(u_0, u_1, \mu_1) > 0 (i = 1, 2, 3)$ such that for any $t \in [0, T_0]$

$$
\left\|\nabla_x u^k\right\|^2_{L^2} \leq C_1 \text{ and } \left\|\partial_t u^k\right\|^2 \leq C_2.
$$

So, we obtain

$$
\begin{aligned}
&\|\partial_t u^k\|^2_{L^2} + \|\nabla_x u^k\|^2_{L^2} + \|\nabla_x\partial_t u^k\|^2_{L^2} + \int_{\tau_1}^{\tau_2}\int_0^t |z^k(x, 1, s, \rho)|^2 dsdxd\rho \\
&+ \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(|\mu_2| + \xi)|z^k(x, k, s, t)|^2 dsdkdx \\
&\leq C^{te}.
\end{aligned} \tag{33}
$$

The estimate implies that the solution $\left(u^k, z^k\right)$ exists in $[0, T)$ and it yields

$$
\begin{gathered}
u^k \text{ is bounded in } L^\infty\left(0, T; H^1_0(\Omega)\right), \\
\partial_t u^k \text{ is bounded in } L^\infty\left(0, T; L^2(\Omega)\right), \\
s(|\mu_2(s)| + \xi)z^k(x, \kappa, s, t) \text{ is bounded in } L^\infty\left(0, T, L^2(\Omega \times (0, 1) \times (\tau_1, \tau_2))\right), \\
z^k(x, 1, s, t) \text{ is bounded in } L^2(\Omega \times (\tau_1, \tau_2) \times (0, T)).
\end{gathered}
$$

3.2.2. Second Estimate

We replace φ^j by $-\Delta_x\varphi^j$ in (24), multiply by η_t^{jk} and sum up over j to k, such that

$$\frac{1}{2}\left[\|\nabla_x\partial_t u^k\|_{L^2}^2+\|\Delta_x u^k\|_{L^2}^2+\|\Delta_x\partial_t u^k\|_{L^2}^2\right]+\mu_1\|\nabla_x\partial_t u^k\|_{L^2}^2+\int_0^1 h(t-s)\Delta_x u^k\Delta_x\partial_t u^k dxds$$
$$+\int_{\tau_1}^{\tau_2}\int_\Omega \nabla_x z^k(x,1,s1t)\nabla_x\partial_t u^k dsdx=-b\int_\Omega \Delta_x\partial_t u^k u^k|u^k|^{p(x)-1}dx. \tag{34}$$

Replacing ϕ^j by $-\Delta_x\phi^j$ in (26), we multiply by $(|\mu_2(s)|+\xi)c^{jk}$ and sum up over j from 1 to k, and we obtain

$$s(|\mu_2(s)|+\xi)\int_\Omega \nabla_x z_t^k\nabla_x z^k dx+(|\mu_2(s)|+\xi)\int_\Omega \nabla_x z_\kappa^k\nabla_x z^k dx=0.$$

Then, we obtain

$$\frac{s(|\mu_2(s)|+\xi)}{2}\frac{d}{dt}\left\|\nabla_x z^k\right\|_{L^2}^2+\frac{|\mu_2(s)|+\xi}{2}\frac{d}{d\kappa}\left\|\nabla_x z^k\right\|_{L^2}^2=0.$$

Integrating over $(0,1)\times(\tau_1,\tau_2)$ to find that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\int_{\tau_1}^{t_2} s(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z^k(x,\kappa,s,t)\right|^2 dsd\kappa dx\\&+\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z^k(x,1,s,t)\right|^2 dsdx\\&-\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x\partial_t u^k\right|^2 dsdx\\&=0.\end{aligned}\tag{35}$$

Combining (35) and (34), taking into consideration Lemma 9, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Bigg[\|\nabla_x u^k\|_{L^2}^2+\left(1-\int_0^t h(\tau)d\tau\right)\left\|\Delta_x u^k\right\|_{L^2}^2+\left\|\Delta_x\partial_t u^k\right\|_{L^2}^2+\left(h\circ\Delta_x u^k\right)\\&+\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z^k(x,k,s,t)\right|^2 dsd\kappa dx\Bigg]\\&+\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z^k(x,1,s,t)\right|^2 dsdx\\=&-\int_{\tau_1}^{t_2}\mu_2(s)\int_\Omega \nabla_x z^k(x,1,s,t)\nabla_x\partial_t u^k dsdx-\mu_1\left\|\nabla_x\partial_t u^k\right\|^2\\&+\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x\partial_t u^k\right|^2 dsdx\\&-\frac{1}{2}h(t)\left\|\Delta_x u^k\right\|_{L^2}^2+\frac{1}{2}\left(h'\circ\Delta_x u^k\right)-b\int_\Omega \Delta_x\partial_t u^k u^k\left|u^k\right|^{p(x)-1}dx.\end{aligned}\tag{36}$$

By using Young's inequality and the first estimation, we have

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} \mu_2(s) \int_\Omega \nabla_x z^k(x,1,s,t) \nabla_x \partial_t u^k ds dx \\
&\leq \frac{1}{4\eta} \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega |\nabla_x \partial_t u|^2 ds dx + \eta \int_\Omega \int_{\tau_1}^{t_2} |\mu_2(s)| \left|\nabla_x z^k(x,1,s,t)\right|^2 ds dx \\
\leq& \frac{\mu_1}{4\eta} \|\nabla_x \partial_t u\|^2 + \eta \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \left|\nabla_x z^k(x,1,s,t)\right|^2 ds dx \\
\leq& \frac{\mu_1}{4\eta} C_2 + \varepsilon \int_\Omega \int_{\tau_1}^{t_2} |\mu_2(s)| \left|\nabla_x z^k(x,1,s,t)\right|^2 ds dx \\
\leq& C(\varepsilon) + \varepsilon \int_\Omega \int_{\tau_1}^{t_2} |\mu_2(s)| \left|\nabla_x z^k(x,1,s,t)\right|^2 ds dx,\ \varepsilon > 0.
\end{aligned}
\tag{37}
$$

The first estimation and Young's inequality give us

$$
\begin{aligned}
\left|\int_\Omega \Delta_x u^k u^k |u^k|^{p(x)-1} dx\right| &\leq \frac{1}{2} b\varepsilon \|\Delta_x \partial_t u^k\|_{L^2}^2 + \frac{1}{2} C_\varepsilon \int_\Omega |u^k|^{2p(x)} dx \\
&\leq \frac{1}{2} b\varepsilon \|\Delta_x \partial_t u^k\|_{L^2}^2 + \frac{1}{2} C_\varepsilon \max\left(\int_\Omega |u^k|^{2p^+} dx, \int_\Omega |u^k|^{2p^-} dx\right) \\
&\leq \frac{b\varepsilon}{2} \|\Delta_x \partial_t u^k\|_{L^2}^2 + C(\varepsilon),\ \varepsilon > 0.
\end{aligned}
\tag{38}
$$

Combining (32)–(38) with (31), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg[\|\nabla_x u^k\|_{L^2}^2 + \left(1 - \int_0^t h(\tau) d\tau\right) \left\|\Delta_x u^k\right\|_{L^2}^2 + \left\|\Delta_x \partial_t u^k\right\|_{L^2}^2 + \left(h \circ \Delta_x u^k\right) \\
&\int_{\tau_1}^{\tau_2} \int_0^1 s(|\mu_2(s)| + \xi) \int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 ds d\kappa dx\Bigg] \\
&+ \frac{1}{2} \int_{\tau_1}^{t_2} (|\mu_2(s)| + \xi - 2\varepsilon) \int_\Omega \left|\nabla_x z^k(x,1,s,t)\right|^2 ds dx \\
\leq& -\frac{1}{2} h(t) \left\|\Delta_x u^k\right\|_{L^2}^2 + \frac{1}{2}\left(h' \circ \Delta_x u^k\right) + \frac{b\varepsilon}{2} \|\Delta_x \partial_t u^k\|_{L^2}^2 + C_\varepsilon,\ \varepsilon > 0.
\end{aligned}
\tag{39}
$$

We multiply (24) by η_{tt}^{jk} and summing over j from 1 to k, we obtain

$$
\begin{aligned}
\|\partial_{tt} u\|_{L^2}^2 + \left\|\nabla_x \partial_{tt} u^k\right\|_{L^2}^2 &= \int_\Omega \Delta_x u^k \partial_{tt} u^k dx \\
&- \int_0^t h(t-\tau) \int_\Omega \nabla_x u^k(\tau) \nabla_x \partial_{tt} u^k dx d\tau \\
&- \mu_1 \int_\Omega \partial_t u^k \partial_{tt} u^k dx - \int_\Omega \int_{t_1}^{t_2} \mu_2(s) z^k(x,1,s,t) \partial_{tt} u^k ds dx \\
&+ b \int_\Omega \partial_{tt} u^k u^k \left|u^k\right|^{p(x)-1} dx.
\end{aligned}
\tag{40}
$$

Differentiating (28) with respect to t, we obtain

$$
\left(s z_{tt}^k + z_{t\rho}^k, \phi^f\right) = 0.
$$

We multiply by $(|\mu_2(s)| + \xi) d_t^k$ and sum up over j from 1 to k, to have

$$
\frac{s(|\mu_2(s)| + \xi)}{2} \frac{d}{dt} \left\|\nabla_x z_t^k\right\|_{L^2}^2 + \frac{|\mu_2(s)| + \xi}{2} \frac{d}{dk} \left\|\nabla_x z_t^k\right\|_{L^2}^2 = 0.
$$

We integrate over $(0,1)\times(\tau_1,\tau_2)$ with respect to κ and s, to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\tau_1}^{t_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z_t^k(x,\kappa,s,t)\right|^2 ds d\kappa dx\\
+\ &\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z_t^k(x,1,s,t)\right|^2 dsdx\\
-\ &\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x\partial_{tt}u^k\right|^2 dsdx\\
=\ &0.
\end{aligned}\tag{41}$$

Summing (40) and (41), we have

$$\begin{aligned}
&\|\partial_{tt}u^k\|_{L^2}^2+\left\|\nabla_x\partial_{tt}u^k\right\|_{L^2}^2+\frac{1}{2}\frac{d}{dt}\int_{\tau_1}^{t_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z_t^k(x,\kappa,s,t)\right|^2 dsd\kappa dx\\
+\ &\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z_t^k(x,1,s,t)\right|^2 dsdx\\
=\ &\int_\Omega\Delta_x\partial_{tt}u^k dx-\int_0^t h(t-\tau)\int_\Omega\nabla_x u^k(\tau)\nabla_x\partial_{tt}u^k(t)dxd\tau\\
-\ &\mu_1\int_\Omega\partial_t u^k\partial_{tt}u^k dx-\int_\Omega\int_{t_1}^{t_2}\mu_2(s)z^k(x,1,s,t)\partial_{tt}u^k dsdx\\
+\ &b\int_\Omega\partial_{tt}u^k u^k\left|u^k\right|^{p(x)-1}dx+\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x\partial_{tt}u^k\right|^2 dsdx.
\end{aligned}\tag{42}$$

Utilizing Young's inequality, the right hand side of (42) can be written as

$$\left|\int_\Omega\Delta_x\partial_{tt}u^k\right|\le\varepsilon\|\nabla_x\partial_{tt}u^k\|_{L^2}^2+C(\varepsilon)\|\nabla_x u^k\|_{L^2}^2,\ \ \varepsilon>0,\tag{43}$$

and

$$\begin{aligned}
&\int_0^t h(t-\tau)\int_\Omega\Delta_x u^k(\tau)\partial_{tt}u^k(t)dxd\tau\\
&=-\int_0^t h(t-\tau)\int_\Omega\nabla_x u^k(\tau)\nabla_x u_{tt}^k(t)dxd\tau\\
&\le\varepsilon\left\|\nabla_x\partial_{tt}u^k\right\|_{L^2}^2+\frac{\beta^2}{4\eta}(1+\varepsilon)\left\|\nabla_x u^k\right\|_{L^2}^2\\
&\quad+\frac{\beta}{4\varepsilon}\left(1+\frac{1}{\varepsilon}\right)\left(h\circ\nabla_x u^k\right),\ \ \varepsilon>0.
\end{aligned}\tag{44}$$

Thanks to the Young, Poincaré's inequalities and the first estimate, we have

$$\begin{aligned}
\mu_1\int_\Omega\partial_t u^k\partial_{tt}u^k dx&\le\mu_1\varepsilon\left\|\partial_{tt}u^k\right\|_{L^2}^2+\frac{\mu_1^2}{4\varepsilon}\left\|\partial_t u^k\right\|_{L^2}^2\\
&\le\varepsilon\mu_1\left\|\partial_{tt}u^k\right\|_{L^2}^2+C(\varepsilon),
\end{aligned}\tag{45}$$

and

$$
\begin{aligned}
&\int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z^k(x,1,s,t)\partial_{tt}u^k dsdx \\
\leq & \varepsilon C_s^2 \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega \left|\nabla_x \partial_{tt} u^k\right|^2 dsdx \\
&+ \frac{1}{4\varepsilon}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdx \\
\leq & \varepsilon C_s^2 \mu_1 \int_\Omega \left|\nabla_x \partial_{tt} u^k\right|^2 + \frac{1}{4\varepsilon}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdx.
\end{aligned}
\tag{46}
$$

So, thanks to Young's inequality, the nonlinear term can be estimated as

$$
|b\int_\Omega \partial_{tt}u^k u^k |u^k|^{p(x)-1}| \leq C_p \eta \|\nabla_x \partial_{tt} u\|^2 + C(\eta), \;\; \eta > 0. \tag{47}
$$

Taking into account (38)–(47) into (42) satisfies

$$
\begin{aligned}
&\|\partial_{tt}u\|_{L^2}^2 + \left\|\nabla_x \partial_{tt} u^k\right\|_{L^2}^2 + \frac{1}{2}\int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \int_\Omega \left|\nabla_x \partial_t z^k(x,1,s,t)\right|^2 dsdx \\
&+ \frac{1}{2}\frac{d}{dt}\int_{\tau_1}^{t_2}\int_0^1 s(|\mu_2(s)| + \xi)\int_\Omega \left|\nabla_x \partial_t z^k(x,\kappa,s,t)\right|^2 dsd\kappa dx \\
&\leq \varepsilon \|\nabla_x \partial_{tt} u^k\|_{L^2}^2 + C(\varepsilon)\|\nabla_x u^k\|_{L^2}^2 + \varepsilon\left\|\nabla_x \partial_{tt} u^k\right\|_{L^2}^2 + \frac{\beta^2}{4\eta}(1+\varepsilon)\left\|\nabla_x u^k\right\|_{L^2}^2 \\
&+ \frac{\beta}{\varepsilon}\left(1+\frac{1}{\varepsilon}\right)\left(h \circ \nabla_x u^k\right) \\
&\varepsilon\mu_1 \left\|\partial_{tt} u^k\right\|_{L^2}^2 + C(\varepsilon) + \varepsilon C_s^2 \mu_1 \int_\Omega \left|\nabla_x \partial_{tt} u^k\right|^2 + \frac{1}{4\varepsilon}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdx \\
&+ C_p \eta \|\nabla_x \partial_{tt} u\|_{L^2}^2 + C(\eta).
\end{aligned}
\tag{48}
$$

Then, let the first estimate hold, then (48) will be

$$
\begin{aligned}
&(1-\varepsilon\mu_1)\|\partial_{tt}u\|_{L^2}^2 + (1-(2+C_s^2\mu_1)\varepsilon - C_p\eta)\left\|\nabla_x \partial_{tt} u^k\right\|_{L^2}^2 \\
&+ \frac{1}{2}\frac{d}{dt}\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)| + \xi)\int_\Omega \left|\nabla_x \partial_t z^k(x,\kappa,s,t)\right|^2 dsd\kappa dx \\
&+ \frac{1}{2}\int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi)\int_\Omega \left|\nabla_x \partial_t z^k(x,1,s,t)\right|^2 dsdx \\
&\leq \frac{\beta}{\varepsilon}\left(1+\frac{1}{\varepsilon}\right)\left(h \circ \nabla_x u^k\right) + \frac{1}{4\varepsilon}\int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_\Omega \left|z^k(x,1,s,t)\right|^2 dsdx + C(\varepsilon,\eta).
\end{aligned}
\tag{49}
$$

Therefore, by (39) and (49)

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Bigg[\|\nabla_x u^k\|_{L^2}^2 + \left(1-\int_0^t h(\tau)d\tau\right)\left\|\Delta_x u^k\right\|_2^2 + \left\|\Delta_x \partial_t u^k\right\|_{L^2}^2 + \left(h\circ \Delta u^k\right) \\
&\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 dsd\kappa dx \\
&+\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 dsd\kappa dx\Bigg] \\
&+\frac{1}{2}\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi-2\varepsilon)\int_\Omega\left|\nabla_x z^k(x,1,s,t)\right|^2 dsdx \\
&+\frac{1}{2}\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z_t^k(x,1,s,t)\right|^2 dsdx \\
&+(1-\varepsilon\mu_1)\|\partial_{tt}u\|_{L^2}^2 + (1-(2+C_s^2\mu_1)\varepsilon - C_p\eta)\left\|\nabla_x\partial_{tt}u^k\right\|_2^2 \\
&\le -\frac{1}{2}h(t)\left\|\Delta_x u^k\right\|_{L^2}^2 + \frac{1}{2}\left(h'\circ\Delta_x u^k\right) + \frac{b\varepsilon}{2}\|\Delta_x\partial_t u^k\|_2^2 + \frac{\beta}{\varepsilon}\left(1+\frac{1}{\varepsilon}\right)\left(h\circ\nabla_x u^k\right) \\
&+\frac{1}{4\varepsilon}\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_\Omega\left|z^k(x,1,s,t)\right|^2 dsdx + C(\varepsilon,\eta),\ \varepsilon,\eta>0.
\end{aligned} \tag{50}$$

Choosing ε,η tow positive small enough such that $(1-\varepsilon\mu_1)>0$ and $(1-(2+C_s^2)\varepsilon - C_p\eta)>0$ and integrating over $(0,t)$, we obtain

$$\begin{aligned}
&\frac{1}{2}\Bigg[\|\nabla_x u^k\|_{L^2}^2 + \left(1-\int_0^t h(\tau)d\tau\right)\left\|\Delta_x u^k\right\|_{L^2}^2 + \left\|\Delta_x \partial_t u^k\right\|_{L^2}^2 + \left(h\circ \Delta_x u^k\right) \\
&+\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 dsd\kappa dx \\
&+\int_{\tau_1}^{\tau_2}\int_0^1 s(|\mu_2(s)|+\xi)\int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 dsd\kappa dx\Bigg] \\
&+\frac{1}{2}\int_0^t\int_{\tau_1}^{t_2}(|\mu_2(s)|+\xi-2\varepsilon)\int_\Omega\left|\nabla_x z^k(x,1,s,\rho)\right|^2 dsdxd\rho \\
&+\frac{1}{2}\int_0^t\int_{\tau_1}^{\tau_2}(|\mu_2(s)|+\xi)\int_\Omega\left|\nabla_x z_t^k(x,1,s,\rho)\right|^2 dsdxd\rho \\
&+(1-\varepsilon\mu_1)\int_0^t\|\partial_{tt}u\|_{L^2}^2 + (1-(2+C_s^2\mu_1)\varepsilon - C_p\eta)\int_0^t\left\|\nabla_x\partial_{tt}u^k\right\|_{L^2}^2 ds \\
&\le -\frac{1}{2}\int_0^t h(\tau)\left\|\Delta_x u^k\right\|_{L^2}^2 d\tau + \frac{1}{2}\int_0^t\left(h'\circ\Delta_x u^k\right)ds + \frac{\beta}{\varepsilon}\left(1+\frac{1}{\varepsilon}\right)\int_0^t\left(h\circ\nabla_x u^k\right)ds \\
&+\frac{b\varepsilon}{2}\int_0^t\|\Delta_x\partial_t u^k\|_{L^2}^2 ds + \frac{1}{4\varepsilon}\int_0^t\int_{\tau_1}^{\tau_2}|\mu_2(s)|\int_\Omega\left|z^k(x,1,s,\rho)\right|^2 dsdxd\rho + TC(\varepsilon,\eta),\ \varepsilon,\eta>0,
\end{aligned} \tag{51}$$

By using Gronwall's lemma and taking $h_1 = \{h(t)| \text{ for all } t \geq t_0\}$, we have

$$\begin{aligned}
&\int_0^t \|\partial_{tt}u\|_{L^2}^2 + \int_0^t \left\|\nabla_x \partial_{tt} u^k\right\|_{L^2}^2 ds + \|\nabla_x u^k\|_{L^2}^2 + \left\|\Delta u^k\right\|_{L^2}^2 + \left\|\Delta_x \partial_t u^k\right\|_2^2 + \left(h \circ \Delta_x u^k\right) \\
+ \; & \int_{\tau_1}^{\tau_2} \int_0^1 s(|\mu_2(s)| + \xi) \int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 ds d\kappa dx \\
+ \; & \int_{\tau_1}^{\tau_2} \int_0^1 s(|\mu_2(s)| + \xi) \int_\Omega \left|\nabla_x z^k(x,k,s,t)\right|^2 ds d\kappa dx \\
+ \; & \frac{1}{2} \int_0^t \int_{\tau_1}^{t_2} (|\mu_2(s)| + \xi - 2\varepsilon) \int_\Omega \left|\nabla_x z^k(x,1,s,\rho)\right|^2 ds dx d\rho \\
+ \; & \frac{1}{2} \int_0^t \int_{\tau_1}^{\tau_2} (|\mu_2(s)| + \xi) \int_\Omega \left|\nabla_x z_t^k(x,1,s,\rho)\right|^2 ds dx d\rho \\
\leq \; & C'.
\end{aligned} \tag{52}$$

The estimate (52) yields

$$\begin{aligned}
&\left(u^k\right) \text{ is uniformly bounded in } L^\infty\left(0,T;,H_0^2(\Omega)\right), \\
&\left(\partial_t u^k\right) \text{ is uniformly bounded in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\
&\left(\partial_{tt} u^k\right) \text{ is uniformly bounded in } L^2\left(0,T;H_0^1(\Omega)\right).
\end{aligned} \tag{53}$$

We see that, by the estimates (26) and (47), we have a subsequence $\{u^m\}$ of $\left\{u^k\right\}$ and a function u where

$$\begin{aligned}
&u^m \to u \text{ weakly star in } L^\infty\left(0,T;H^2(\Omega)\right), \\
&\partial_t u^m \to \partial_t u \text{ weakly star in } L^\infty\left(0,T;H_0^2(\Omega)\right), \\
&u_{tt}^m \to \partial_{tt} u \text{ weakly star in } L^2\left(0,T;H_0^1(\Omega)\right).
\end{aligned} \tag{54}$$

Since $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact and from the Aubin–Lions theorem, we deduce that

$$\begin{aligned}
&u^m \longrightarrow u \text{ strongly in } L^2\left(0,T;H_0^1(\Omega)\right), \\
&u_i^m \longrightarrow \partial_t u \text{ strongly in } L^2\left(0,T;L^2(\Omega)\right),
\end{aligned}$$

and consequently, by making use of Lion's lemma ([12], Lemma 1.3), we have

$$|u^m(t)|^{p(.)-1} u^m(t) \to |u|^{p(\cdot)-1} u \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right). \tag{55}$$

We multiply (24) by $\theta(t) \in D(0,T)$ and integrate over $(0,T)$, we have

$$\begin{aligned}
&\int_0^T \left(\partial_t u^k(t), w^j\right) \theta'(t) dt + \int_0^T \left(\nabla_x u^k(t), \nabla_x w^j\right) \theta(t) dt \\
&+ \int_0^T \left(\nabla_x \partial_{tt} u^k, \nabla_x w^j\right) \theta(t) dt - \int_0^T \int_0^t h(t-\tau) \left(\Delta_x u^k(\tau), \Delta_x w^j\right) \theta(t) d\tau dt \\
&+ \mu_1 \int_0^T \left(\partial_t u^k, w^j\right) \theta(t) dt + \int_0^T \int_{\tau_1}^{t_2} \mu_2(s) \left(z^k(x,1,s,t), w^j\right) \theta(t) ds dt \\
&= \int_0^T \left(u^k(s) \left|u^k(s)\right|^{p(x)-1}, w^j\right) \theta(t) dx dt,
\end{aligned} \tag{56}$$

we multiply (28) by $\theta(t) \in D(0,T)$ and integrate over $(0,T) \times (0,1)$, to obtain

$$\int_0^T \int_0^1 \Big(s\partial_t z^k + z^k_\kappa, \phi^j\Big)\theta(t)dxd\kappa = 0. \tag{57}$$

The convergence of (54) and (55) are sufficient to pass the limit in (56) and (57) to obtain

$$\begin{aligned}
&\int_0^T (\partial_t u, w)\theta'(t)dt + \int_0^T (\nabla_x u, \nabla_x w)\theta(t)dt \\
&+ \int_0^T (\nabla_x \partial_{tt} u, \nabla_x w)\theta(t)dt - \int_0^T \int_0^t h(t-\tau)(\nabla_x u(\tau), \nabla_x w)\theta(t)d\tau dt \\
&+ \mu_1 \int_0^T (\partial_t u, w)\theta(t)dt + \int_0^T \int_{\tau_1}^{\tau_2} \mu_2(s)(z(x,1,s,t), w)\theta(t)dsdt \\
&= b\int_0^T (u(s)|u(s)|^{p(x)-1}, w)\theta(t)dxdt,
\end{aligned}$$

and

$$\int_0^T \int_0^1 (sz_t + z_\kappa, \phi)\theta(t)dtd\kappa = 0.$$

Integrating over $(0,T)$, we have

$$\begin{aligned}
&\int_0^T \Big(\partial_{tt} u + \Delta_x u - \Delta_x \partial_{tt} u - \int_0^t h(t-\tau)\Delta_x u(\tau)d\tau \\
&+\mu_1 \partial_t u + \int_{\tau_1}^{\tau_2} \mu_2(s)(z(x,1,s,t)ds, w)\Big)\theta(t)dt \\
=&b\int_0^T (u|u|^{p(x)-1}, w)\theta(t)dxdt.
\end{aligned}$$

Consequently, we find the local existence of the problem. □

3.3. Global Existence

We are now ready to treat the global existence result.
Firstly, we define the following functionals:

$$I(t) = \Big(1 - \int_0^t h(r)dr\Big)\|\nabla_x u\|^2_{L^2} + \|\nabla_x \partial_t u\|^2_{L^2} + (h \circ \nabla_x u) - b\int_\Omega |u|^{p(x)}dx, \tag{58}$$

$$J(t) = \frac{1}{2}\Big(1 - \int_0^t h(r)dr\Big)\|\nabla_x u\|^2_{L^2} + \frac{1}{2}\|\nabla_x \partial_t u\|^2_{L^2} + \frac{1}{2}(h \circ \nabla_x u) - \int_\Omega \frac{b}{p(x)}|u|^{p(x)}dx. \tag{59}$$

We note that

$$\mathcal{E}(t) = \frac{1}{2}\|\partial_t u\|^2_{L^2} + \frac{1}{2}\|\nabla_x \partial_t u\|^2_{L^2} + J(t) + \frac{1}{2}\int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s(\mu_2(s) + \zeta)z^2(x,\rho,s,t)dsd\rho dx, \tag{60}$$

Lemma 12. *Suppose that (A1)–(A2). Assume that $(u_0, u_1) \in H^1_0(\Omega) \times L^2(\Omega)$ such that*

$$I(0) > 0,$$

and

$$\theta < 1, \tag{61}$$

where

$$\theta = \max\left\{ c_*^{p_1}\left(\frac{2p_1}{p_1-2}\mathcal{E}(0)\right)^{\frac{p_1-2}{2}}, c_*^{p2}\left(\frac{2p_2}{p_2-2}\mathcal{E}(0)\right)^{\frac{p_2-2}{2}} \right\},$$

with c_* as the best embedding constant of $H_0^1(\Omega) \hookrightarrow L_{p(.)}(\Omega)$, then $I(t) > 0$ for all $t \in [0,T]$.

Proof. By continuity, there exists T^*, such that

$$I(t) \geq 0, \ for\ all\ t \in [0,T^*]. \tag{62}$$

Now, we have for all $t \in [0,T^*]$

$$\begin{aligned}
J(t) = J(u) &= \frac{1}{2}(1 - \int_0^t h(r)dr)\|\nabla_x u\|_{L^2}^2 + \frac{1}{2}\|\nabla_x \partial_t u\|_{L^2}^2 \\
&+ \frac{1}{2}(h \circ \nabla_x u) - \int_\Omega \frac{b}{p(x)}|u|^{p(x)}dx \\
&\geq \frac{1}{2}(1 - \int_0^t h(r)dr)\|\nabla_x u\|_2^2 + \frac{1}{2}\|\nabla_x \partial_t u\|_{L^2}^2 + \frac{1}{2}(h \circ \nabla_x u) \\
&- \frac{b}{p_1}\left((1 - \int_0^t h(r)dr)\|\nabla_x u\|_{L^2}^2 + \|\nabla_x \partial_t u\|_{L^2}^2 + (h \circ \nabla_x u) - I(t)\right) \\
&\geq \frac{p_1 - 2b}{2p_1}\left((1 - \int_0^t h(r)dr)\|\nabla_x u\|_{L^2}^2 + \|\nabla_x \partial_t u\|_{L^2}^2 + (h \circ \nabla_x u)\right) + \frac{b}{p_1}I(t).
\end{aligned}$$

Using (62), we obtain

$$\|\nabla_x \partial_t u\|_{L^2}^2 + \|\nabla_x u\|_2^2 \leq \frac{2p_1 l}{p_1 - 2b}J(t) \quad for\ all\ t \in [0,T^*]. \tag{63}$$

By the definition of E, we have

$$\|\nabla_x \partial_t u\|_{L^2}^2 + \|\nabla_x u\|_2^2 \leq \frac{2p_1 l}{p_1 - 2b}\mathcal{E}(t) \leq \frac{2p_1 l}{p_1 - 2b}\mathcal{E}(0) \quad for\ all\ t \in [0,T^*]. \tag{64}$$

On the other hand, we obtain

$$\begin{aligned}
\int_\Omega |u|^{p(x)}dx &\leq \max\left\{c_*^{p_1}\|\nabla_x u\|_{L^2}^{p_1}, c_*^{p2}\|\nabla_x u\|_{L^2}^{p_2}\right\} \\
&\leq \max\left\{c_*^{p_1}\|\nabla_x u\|_{L^2}^{p_1-2}, c_*^{p2}\|\nabla_x u\|_{L^2}^{p_2-2}\right\} \times \|\nabla_x u\|_{L^2}^2 \\
&\leq \max\left\{ c_*^{p_1}\left(\frac{2p_1}{p_1-2}\mathcal{E}(0)\right)^{\frac{p_1-2}{2}}, c_*^{p2}\left(\frac{2p_2}{p_2-2}\mathcal{E}(0)\right)^{\frac{p_2-2}{2}} \right\} \times \|\nabla_x u\|_{L^2}^2.
\end{aligned}$$

Then, we have

$$\int_\Omega |u|^{p(x)}dx \leq \theta\|\nabla_x u\|_{L^2}^2, \quad for\ all\ t \in [0,T^*].$$

Since $\theta < 1$, then

$$\int_\Omega |u|^{p(x)}dx \leq \|\nabla_x u\|_{L^2}^2, \quad for\ all\ t \in [0,T^*].$$

This implies that

$$I(t) > 0, \quad for\ all\ t \in [0,T^*].$$

By repeating the above procedure, we can extend T^* to T.
Consequently, the local solution can be extend to be global in time. □

4. Asymptotic Behavior

In this section, by constructing a suitable Lyapunov function, we obtain an asymptotic behavior result for our problem.

Theorem 2. *Suppose that* $(A1)-(A2)$ *hold. Then,* $\mathcal{E}(t)$ *energy functional (15) satisfies,*

$$\mathcal{E}(t) \leq C_1 e^{-k_1 t} + C_2 \quad \forall t > 0, \tag{65}$$

where C_1, C_2 *and* k_1 *are positive constants.*

Proof. Firstly, we defined the function of Lyapunov as follows:

$$L(t) = \mathcal{E}(t) + \varepsilon\bigg(\int_\Omega \partial_t u u \, dx + \int_\Omega \nabla_x u \partial_t u \nabla_x u \, dx\bigg), \tag{66}$$

where ε is a positive real number.

We prove that $L(t)$ and $\mathcal{E}(t)$ are equivalent, meaning that there exist two positive constants B_1 and B_2 depending on such that for $t \geq 0$

$$B_1\mathcal{E}(t) \leq L(t) \leq B_2\mathcal{E}(t). \tag{67}$$

From the Young's inequality, we obtain

$$L(t) \leq \mathcal{E}(t) + \varepsilon\left[\frac{1}{2\delta}\|\partial_t u\|_2^2 + \delta\|u\|_{L^2}^2\right] + \varepsilon\left[\frac{1}{2\delta}\|\nabla_x\partial_t u\|_{L^2}^2 + \delta\|\nabla_x u\|_{L^2}^2\right].$$

By using the Poincaré inequality, we obtain

$$L(t) \leq \mathcal{E}(t) + \varepsilon\left[\frac{1}{2\delta}\|\partial_t u\|_2^2 + \delta C_P\|\nabla_x u\|_{L^2}^2\right] + \varepsilon\left[\frac{1}{2\delta}\|\nabla_x\partial_t u\|_2^2 + \delta\|\nabla_x u\|_{L^2}^2\right].$$

From (15), we have

$$\begin{aligned} L(t) &\leq \mathcal{E}(t) + \varepsilon\left[\frac{1}{2\delta}\mathcal{E}(t) + \delta C_P\mathcal{E}(t)\right] + \varepsilon\left[\frac{1}{2\delta}\mathcal{E}(t) + \delta\mathcal{E}(t)\right] \\ &\leq B_2\mathcal{E}(t), \end{aligned}$$

with $B_2 = 1 + \delta\varepsilon(1 + C_P) + \frac{\varepsilon}{\delta}$.

On the other hand, we have

$$\begin{aligned} L(t) &\geq \mathcal{E}(t) - \varepsilon\left[\frac{1}{2\delta}\|\partial_t u\|_{L^2}^2 + \delta\|u\|_{L^2}^2\right] - \varepsilon\left[\frac{1}{2\delta}\|\nabla_x\partial_t u\|_{L^2}^2 + \delta\|\nabla_x u\|_{L^2}^2\right] \\ &\geq \mathcal{E}(t) - \varepsilon\left[\frac{1}{2\delta}\|\partial_t u\|_2^{L^2} + \delta C_P\|\nabla_x u\|_{L^2}^2\right] - \varepsilon\left[\frac{1}{2\delta}\|\nabla_x\partial_t u\|_{L^2}^2 + \delta\|\nabla_x u\|_{L^2}^2\right] \\ &\geq B_1\mathcal{E}(t), \end{aligned}$$

such that $B_1 = 1 - \frac{\varepsilon}{\delta} - \delta\varepsilon(1 + C_P)$.

Now, we have

$$\frac{dL(t)}{dt} = \frac{d\mathcal{E}(t)}{dt} + \varepsilon\bigg(\int_\Omega \partial_{tt} u u \, dx + \|\partial_t u\|_{L^2}^2 + \int_\Omega \nabla_x u \partial_{tt} u \nabla_x u \, dx + \|\nabla_x\partial_t u\|_{L^2}^2\bigg), \tag{68}$$

and

$$\begin{aligned}
&\int_\Omega (\partial_{tt}u - \Delta_x\partial_{tt}u)u\,dx \\
&= \int_\Omega \Delta_x u(x,t)dx + \int_\Omega u\int_0^t h(t-\tau)\Delta_x u(\tau)d\tau dx \\
&+ \int_\Omega \mu_1\partial_t u(x,t)u(x,t)dx - \int_\Omega u\int_0^t \mu_2(s)z(x,t,1,s,t)dtdx + b\int_\Omega |u|^{p(x)}dx, \\
&\leq -\|\nabla_x u\|_{L^2}^2 - \int_\Omega \nabla_x u\int_0^t h(t-\tau)\nabla_x u(\tau)dsdx + \mu_1\frac{1}{2\delta}\|\partial_t u\|_{L^2}^2 + \delta C\mu_1\|\nabla_x u\|_{L^2}^2 \\
&+ \frac{2p_1 l}{p_1-2b}C\mathcal{E}(0) - \delta\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\mu_2(s)+\zeta)z^2(x,\rho,s,t)dsd\rho dx.
\end{aligned} \tag{69}$$

The last term of relation (69) can be estimated as follows.

$$\begin{aligned}
&\left|\int_\Omega \nabla_x u\int_0^t h(t-\tau)\nabla_x u(\tau)d\tau dx\right| \\
&\leq \int_\Omega\left(\int_0^t h(t-\tau)|\nabla_x u(\tau) - \nabla_x u(t)|ds\right)dx + \int_0^t h(\tau)d\tau|\nabla_x u|_{L^2}^2 \\
&\leq (1+\eta)(1-l)\|\nabla_x u\|_{L^2}^2 + \frac{1}{4\eta}(h\circ\nabla_x u) \quad \text{for } \eta > 0.
\end{aligned} \tag{70}$$

So,

$$\begin{aligned}
\frac{dL(t)}{dt} &\leq -\varepsilon\|\nabla_x u\|_{L^2}^2 + (1+\varepsilon)\|\partial_t u\|_{L^2}^2 + (1+\varepsilon)\|\nabla_x\partial_t u\|_{L^2}^2 \\
&- \varepsilon\int_\Omega \nabla_x u\int_0^t h(t-\tau)\nabla_x u(\tau)dsdx \\
&+ \mu_1\frac{\varepsilon}{2\delta}\|\partial_t u\|_{L^2}^2 + \delta\varepsilon C\mu_1\|\nabla_x u\|_{L^2}^2 + \frac{2\varepsilon p_1 l}{2p_1-2b}C\mathcal{E}(0) - \|\partial_t u\|_{L^2}^2 - \|\nabla_x\partial_t u\|_{L^2}^2 \\
&- \delta\varepsilon\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\mu_2(s)+\zeta)z^2(x,\rho,s,t)dsd\rho dx \\
&\leq [(1+\varepsilon(1+\eta))(1-l) + \varepsilon\delta C_P]\|\nabla_x u\|_{L^2}^2 \\
&+ (1+\varepsilon+\mu_1\frac{\varepsilon}{2\delta})\|\partial_t u\|_{L^2}^2 + \frac{2\varepsilon p_1 l}{2p_1-2b}C\mathcal{E}(0) \\
&- \delta\varepsilon\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s(\mu_2(s)+\zeta)z^2(x,\rho,s,t)dsd\rho dx \\
&+ (1+\frac{\varepsilon}{4\eta})(h\circ\nabla_x u) - \|\partial_t u\|_{L^2}^2 - \|\nabla_x\partial_t u\|_{L^2}^2 \\
&+ (1+\varepsilon)\|\nabla_x\partial_t u\|_{L^2}^2 + \int_\Omega \frac{b}{p(x)}|u|^{p(x)}dx - (\varepsilon+(1-l))\|\nabla_x u\|_{L^2}^2 - (h\circ\nabla_x u),
\end{aligned} \tag{71}$$

then

$$\frac{dL(t)}{dt} \leq \lambda\mathcal{E}(t) + \varrho, \tag{72}$$

with

$$\lambda = \min\{-1; -\delta\varepsilon; -(\varepsilon+(1-l)\} < 0, \tag{73}$$

and

$$\begin{aligned}\varrho =& [(1+\varepsilon(1+\eta))(1-l)+\varepsilon\delta C_P]\|\nabla_x u\|_{L^2}^2\\ &+(1+\varepsilon+\mu_1\frac{\varepsilon}{2\delta})\|\partial_t u\|_{L^2}^2+\frac{2\varepsilon p_1 l}{2p_1-2b}C\mathcal{E}(0)\\ &+(1+\frac{\varepsilon}{4\eta})(h\circ\nabla_x u)+(1+\varepsilon)\|\nabla_x\partial_t u\|_{L^2}^2\\ &< C^{te}.\end{aligned}$$

From (67), we have

$$\frac{dL(t)}{dt}\leq -k_1L(t)+\varrho, \tag{74}$$

where $k_1=\frac{-\lambda}{B_2}$. Thus, with a simple integration of differential Inequality (74) between 0 and t, we obtain the following estimate for the function L:

$$L(t)\leq C_0e^{-k_1t}+\frac{\varrho}{k_1},\qquad \forall t>0. \tag{75}$$

Finally, by combining (67) and (75), we obtain

$$\mathcal{E}(t)\leq C_1e^{-k_1t}+\frac{\varrho}{k_1B_1},\qquad \forall t>0. \tag{76}$$

This completes the proof of Theorem 2. □

5. Conclusions

This manuscript examines the existence (in time) of a weak solution and the derivation of qualitative properties of that solution for an attractive topic introduced as a nonlinear viscoelastic wave equation with a variable exponent and a minor damping component. Here, using the energy method in conjunction with the Faedo–Galerkin method, both the local and global existence of the solution are established. The estimate of the solution's stability is then obtained by introducing an adequate Lyapunov functional.

First, the initial BVP (1) is considered. Next, it is transformed to an associate BVP (14) in order to deal with distributed delay. As the main results of the manuscript, Theorem 1 includes sufficient conditions such that the Problem (14) has a weak solution. Theorem 2 includes sufficient conditions such that the energy function $\mathcal{E}(t)$ satisfies the estimate (15) to extend the results in [13,14]. The existence of different types of damping terms makes the problem very interesting in the application point of view. We showed the interaction between them to find a sharp decay rate.

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