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Nonuniform Dichotomy with Growth Rates of Skew-Evolution Cocycles in Banach Spaces

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Abstract: This paper presents integral charaterizations for nonuniform dichotomy with growth rates and their correspondents for the particular cases of nonuniform exponential dichotomy and nonuniform polynomial dichotomy of skew-evolution cocycles in Banach spaces. The connections between these three concepts are presented.

Keywords: skew-evolution cocycle; nonuniform dichotomy with growth rates; nonuniform exponential dichotomy; nonuniform polynomial dichotomy

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1. Introduction

In recent years, the theory of asymptotic behavior for evolution equations has become a topic of large interest that has witnessed significant progress. This concept has been used in studying problems concerning aeronautics, applied engineering subjects, fluid mechanics and acoustics. The connections between functional analysis, control theory, differential equations and fractional differential equations are given in the work of Ben-Artzi and Gohberg [1], Elaydi and Hajek [2], Foiaş, Sell and Temam [3], Lidskii [4] and Kukushkin [5].

One of the most famous theorems that has had an important role in the development of the stability theory of evolution operator is due to Datko–Pazy, who proved that an evolution operator $\mathcal{U}$ on a Banach space X is uniformly exponentially stable if and only if there exists a constant $p \in [1, \infty)$ with

$$\sup_{s\geq 0} \int_s^\infty ||U(t,s)x||^p dt < \infty, \text{ for all } x \in X.$$

Datko's article [6] from 1972 refers to the integral characterization of the uniform exponential stability property. We generalize this result in three directions: a variant of Datko's theorem is given for the general concept of nonuniform dichotomy with growth rates which has as particular cases the concepts of nonuniform exponential dichotomy and nonuniform polynomial dichotomy that obviously contain the concepts of uniform exponential, respectively polynomial stability and dichotomy. For the case $p \in [1, \infty)$, Pazy used different techniques in order to prove that the above result holds for any $p \in [1, \infty)$, in [7] for C_0- semigroups in Banach spaces. In 1984, Ichikawa presented in [8] an interesting result in the nonlinear case; more precisely, the Datko type theorem was proved for a family of nonlinear operators $T = \{T(t,s)\}_{t\geq s\geq 0}$ on a Banach space X. Moreover, Reghiş obtained several important applications using this concept in [9]. The idea was continued by Rolewicz in [10], where the author gave a new nonlinear condition.

The notion of skew-evolution cocycle has become a front-line topic in the modern theory of dynamical systems and differential equations. It can be traced back to the works of Megan, Stoica and Buliga [11], continued by Hai [12] and Stoica and Megan [13]. Moreover, several results were obtained by Huy [14],Megan, Sasu and Sasu [15], Mihiţ [16] and Sacker and Sell [17] for skew-product semiflows or evolution operators which depend on two variables compared to skew-evolution cocycles which have three variables. The class of skew-product flows became a great topic of interest because it arises from linearization of nonlinear equations with multiple applications in nonlinear problems (see [3,18,19] and the references therein). For the future, the authors would like to extend these results for the nonlinear case.

The present paper gives three types of integral characterizations for the concepts of nonuniform dichotomy with growth rates, nonuniform exponential dichotomy and nonuniform polynomial dichotomy for the general case of skew-evolution cocycles. The first theorem that we present has as particular case the classical result of Datko [6] for uniform exponential stability. Other important approaches for these characterizations have been presented by Boruga and Megan [20], Megan, Sasu and Sasu [15], Sasu [21] and Stoica [22].

Besides the concepts of stability and instability, special attention has been given to the study of dichotomy. In 1920, Perron [23] introduced the notion of exponential dichotomy and it gained prominence since the appearance of the monographs of Massera and Schäffer [24], Daleckii and Krein [25] and the books of Chicone and Latushkin [26] and Barreira and Valls [27]. Regarding the practical examples for dichotomy concepts, we refer to the above works and references therein. The concept of exponential dichotomy can also be found in the works of Bento, Lupa, Megan and Silva [28], Dragičević, Sasu and Sasu [29], Stoica and Megan [13], Stoica [22], Boruga and Megan [20] for both nonuniform and uniform behavior.

The notion of exponential dichotomy may be considered too restrictive for dynamical systems so, lately, a lot of researchers have focused on polynomial dichotomy, a concept that was introduced by Barreira and Valls in [30] and it was also studied by Dragičević, Sasu and Sasu [31], Stoica [22], Bento and Silva [32], Boruga and Megan [20].

Throughout the years a more general concept of dichotomy has been considered called dichotomy with growth rates (or h-dichotomy) which is an important extension of exponential dichotomy and polynomial dichotomy. Pinto studied for the first time in [33] the notion of a growth rate, which is a nondecreasing and bijective function $h : \mathbb{R}_+ \to [1, \infty)$.

The purpose of this paper is to study integral characterizations of nonuniform dichotomy with growth rates with the particular cases of nonuniform exponential dichotomy and nonuniform polynomial dichotomy for skew-evolution cocycles in Banach spaces. We also establish the relation between these three concepts of dichotomy.

2. Preliminaries

Throughout this paper, we consider X a metric space, V a Banach space and $\mathcal{B}(V)$ the Banach algebra of all bounded linear operators acting on V. The norm of vectors on V and $\mathcal{B}(V)$ is denoted by $||\cdot||$. Moreover, we consider the following sets

$$\Delta = \{(t,s) \in \mathbb{R}^2_+ : t \geq s\}$$

$$T = \{(t,s,t_0) \in \mathbb{R}^3_+ : t \geq s \geq t_0\}.$$

Definition 1. *A mapping $\varphi : \Delta \times X \to X$ is called an evolution semiflow on X if:*

(es_1) $\varphi(s,s,x) = x$, *for any* $(s,x) \in \mathbb{R}_+ \times X$;
(es_2) $\varphi(t,s,\varphi(s,t_0,x_0)) = \varphi(t,t_0,x_0)$, *for any* $(t,s,t_0,x_0) \in T \times X$.

Definition 2. *A mapping $\Phi : \Delta \times X \to \mathcal{B}(V)$ is called a skew-evolution cocycle on $X \times V$ over the evolution semiflow φ if :*

(ses_1) $\quad \Phi(s,s,x) = I$ *(the identity operator on X), for any* $(s,x) \in \mathbb{R}_+ \times X$;
(ses_2) $\quad \Phi(t,s,\varphi(s,t_0,x_0))\Phi(s,t_0,x_0) = \Phi(t,t_0,x_0)$, *for any* $(t,s,t_0,x_0) \in T \times X$.

Example 1. *Let us consider* $X = \mathbb{R}_+$. *The mapping* $\varphi : \Delta \times \mathbb{R}_+ \to \mathbb{R}_+$ *defined by* $\varphi(t,s,x) = t - s + x$ *is an evolution semiflow on* $\mathbb{R}_+$. *For every evolution operator* $E : \Delta \to \mathcal{B}(V)$, *we have*

$$\Phi_E : \Delta \times X \to \mathcal{B}(V), \Phi_E(t,s,x) = E(t-s+x,x)$$

is a skew-evolution cocycle on $X \times V$ *over the evolution semiflow* φ. *So, a skew-evolution cocycle on* $X \times V$ *is generated by an evolution operator on* V.

Definition 3. *We say that a skew-evolution cocycle* Φ *is strongly measurable if for any* $(s,x,v) \in \mathbb{R}_+ \times X \times V$ *the mapping* $t \mapsto ||\Phi(t,s,x)v||$ *is measurable on* $[s,\infty)$.

Definition 4. *A mapping* $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ *is called a family of projectors if*

$$P^2(s,x) = P(s,x), \text{ for any } (s,x) \in \mathbb{R}_+ \times X.$$

Remark 1. *If* $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ *is a family of projectors, then* $Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$, $Q(s,x) = I - P(s,x)$ *is also a family of projectors, which is called the complementary family of projectors of P.*

Definition 5. *A family of projectors* $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ *is called invariant to the skew-evolution cocycle* Φ *if*

$$\Phi(t,s,x)P(s,x) = P(t,\varphi(t,s,x))\Phi(t,s,x),$$

for any $(t,s,x) \in \Delta \times X$.

Definition 6. *The pair* (Φ, P) *is called nonuniformly h-dichotomic if there are a nondecreasing function* $N : \mathbb{R}_+ \to [1,\infty)$ *and a constant* $\nu > 0$ *such that:*

(nhd_1) $\quad h(t)^\nu ||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \le N(t_0)h(s)^\nu ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$;
(nhd_2) $\quad h(t)^\nu ||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \le N(t_0)h(s)^\nu ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$,

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Remark 2. *The particular cases for Definition 6 are:*

1. $h(t) = e^t$, *then the concept of nonuniform exponential dichotomy is obtained.*
2. $h(t) = t + 1$, *then we have the nonuniform polynomial dichotomy concept.*
3. $N(t_0)$ *constant, we obtain the concepts of uniform h-dichotomy, uniform exponential dichotomy and uniform polynomial dichotomy.*

Example 2. *Let* $\mathcal{C}$ *be the metric space of all continuous functions* $x : \mathbb{R} \to \mathbb{R}$ *with the topology of uniform convergence on compact subsets of* $\mathbb{R}$.

Let $f : \mathbb{R}_+ \to (0,\infty)$ *be a decreasing function with the property that there exists* $\lim_{t\to\infty} f(t) = l > 0$.

We denote X the closure in $\mathcal{C}$ *of the set* $\{f_t, t \in \mathbb{R}_+\}$, *where* $f_t(\tau) = f(t+\tau)$, *for any* $\tau \in \mathbb{R}_+$. *Then* (X,d) *is a metric space. The mapping*

$$\varphi : T \times X \to X, \varphi(t,s,x)(\tau) = x_{t-s}(\tau) = x(t-s+\tau)$$

is an evolution semiflow on X. We consider $V = \mathbb{R}^2$ *the Banach space with the norm* $||v|| = |v_1| + |v_2|$, *where* $v = (v_1,v_2) \in V$. *The mapping*

$$\Phi : T \times X \to \mathcal{B}(V),$$

$$\Phi(t,s,x)v = \left(e^{\frac{\sin(t+1)}{t+1} - 2t - \frac{\sin(s+1)}{s+1} + 2s - \int_s^t x(\tau - s)d\tau} v_1, e^{\frac{\sin(s+1)}{s+1} - 2s - \frac{\sin(t+1)}{t+1} + 2t + \int_s^t x(\tau - s)d\tau} v_2 \right),$$

where $(t,s,x,v) \in \Delta \times X \times V$, *is a skew-evolution cocycle over the evolution semiflow* φ. *Let us consider the families of projectors* $P, Q : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ *defined by* $P(t,x)v = (v_1, 0)$ *and* $Q(t,x)v = (0, v_2)$, *for any* $(t,x,v) \in \mathbb{R}_+ \times X \times V$.

Using the properties of x, *we obtain*

$$\begin{aligned} ||\Phi_1(t,t_0,x_0)v|| &= e^{\frac{\sin(t+1)}{t+1} - 2t - \frac{\sin(t_0+1)}{t_0+1} + 2t_0 - \int_{t_0}^t x(\tau - t_0)d\tau} |v_1| = \\ &= ||\Phi_1(s,t_0,x_0)v|| e^{\frac{\sin(t+1)}{t+1} - 2t - \frac{\sin(s+1)}{s+1} + 2s - \int_s^t x(\tau - t_0)d\tau} \leq \\ &\leq ||\Phi_1(s,t_0,x_0)v|| e^{-l(t-s)} e^{\frac{\sin(t+1)}{t+1} - 2t - \frac{\sin(s+1)}{s+1} + 2s} \leq \\ &\leq e^{\frac{2}{t_0+1}} e^{-(l+2)(t-s)} ||\Phi_1(s,t_0,x_0)v||, \end{aligned}$$

where $||\Phi_1(t,t_0,x_0)v|| = ||\Phi(t,t_0,x_0)P(t_0,x_0)v||$.

Moreover,

$$\begin{aligned} ||\Phi_2(t,t_0,x_0)v|| &= e^{\frac{\sin(t+1)}{t+1} - 2t - \frac{\sin(t_0+1)}{t_0+1} + 2t_0 + \int_{t_0}^t x(\tau - t_0)d\tau} |v_2| = \\ &= ||\Phi_2(s,t_0,x_0)v|| e^{\frac{\sin(s+1)}{s+1} - 2s - \frac{\sin(t+1)}{t+1} + 2t + \int_s^t x(\tau - t_0)d\tau} \geq \\ &\geq ||\Phi_2(s,t_0,x_0)v|| e^{(2+l)(t-s)} e^{-\frac{1}{t+1} - \frac{1}{s+1}}, \end{aligned}$$

where $||\Phi_2(t,t_0,x_0)v|| = ||\Phi(t,t_0,x_0)Q(t_0,x_0)v||$.

We obtain that

$$\begin{aligned} ||\Phi_2(s,t_0,x_0)v|| &\leq e^{-(2+l)(t-s)} e^{\frac{1}{t+1} + \frac{1}{s+1}} ||\Phi_2(t,t_0,x_0)v|| \leq \\ &\leq e^{-(2+l)(t-s)} e^{\frac{2}{t_0+1}} ||\Phi_2(t,t_0,x_0)v||. \end{aligned}$$

In conclusion, the skew-evolution cocycle Φ *is nonuniformly exponentially dichotomic with* $\nu = l + 2$ *and* $N(t_0) = e^{\frac{2}{t_0+1}}$.

Remark 3. *Using the inequality*

$$\frac{e^t}{t+1} \geq \frac{e^s}{s+1},$$

for any $(t,s) \in \Delta$, *it results that if* (Φ, P) *is nonuniformly exponentially dichotomic, then it is also nonuniformly polynomially dichotomic. The converse implication is not true, as we can see in the following.*

Example 3. *We consider the metric space* $\mathcal{C}$, *the decreasing function* f, *the Banach space* V, *the evolution semiflow* φ *and the family of projectors* P *and* Q *defined as in Example 2.*

Let

$$\Phi : T \times X \to \mathcal{B}(V),$$

$$\Phi(t,s,x)v = \left(\left(\frac{s+3}{t+3}\right)^2 e^{-\int_s^t x(\tau - s)d\tau} v_1, \left(\frac{t+3}{s+3}\right)^2 e^{\int_s^t x(\tau - s)d\tau} v_2 \right),$$

where $(t,s,x,v) \in \Delta \times X \times V$, *be a skew-evolution cocycle over* φ.

We observe that

$$\begin{aligned}
\left(\frac{t+1}{s+1}\right)^2 ||\Phi_1(t,t_0,x_0)v|| &= \left(\frac{t+1}{s+1}\right)^2 \left(\frac{t_0+3}{t+3}\right)^2 e^{-\int\limits_{t_0}^{t} x(\tau-t_0)d\tau} |v_1| = \\
&= ||\Phi_2(t,t_0,x_0)v|| \left(\frac{s+3}{s+1}\right)^2 \left(\frac{t+1}{t+3}\right)^2 (t_0+1) e^{-\int\limits_{s}^{t} x(\tau-t_0)d\tau} \leq \\
&\leq 9(t_0+1)||\Phi_2(t,t_0,x_0)v|| e^{-l(t-s)} \leq \\
&\leq 9(t_0+1)||\Phi_2(s,t_0,x_0)v|| \left(\frac{s+1}{t+1}\right)^l
\end{aligned}$$

Thus

$$(t+1)^{2+l} ||\Phi_2(s,t_0,x_0)v|| \leq N(t_0)(s+1)^{2+l} ||\Phi_2(t,t_0,x_0)v||,$$

where $||\Phi_2(t,t_0,x_0)v|| = ||\Phi(t,t_0,x_0)Q(t_0,x_0)v||$, $N(t_0) = 9(t_0+1)$ *and* $\nu = 2+l$, *for any* $(t,s,t_0,x_0,v) \in T \times X \times V$ *and also*

$$\begin{aligned}
\left(\frac{t+1}{s+1}\right)^2 ||\Phi_2(s,t_0,x_0)v|| &= \left(\frac{t+1}{s+1}\right)^2 \left(\frac{s+3}{t_0+3}\right)^2 e^{\int\limits_{t_0}^{s} x(\tau-t_0)d\tau} |v_2| = \\
&= ||\Phi_1(s,t_0,x_0)v|| \left(\frac{s+3}{s+1}\right)^2 \left(\frac{t+1}{t+3}\right)^2 (t_0+1) e^{-\int\limits_{s}^{t} x(\tau-t_0)d\tau} \leq \\
&\leq 9(t_0+1)||\Phi_1(s,t_0,x_0)v|| e^{-l(t-s)} \leq \\
&\leq 9(t_0+1)||\Phi_1(s,t_0,x_0)v|| \left(\frac{s+1}{t+1}\right)^l
\end{aligned}$$

So

$$(t+1)^{2+l} ||\Phi_1(t,t_0,x_0)v|| \leq N(t_0)(s+1)^{2+l} ||\Phi_1(s,t_0,x_0)v||,$$

where $||\Phi_1(t,t_0,x_0)v|| = ||\Phi(t,t_0,x_0)P(t_0,x_0)v||$, $N(t_0) = 9(t_0+1)$ *and* $\nu = 2+l$, *for any* $(t,s,t_0,x_0,v) \in T \times X \times V$.

Then the pair (Φ, P) *is nonuniformly polynomially dichotomic.*

If we suppose that (Φ, P) *is nonuniformly exponentially dichotomic, then there is a nondecreasing function* $N : \mathbb{R}_+ \to [1,\infty)$ *and* $\nu > 0$ *with*

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq N(t_0)e^{-\nu(t-s)} ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

For (Φ, P) *defined as above, we have*

$$\left(\frac{t_0+3}{t+3}\right)^2 e^{-\int\limits_{t_0}^{t} x(\tau-t_0)d\tau} \leq N(t_0)e^{-\nu(t-s)} \left(\frac{t_0+3}{s+3}\right)^2 e^{-\int\limits_{t_0}^{s} x(\tau-t_0)d\tau}$$

meaning

$$\frac{1}{(t+3)^2} e^{-\int\limits_{s}^{t} x(\tau-t_0)d\tau} \leq N(s)e^{-\nu(t-s)} \frac{1}{(s+3)^2}$$

$$\frac{1}{(t+3)^2} e^{-l(t-s)} \leq N(s)e^{-\nu(t-s)} \frac{1}{(s+3)^2}$$

For $s = 0$, *we have*

$$\frac{e^{(\nu-l)t}}{(t+3)^2} \leq \frac{N(0)}{9}$$

and for $t \to \infty$ *we obtain a contradiction.*

In conclusion, (Φ, P) *is not nonuniformly exponentially dichotomic.*

Remark 4. *It is obvious that the pair (Φ, P) is uniformly h-dichotomic, then it is nonuniformly h-dichotomic. The converse implication is not generally valid. For an example, see [13].*

Definition 7. *The pair (Φ, P) has nonuniform h-growth if there are a nondecreasing function $M : \mathbb{R}_+ \to [1, \infty)$ and a constant $\omega > 0$ with:*

(nhg_1) $\quad h(s)^\omega ||\Phi(t, t_0, x_0)P(t_0, x_0)v_0|| \leq M(t_0)h(t)^\omega ||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||;$

(nhg_2) $\quad h(s)^\omega ||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0|| \leq M(t_0)h(t)^\omega ||\Phi(t, t_0, x_0)Q(t_0, x_0)v_0||,$

for any $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

Remark 5. *In Definition 7, if we consider*

1. *$h(t) = e^t$, then we obtain the nonuniform exponential growth concept.*
2. *$h(t) = t + 1$, then the skew-evolution cocycle has nonuniform polynomial growth.*
3. *$M(t_0)$ constant, we obtain the concepts of uniform h-growth, uniform exponential growth and uniform polynomial growth.*

3. Results

In what follows, we present the relation between nonuniform h-dichotomy and nonuniform exponential dichotomy, respectively, between nonuniform polynomial dichotomy and nonuniform exponential dichotomy for skew-evolution cocycles in Banach spaces.

Let $\Phi : \Delta \times X \to \mathbb{B}(V)$ be a skew-evolution cocycle over the evolution semiflow, let $\varphi : \Delta \times X \to X$, $P : \mathbb{R}_+ \times X \to \mathcal{B}(V)$ be the invariant family of projectors and let $h : \mathbb{R}_+ \to [1, \infty)$ be a growth rate.

Theorem 1. *Let*

$$\varphi_h : \Delta \times X \to X, \varphi_h(t, s, x) = \varphi(h^{-1}(e^t), h^{-1}(e^s), x)$$

be the evolution semiflow on X,

$$\Phi_h : \Delta \times X \to \mathcal{B}(V), \Phi_h(t, s, x) = \Phi(h^{-1}(e^t), h^{-1}(e^s), x)$$

the skew-evolution cocycle over φ_h and the families of projectors

$$P_h : \mathbb{R}_+ \times X \to \mathcal{B}(V), P_h(t, x) = P(h^{-1}(e^t), x),$$

$$Q_h : \mathbb{R}_+ \times X \to \mathcal{B}(V), Q_h(t, x) = Q(h^{-1}(e^t), x).$$

The pair (Φ, P) is nonuniformly h-dichotomic if and only if the pair (Φ_h, P_h) is nonuniformly exponentially dichotomic.

Proof. *Necessity.* If we suppose that the pair (Φ, P) is nonuniformly h-dichotomic, then we obtain

$$\begin{aligned} & ||\Phi_h(t, t_0, x_0)P_h(t_0, x_0)v_0|| = ||\Phi(h^{-1}(e^t), h^{-1}(e^{t_0}), x_0)P(h^{-1}(e^{t_0}), x_0)v_0|| \leq \\ \leq \quad & N(h^{-1}(e^{t_0}))e^{-\nu(t-s)}||\Phi(h^{-1}(e^s), h^{-1}(e^{t_0}), x_0)P(h^{-1}(e^{t_0}), x_0)v_0|| = \\ = \quad & N(h^{-1}(e^{t_0}))e^{-\nu(t-s)}||\Phi_h(s, t_0, x_0)P_h(t_0, x_0)v_0|| \leq \\ \leq \quad & N_1(t_0)e^{-\nu(t-s)}||\Phi_h(s, t_0, x_0)P_h(t_0, x_0)v_0||, \end{aligned}$$

where $N_1(t_0) = N(h^{-1}(e^{t_0})) + 1$, for any $(t, s, t_0, x_0, v_0) \in T \times X \times V$.

Analogously, we have

$$\begin{aligned}
& ||\Phi_h(s,t_0,x_0)Q_h(t_0,x_0)v_0|| = ||\Phi(h^{-1}(e^s),h^{-1}(e^{t_0}),x_0)Q(h^{-1}(e^{t_0}),x_0)v_0|| \leq \\
\leq\ & N(h^{-1}(e^{t_0}))e^{-\nu(t-s)}||\Phi(h^{-1}(e^t),h^{-1}(e^{t_0}),x_0)Q(h^{-1}(e^{t_0}),x_0)v_0|| = \\
=\ & N(h^{-1}(e^{t_0}))e^{-\nu(t-s)}||\Phi_h(t,t_0,x_0)Q_h(t_0,x_0)v_0|| \leq \\
\leq\ & N_1(t_0)e^{-\nu(t-s)}||\Phi_h(t,t_0,x_0)Q_h(t_0,x_0)v_0||
\end{aligned}$$

where $N_1(t_0) = N(h^{-1}(e^{t_0})) + 1$, for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Sufficiency. We suppose that the pair (Φ_h, P_h) is nonuniformly exponentially dichotomic.

$$||\Phi_h(t,t_0,x_0)P_h(t_0,x_0)v_0|| \leq N(t_0)e^{-\nu(t-s)}||\Phi_h(s,t_0,x_0)P_h(t_0,x_0)v_0||$$

$$\begin{gathered}
||\Phi(h^{-1}(e^t),h^{-1}(e^{t_0}),x_0)P(h^{-1}(e^{t_0}),x_0)v_0|| \leq \\
\leq N(t_0)e^{-\nu(t-s)}||\Phi(h^{-1}(e^s),h^{-1}(e^{t_0}),x_0)P(h^{-1}(e^{t_0}),x_0)v_0||
\end{gathered}$$

Denoting $h^{-1}(e^t) = w, h^{-1}(e^{t_0}) = w_0$ and $h^{-1}(e^s) = u$, we have

$$||\Phi(w,w_0,x_0)P(w_0,x_0)v_0|| \leq N(\ln h(w_0))\left(\frac{h(w)}{h(u)}\right)^{-\nu}||\Phi(u,w_0,x_0)P(w_0,x_0)v_0|| \leq$$

$$\leq N_2(w_0)\left(\frac{h(w)}{h(u)}\right)^{-\nu}||\Phi(u,w_0,x_0)P(w_0,x_0)v_0||,$$

where $N_2(u_0) = N(\ln h(w_0))$.

Similarly, we prove (nhd_2):

$$||\Phi_h(s,t_0,x_0)Q_h(t_0,x_0)v_0|| \leq N(t_0)e^{-\nu(t-s)}||\Phi_h(t,t_0,x_0)Q_h(t_0,x_0)v_0||$$

$$\begin{gathered}
||\Phi(h^{-1}(e^s),h^{-1}(e^{t_0}),x_0)Q(h^{-1}(e^{t_0}),x_0)v_0|| \leq \\
\leq N(t_0)e^{-\nu(t-s)}||\Phi(h^{-1}(e^t),h^{-1}(e^{t_0}),x_0)Q(h^{-1}(e^{t_0}),x_0)v_0||
\end{gathered}$$

For $h^{-1}(e^t) = w, h^{-1}(e^{t_0}) = w_0$ şi $h^{-1}(e^s) = u$, we obtain

$$||\Phi(u,w_0,x_0)Q(w_0,x_0)v_0|| \leq N(\ln h(w_0))\left(\frac{h(w)}{h(u)}\right)^{-\nu}||\Phi(w,w_0,x_0)Q(w_0,x_0)v_0|| \leq$$

$$\leq N_2(w_0)\left(\frac{h(w)}{h(u)}\right)^{-\nu}||\Phi(w,w_0,x_0)Q(w_0,x_0)v_0||,$$

where $N_2(u_0) = N(\ln h(w_0))$.

In conclusion, (Φ, P) is nonuniformly h-dichotomic. □

Corollary 1. *We consider the evolution semiflow on X*

$$\varphi_1 : \Delta \times X \to X, \varphi_1(t,s,x) = \varphi(e^t - 1, e^s - 1, x),$$

the skew-evolution cocycle over φ_1

$$\Phi_1 : \Delta \times X \to \mathcal{B}(V), \Phi_1(t,s,x) = \Phi(e^t - 1, e^s - 1, x)$$

and the families of projectors

$$P_1 : \mathbb{R}_+ \times X \to \mathcal{B}(V), P_1(s,x) = P(e^s - 1, x),$$

$$Q_1 : \mathbb{R}_+ \times X \to \mathcal{B}(V), Q_1(s,x) = Q(e^s - 1, x).$$

The pair (Φ, P) is nonuniformly polynomially dichotomic if and only if the pair (Φ_1, P_1) is nonuniformly exponentially dichotomic.

Proof. *Necessity.* We suppose that the pair (Φ, P) is nonuniformly polynomially dichotomic:

$$\begin{aligned} & ||\Phi_1(t,t_0,x_0)P_1(t_0,x_0)v_0|| = ||\Phi(e^t-1,e^{t_0}-1,x_0)P(e^{t_0}-1,x_0)v_0|| \leq \\ \leq\ & N(e^{t_0}-1)e^{-\nu(t-s)}||\Phi(e^s-1,e^{t_0}-1,x_0)P(e^{t_0}-1,x_0)v_0|| \leq \\ \leq\ & N(t_0)e^{-\nu(t-s)}||\Phi_1(s,t_0,x_0)P_1(t_0,x_0)v_0||. \end{aligned}$$

In a similar manner, we obtain:

$$\begin{aligned} & ||\Phi_1(s,t_0,x_0)Q_1(t_0,x_0)v_0|| = ||\Phi(e^s-1,e^{t_0}-1,x_0)Q(e^{t_0}-1,x_0)v_0|| \leq \\ \leq\ & N(e^{t_0}-1)e^{-\nu(t-s)}||\Phi(e^t-1,e^{t_0}-1,x_0)Q(e^{t_0}-1,x_0)v_0|| \leq \\ \leq\ & N(t_0)e^{-\nu(t-s)}||\Phi_1(t,t_0,x_0)Q_1(t_0,x_0)v_0||. \end{aligned}$$

Sufficiency. If the pair (Φ_1, P_1) is nonuniformly exponentially dichotomic, then

$$||\Phi_1(t,t_0,x_0)P_1(t_0,x_0)v_0|| \leq N(t_0)e^{-\nu(t-s)}||\Phi_1(s,t_0,x_0)P_1(t_0,x_0)v_0||$$
$$||\Phi(e^t-1,e^{t_0}-1,x_0)P(e^{t_0}-1,x_0)v_0|| \leq$$
$$\leq N(t_0)e^{-\nu(t-s)}||\Phi(e^s-1,e^{t_0}-1,x_0)P(e^{t_0}-1,x_0)v_0||.$$

For $e^t-1=u, e^{t_0}-1=u_0$ and $e^s-1=w$, we obtain

$$||\Phi(u,u_0,x_0)P(u_0,x_0)v_0|| \leq N(\ln(u_0+1))\left(\frac{u+1}{w+1}\right)^{-\nu}||\Phi(w,u_0,x_0)P(u_0,x_0)v_0||$$

$$||\Phi(u,u_0,x_0)P(u_0,x_0)v_0|| \leq N_1(u_0)\left(\frac{u+1}{w+1}\right)^{-\nu}||\Phi(w,u_0,x_0)P(u_0,x_0)v_0||,$$

where $N_1(u_0) = N(\ln(u_0+1))$.

Similarly, we prove (npd_2)

$$||\Phi_1(s,t_0,x_0)Q_1(t_0,x_0)v_0|| \leq N(t_0)e^{-\nu(t-s)}||\Phi_1(t,t_0,x_0)Q_1(t_0,x_0)v_0||$$
$$||\Phi(e^s-1,e^{t_0}-1,x_0)Q(e^{t_0}-1,x_0)v_0|| \leq$$
$$\leq N(t_0)e^{-\nu(t-s)}||\Phi(e^t-1,e^{t_0}-1,x_0)Q(e^{t_0}-1,x_0)v_0||.$$

If we denote $e^t-1=u, e^{t_0}-1=u_0$ and $e^s-1=w$, then the following relation takes place:

$$||\Phi(w,u_0,x_0)Q(u_0,x_0)v_0|| \leq N(\ln(u_0+1))\left(\frac{u+1}{w+1}\right)^{-\nu}||\Phi(u,u_0,x_0)Q(u_0,x_0)v_0||.$$

$$||\Phi(w,u_0,x_0)Q(u_0,x_0)v_0|| \leq N_1(u_0)\left(\frac{u+1}{w+1}\right)^{-\nu}||\Phi(u,u_0,x_0)Q(u_0,x_0)v_0||.$$

In conclusion, the pair (Φ, P) is nonuniformly polynomially dichotomic. □

Necessary and sufficient conditions of nonuniform h-dichotomy, including the particular cases of nonuniform exponential dichotomy and nonuniform polynomial dichotomy for skew-evolution cocycles, are given by the following theorems.

In what follows, we consider some classes of functions:

- $\mathcal{H}$ the set of functions $h : \mathbb{R}_+ \to [1, \infty)$ with the properties:

  (h_1) exists $H_1 > 1$ such that $h(t+1) \leq H_1 h(t)$, for any $t \geq 0$;

(h_2) for any $\alpha \in (0,1)$, there exists $H_2 > 1$ such that $\int\limits_s^\infty \frac{dt}{h(t)^\alpha} \leq \frac{H_2}{h(s)^\alpha}$, for any $s \geq 0$;

(h_3) for any $\alpha \in (-1,0)$, there exists $H_3 > 1$ such that $\int\limits_0^t h(s)^\alpha ds \leq H_3 h(t)^\alpha$, for any $t \geq 0$;

- $\mathcal{H}_0$ the set of functions $h : \mathbb{R}_+ \to [1,\infty)$ with $h(t) \geq t+1$, for any $t \geq 0$.

Remark 6. *For the particular case when $h(t) = e^t$, we obtain that $h \in \mathcal{H}$.*

Theorem 2. *If the pair (Φ, P) has nonuniform h-growth, with Φ being a strongly measurable skew-evolution cocycle and $h \in \mathcal{H}$, then (Φ, P) is nonuniformly h-dichotomic if and only if there are a nondecreasing function $D : \mathbb{R}_+ \to [1,\infty)$ and a constant $d \in (0,1)$ such that*

(nhD_1) $\int\limits_s^\infty h(t)^d ||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt \leq D(t_0)h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$, *for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$.*

(nhD_2) $\int\limits_{t_0}^t h(s)^{-d} ||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||ds \leq D(t_0)h(t)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$, *for any $(t,t_0,x_0,v_0) \in \Delta \times X \times V$.*

Proof. *Necessity.* Let $d \in (0,\nu)$.

$$\begin{aligned}
&\int\limits_s^\infty h(t)^d ||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt \leq \\
\leq\ & \int\limits_s^\infty N(t_0)h(t)^d \left(\frac{h(t)}{h(s)}\right)^{-\nu} ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||dt = \\
=\ & N(t_0)h(s)^\nu ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| \int\limits_s^\infty h(t)^{d-\nu}dt \leq \\
\leq\ & N(t_0)H_2 h(s)^\nu ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| h(s)^{d-\nu} = \\
=\ & N(t_0)H_2 h(s)^d ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| = \\
=\ & D(t_0)h(s)^d ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,
\end{aligned}$$

where $D(t_0) = N(t_0)H_2$.

Analogously, we have

$$\begin{aligned}
&\int\limits_{t_0}^t h(s)^{-d} ||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||ds \leq \\
\leq\ & \int\limits_{t_0}^t N(t_0)h(s)^{-d} \left(\frac{h(t)}{h(s)}\right)^{-\nu} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||ds = \\
=\ & N(t_0)h(t)^{-\nu} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \int\limits_{t_0}^t h(s)^{\nu-d}ds \leq \\
\leq\ & N(t_0)H_3 h(t)^{-\nu} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| h(t)^{\nu-d} = \\
=\ & N(t_0)H_3 h(t)^{-d} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \leq \\
\leq\ & D(t_0)h(t)^{-d} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,
\end{aligned}$$

where $D(t_0) = N(t_0)H_3$.

Sufficiency. Case 1.1: $t \geq s+1$.

$$\begin{aligned} h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| &= \int_{t-1}^{t} h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq \\ &\leq \int_{t-1}^{t} h(t)^d M(t_0)\left(\frac{h(t)}{h(\tau)}\right)^{\omega}||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau = \\ &= \int_{t-1}^{t} M(t_0)h(\tau)^d\left(\frac{h(t)}{h(\tau)}\right)^{d+\omega}||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq \\ &\leq H_1^{d+\omega}M(t_0)\int_{s}^{\infty} h(\tau)^d||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq \\ &\leq D(t_0)H_1^{d+\omega}M(t_0)h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||. \end{aligned}$$

So

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq D(t_0)M(t_0)H_1^{d+\omega}\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \quad (1)$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.2: $t \in [s,s+1)$.

$$\begin{aligned} h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| &\leq h(t)^d M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| = \\ &= M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega+d}h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| \leq \\ &\leq M(t_0)H_1^{\omega+d}h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||. \end{aligned}$$

So

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq M(t_0)H_1^{\omega+d}\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \quad (2)$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

From relations (1) and (2), we obtain that there exists $N(t_0) = D(t_0)M(t_0)H_1^{d+\omega}$ with

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq N(t_0)\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

In a similar, way we obtain (nhd_2).

Case 1.1′: $t \geq s+1$.

$$
\begin{aligned}
h(s)^{-d}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| &= \int_s^{s+1} h(s)^{-d}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq \\
&\leq \int_s^{s+1} M(t_0)h(s)^{-d}\left(\frac{h(\tau)}{h(s)}\right)^{\omega}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq \\
&\leq \int_s^{s+1} M(t_0)h(\tau)^{-d}\left(\frac{h(\tau)}{h(s)}\right)^{\omega+d}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq \\
&\leq M(t_0)H_1^{\omega+d}\int_{t_0}^{t} h(\tau)^{-d}||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||d\tau \leq \\
&\leq D(t_0)M(t_0)H_1^{\omega+d}h(t)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.
\end{aligned}
$$

Thus

$$
||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq D(t_0)M(t_0)H_1^{\omega+d}\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||, \tag{3}
$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.2′: $t \in [s,s+1)$.

$$
\begin{aligned}
h(s)^{-d}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| &\leq M(t_0)h(s)^{-d}\left(\frac{h(t)}{h(s)}\right)^{\omega}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| = \\
&= M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega+d}h(t)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \leq \\
&\leq M(t_0)H_1^{\omega+d}h(t)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.
\end{aligned}
$$

So

$$
||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq M(t_0)H_1^{\omega+d}\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||, \tag{4}
$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

It follows from relations (3) and (4) that there exists $N(t_0) = M(t_0)D(t_0)H_1^{\omega+d}$ with

$$
||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq N(t_0)\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,
$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$. □

Corollary 2. *If the pair (Φ, P) has nonuniform exponential growth, with Φ being a strongly measurable skew-evolution cocycle, then (Φ, P) is nonuniformly exponentially dichotomic if and only if there is a nondecreasing function $D : \mathbb{R}_+ \to [1, \infty)$ and $d \in (0, 1)$ with:*

(neD_1) $\int_s^{\infty} e^{dt}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt \leq D(t_0)e^{ds}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||$,
for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$.

(neD_2) $\int_{t_0}^{t} e^{-ds}||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||ds \leq D(t_0)e^{-dt}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||$,
for any $(t,t_0,x_0,v_0) \in \Delta \times X \times V$.

Proof. This results from *Theorem* 2 for $h(t) = e^t$. □

Corollary 3. *If the pair (Φ, P) has nonuniform polynomial growth, with Φ being a strongly measurable skew-evolution cocycle, then (Φ, P) is nonuniformly polynomially dichotomic if and only if there is a nondecreasing function $D : \mathbb{R}_+ \to [1, \infty)$ and $d \in (0, 1)$ with:*

(npD_1) $\int_s^\infty (t+1)^{d-1}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0||dt \le D(t_0)(s+1)^d||\Phi(s, t_0, x)P(t_0, x_0)v_0||$, *for any* $(s, t_0, x_0, v_0) \in \Delta \times X \times V$.

(npD_2) $\int_{t_0}^t (s+1)^{-d-1}||\Phi(s, t_0, x_0)Q(t_0, x_0)v_0||ds \le D(t_0)(t+1)^{-d}||\Phi(t, t_0, x_0)Q(t_0, x_0) v_0||$, *for any* $(t, t_0, x_0, v_0) \in \Delta \times X \times V$.

Proof. From Corollary 1, we have that (Φ, P) is nonuniformly polynomially dichotomic if and only if the pair (Φ_1, P_1) is nonuniformly exponentially dichotomic.

In order to prove $(npd_1) \iff (npD_1)$, we use this equivalence and from Corollary 2 we have

$$\int_s^\infty e^{dt}||\Phi_1(t, t_0, x_0)P_1(t_0, x_0)v_0||dt \le D(t_0)e^{ds}||\Phi_1(s, t_0, x)P_1(t_0, x_0)v_0||,$$

which is

$$\int_s^\infty e^{dt}||\Phi(e^t-1, e^{t_0}-1, x_0)P(e^{t_0}-1, x_0)v_0||dt \le D(t_0)e^{ds}||\Phi(e^s-1, e^{t_0}-1, x_0)P(e^{t_0}-1, x_0)v_0||$$

With the change of variable $e^t - 1 = u$ and the notations $e^{t_0} - 1 = u_0$ and $e^s - 1 = w$, we obtain

$$\int_w^\infty (u+1)^d||\Phi(u, u_0, x_0)P(u_0, x_0)v_0||\frac{du}{u+1} \le D(\ln(u_0+1))(w+1)^d||\Phi_1(w, u_0, x)P_1(u_0, x_0)v_0||$$

meaning

$$\int_w^\infty (u+1)^{d-1}||\Phi(u, u_0, x_0)P(u_0, x_0)v_0||du \le D_1(u_0)(w+1)^d||\Phi_1(w, u_0, x)P_1(u_0, x_0)v_0||,$$

where $D_1(u_0) = D(\ln(u_0+1))$.

For the proof of $(npd_2) \iff (npD_2)$, we follow the same idea as above:

$$\int_{t_0}^t e^{-ds}||\Phi_1(s, t_0, x_0)Q_1(t_0, x_0)v_0||ds \le D(t_0)e^{-dt}||\Phi_1(t, t_0, x_0)Q_1(t_0, x_0)v_0||;$$

more precisely

$$\int_{t_0}^t e^{-ds}||\Phi(e^s-1, e^{t_0}-1, x_0)Q(e^{t_0}-1, x_0)v_0||ds \le$$
$$\le D(t_0)e^{-dt}||\Phi(e^t-1, e^{t_0}-1, x_0)Q(e^{t_0}-1, x_0)v_0||;$$

using the change of variable $e^s - 1 = w$ and the notations $e^t - 1 = u$ and $e^{t_0} - 1 = u_0$, we have

$$\int_{u_0}^u (w+1)^{-d}||\Phi(w, u_0, x_0)Q(u_0, x_0)v_0||\frac{dw}{w+1} \le$$
$$\le D(\ln(u_0+1))(u+1)^{-d}||\Phi(u, u_0, x_0)Q(u_0, x_0)v_0||,$$

which is equivalent to

$$\int_{u_0}^{u}(w+1)^{-d-1}||\Phi(w,u_0,x_0)Q(u_0,x_0)v_0||dw \leq D_1(u_0)(u+1)^{-d}||\Phi(u,u_0,x_0)Q(u_0,x_0)v_0||,$$

where $D_1(u_0) = D(\ln(u_0+1))$. $\square$

Theorem 3. *If the pair (Φ, P) has nonuniform h-growth, with Φ being a strongly measurable skew-evolution cocycle and $h \in \mathcal{H}$, then (Φ, P) is nonuniformly h-dichotomic if and only if there is a nondecreasing function $D : \mathbb{R}_+ \to [1,\infty)$ and a constant $d \in (0,1)$ such that*

(nhD_1') $\int_{t_0}^{t} \frac{h(s)^{-d}}{||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||}ds \leq \frac{D(t_0)h(t)^{-d}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||}$,
for any $(t,t_0,x_0,v_0) \in \Delta \times X \times V$ with $\Phi(t,t_0,x_0)P(t_0,x_0)v_0 \neq 0$.

(nhD_2') $\int_{s}^{\infty} \frac{h(t)^{d}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}dt \leq \frac{D(t_0)h(s)^{d}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}$,
for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$ with $Q(t_0,x_0)v_0 \neq 0$.

Proof. *Necessity.* We consider $d \in (0,\nu)$.

$$\begin{aligned}
\int_{t_0}^{t} \frac{h(s)^{-d}}{||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||}ds &\leq \int_{t_0}^{t} N(t_0)\frac{1}{h(s)^d}\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{ds}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||} = \\
&= \frac{N(t_0)h(t)^{-\nu}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||}\int_{t_0}^{t} h(s)^{\nu-d}ds \leq \\
&\leq \frac{N(t_0)h(t)^{-\nu}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||}H_3 h(t)^{\nu-d} = \\
&= N(t_0)H_3\frac{h(t)^{-d}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||} \leq \\
&\leq D(t_0)\frac{h(t)^{-d}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||},
\end{aligned}$$

where $D(t_0) = N(t_0)H_3$.

Similarly, we have

$$\begin{aligned}
\int_{s}^{\infty} \frac{h(t)^{d}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}dt &\leq \int_{s}^{\infty} N(t_0)h(t)^d\left(\frac{h(t)}{h(s)}\right)^{-\nu}\frac{1}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}dt = \\
&= \frac{N(t_0)h(s)^{\nu}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}\int_{s}^{\infty} h(t)^{d-\nu}dt \leq \\
&\leq \frac{N(t_0)h(s)^{\nu}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}H_2 h(s)^{d-\nu} \leq \\
&\leq \frac{D(t_0)h(s)^{d}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||},
\end{aligned}$$

where $D(t_0) = N(t_0)H_2$.

Sufficiency. Case 1.1: $t \geq s+1$ *and* $\Phi(t,t_0,x_0)P(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}
\frac{h(s)^{-d}}{||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||} &= \int_s^{s+1} \frac{h(s)^{-d}}{||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||} d\tau \leq \\
&\leq \int_s^{s+1} M(t_0)\left(\frac{h(\tau)}{h(s)}\right)^{\omega} \frac{h(s)^{-d}}{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||} d\tau = \\
&= M(t_0)\int_s^{s+1} \left(\frac{h(\tau)}{h(s)}\right)^{\omega+d} \frac{h(\tau)^{-d}}{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||} d\tau \leq \\
&\leq M(t_0)H_1^{\omega+d}\int_{t_0}^{t} \frac{h(\tau)^{-d}}{||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||} d\tau \leq \\
&\leq M(t_0)H_1^{\omega+d}D(t_0)\frac{h(t)^{-d}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||}.
\end{aligned}$$

Thus

$$h(t)^{d}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq M(t_0)D(t_0)H_1^{\omega+d}h(s)^{d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \tag{5}$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.2: $t \in [s,s+1)$ *with* $\Phi(t,t_0,x_0)P(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}
h(t)^{d}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| &\leq M(t_0)h(t)^{d}\left(\frac{h(t)}{h(s)}\right)^{\omega}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| = \\
&= M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega+d}h(s)^{d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| \leq \\
&\leq M(t_0)H_1^{\omega+d}h(s)^{d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||.
\end{aligned}$$

So

$$h(t)^{d}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq M(t_0)H_1^{\omega+d}h(s)^{d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \tag{6}$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

From relation (5) and relation (6), it follows that there exists $N(t_0) = M(t_0)D(t_0)H_1^{\omega+d}$ such that

$$h(t)^{d}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq N(t_0)h(s)^{d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Using the same method as above, we obtain (nhd_2):

Case 1.1′: $t \geq s+1$ *and* $Q(t_0, x_0)v_0 \neq 0$.

$$\begin{aligned}
\frac{h(t)^d}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||} &= \int_{t-1}^{t} \frac{h(t)^d}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||} d\tau \leq \\
&\leq \int_{t-1}^{t} M(t_0)\left(\frac{h(t)}{h(\tau)}\right)^{\omega} \frac{h(t)^d}{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||} d\tau = \\
&= \int_{t-1}^{t} M(t_0)\left(\frac{h(t)}{h(\tau)}\right)^{\omega+d} \frac{h(\tau)^d}{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||} d\tau \leq \\
&\leq M(t_0)H_1^{\omega+d} \int_{s}^{\infty} \frac{h(\tau)^d}{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||} d\tau \leq \\
&\leq M(t_0)H_1^{\omega+d} D(t_0) \frac{h(s)^d}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}.
\end{aligned}$$

So we have

$$h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq M(t_0)D(t_0)H_1^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||, \tag{7}$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.2′: $t \in [s, s+1)$ *with* $Q(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}
h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| &\leq M(t_0)h(t)^d\left(\frac{h(t)}{h(s)}\right)^{\omega} ||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| = \\
&= M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega+d} h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \leq \\
&\leq M(t_0)H_1^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.
\end{aligned}$$

So

$$h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq M(t_0)H_1^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||, \tag{8}$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

From (7) and (8), we obtain that there exists $N(t_0) = M(t_0)D(t_0)H_1^{\omega+d}$ such that

$$h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq N(t_0)h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$. □

Corollary 4. *If the pair* (Φ, P) *has nonuniform exponential growth, with* Φ *being a strongly measurable skew-evolution cocycle, then* (Φ, P) *is nonuniformly exponentially dichotomic if and only if there are a nondecreasing function* $D : \mathbb{R}_+ \to [1,\infty)$ *and a constant* $d \in (0,1)$ *such that:*

(neD_1') $\int_{t_0}^{t} \frac{e^{-ds}}{||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||} ds \leq \frac{D(t_0)e^{-dt}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||}$,
for any $(t,t_0,x_0,v_0) \in \Delta \times X \times V$ *with* $\Phi(t,t_0,x_0)P(t_0,x_0)v_0 \neq 0$.

(neD_2') $\int_{s}^{\infty} \frac{e^{dt}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||} dt \leq \frac{D(t_0)e^{ds}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}$,
for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$ *with* $Q(t_0,x_0)v_0 \neq 0$.

Proof. It follows from *Theorem 3* for $h(t) = e^t$. □

Corollary 5. *If the pair* (Φ, P) *has nonuniform polynomial growth, with* Φ *being a strongly measurable skew-evolution cocycle, then* (Φ, P) *is nonuniformly polynomially dichotomic if and only if there are a nondecreasing function* $D : \mathbb{R}_+ \to [1, \infty)$ *and* $d \in (0, 1)$ *a constant with:*

(npD'_1) $\int\limits_{t_0}^{t} \frac{(s+1)^{-d-1}}{||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||} ds \le \frac{D(t_0)(t+1)^{-d}}{||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||}$,
for any$(t, t_0, x_0, v_0) \in \Delta \times X \times V$ *with* $\Phi(t, t_0, x_0)P(t_0, x_0)v_0 \ne 0$.

(npD'_2) $\int\limits_{s}^{\infty} \frac{(t+1)^{d-1}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||} dt \le \frac{D(t_0)(s+1)^{d}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}$,
for any $(s, t_0, x_0, v_0) \in \Delta \times X \times V$ *with* $Q(t_0, x_0)v_0 \ne 0$.

Proof. From *Corollary 1*, we have

$$\int\limits_{t_0}^{t} \frac{e^{-ds}}{||\Phi_1(s, t_0, x_0)P_1(t_0, x_0)v_0||} ds \le \frac{D(t_0)e^{-dt}}{||\Phi_1(t, t_0, x_0)P_1(t_0, x_0)v_0||},$$

which is

$$\int\limits_{t_0}^{t} \frac{e^{-ds}}{||\Phi(e^s-1, e^{t_0}-1, x_0)P(e^{t_0}-1, x_0)v_0||} ds \le \frac{D(t_0)e^{-dt}}{||\Phi(e^t-1, e^{t_0}-1, x_0)P(e^{t_0}-1, x_0)v_0||}$$

For $e^t - 1 = u, e^{t_0} - 1 = u_0$ and $e^s - 1 = w$

$$\int\limits_{u_0}^{u} \frac{(w+1)^{-d-1}}{||\Phi(w, u_0, x_0)P(u_0, x_0)v_0||} dw \le \frac{D_1(u_0)(u+1)^{-d}}{||\Phi(u, u_0, x_0)P(u_0, x_0)v_0||},$$

where $D_1(u_0) = D(\ln(u_0 + 1))$.
Using the same method, we obtain

$$\int\limits_{s}^{\infty} \frac{e^{dt}}{||\Phi_1(t, t_0, x_0)Q_1(t_0, x_0)v_0||} dt \le \frac{D(t_0)e^{ds}}{||\Phi_1(s, t_0, x_0)Q_1(t_0, x_0)v_0||}$$

$$\int\limits_{s}^{\infty} \frac{e^{dt}}{||\Phi(e^t-1, e^{t_0}-1, x_0)Q(e^{t_0}-1, x_0)v_0||} dt \le \frac{D(t_0)e^{ds}}{||\Phi(e^s-1, e^{t_0}-1, x_0)Q(e^{t_0}-1, x_0)v_0||}$$

For $e^t - 1 = u, e^{t_0} - 1 = u_0$ and $e^s - 1 = w$, we have

$$\int\limits_{w}^{\infty} \frac{(u+1)^{d-1}}{||\Phi(u, u_0, x_0)Q(u_0, x_0)v_0||} du \le \frac{D_1(u_0)(w+1)^{d}}{||\Phi(w, u_0, x_0)Q(u_0, x_0)v_0||},$$

where $D_1(u_0) = D(\ln(u_0 + 1))$. □

Theorem 4. *If the pair* (Φ, P) *has nonuniform h-growth, with* Φ *being a strongly measurable skew-evolution cocycle and* $h \in \mathcal{H} \cap \mathcal{H}_0$*, then* (Φ, P) *is nonuniformly h-dichotomic if and only if there are a nondecreasing function* $D : \mathbb{R}_+ \to [1, \infty)$ *and a constant* $d > 1$ *such that:*

(nhD''_1) $\int\limits_{s}^{\infty} h(t)^{d-1}||\Phi(t, t_0, x_0)P(t_0, x_0)v_0||dt \le D(t_0)h(s)^d||\Phi(s, t_0, x_0)P(t_0, x_0)v_0||$,
for any $(s, t_0, x_0, v_0) \in \Delta \times X \times V$.

(nhD''_2) $\int\limits_{s}^{\infty} \frac{h(t)^{d-1}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||} dt \le \frac{D(t_0)h(s)^{d}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}$,
for any $(s, t_0, x_0, v_0) \in \Delta \times X \times V$ *with* $Q(t_0, x_0)v_0 \ne 0$.

Proof. *Necessity.* From Theorem 2, we obtain

$$\begin{aligned}\int_s^\infty h(t)^{d-1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt &\leq \int_s^\infty h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt \leq \\ &\leq D(t_0)h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||\end{aligned}$$

and from Theorem 3 we have

$$\begin{aligned}\int_s^\infty \frac{h(t)^{d-1}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}dt &\leq \int_s^\infty \frac{h(t)^d}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}dt \leq \\ &\leq \frac{D(t_0)h(s)^d}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}.\end{aligned}$$

Sufficiency. Case 1.1: $h(t) \geq 2s$.

$$\begin{aligned}h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| &= \frac{2}{h(t)}\int_{\frac{h(t)}{2}}^{h(t)} h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq \\ &\leq \frac{2}{h(t)}\int_{\frac{h(t)}{2}}^{h(t)} h(t)^d M(t_0)\left(\frac{h(t)}{h(\tau)}\right)^\omega ||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau = \\ &= 2M(t_0)\int_{\frac{h(t)}{2}}^{h(t)} h(\tau)^{d-1}\left(\frac{h(t)}{h(\tau)}\right)^{d+\omega-1} ||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq \\ &\leq 2^{d+\omega}M(t_0)\int_s^\infty h(\tau)^{d-1}||\Phi(\tau,t_0,x_0)P(t_0,x_0)v_0||d\tau \leq \\ &\leq D(t_0)2^{d+\omega}M(t_0)h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||.\end{aligned}$$

So

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq D(t_0)M(t_0)2^{d+\omega}\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \quad (9)$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.2: $h(t) < 2s$.

$$\begin{aligned}h(t)^d||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| &\leq h(t)^d M(t_0)\left(\frac{h(t)}{h(s)}\right)^\omega ||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| = \\ &= M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega+d} h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0|| \leq \\ &\leq M(t_0)2^{\omega+d}h(s)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||.\end{aligned}$$

So

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq M(t_0)2^{\omega+d}\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||, \quad (10)$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

From relations (9) and (10), we obtain that there exists $N(t_0) = D(t_0)M(t_0)2^{d+\omega}$ with

$$||\Phi(t,t_0,x_0)P(t_0,x_0)v_0|| \leq N(t_0)\left(\frac{h(t)}{h(s)}\right)^{-d}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.1′: $h(t) \geq 2s$ and $Q(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}
\frac{h(t)^d}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||} &= \frac{2}{h(t)}\int_{\frac{h(t)}{2}}^{h(t)} \frac{h(t)^d}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}d\tau \leq \\
&\leq \frac{2}{h(t)}\int_{\frac{h(t)}{2}}^{h(t)} M(t_0)\left(\frac{h(t)}{h(\tau)}\right)^{\omega}\frac{h(t)^d}{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}d\tau = \\
&= 2M(t_0)\int_{\frac{h(t)}{2}}^{h(t)}\left(\frac{h(t)}{h(\tau)}\right)^{\omega+d-1}\frac{h(\tau)^{d-1}}{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}d\tau \leq \\
&\leq M(t_0)2^{\omega+d}\int_{s}^{\infty}\frac{h(\tau)^{d-1}}{||\Phi(\tau,t_0,x_0)Q(t_0,x_0)v_0||}d\tau \leq \\
&\leq M(t_0)2^{\omega+d}D(t_0)\frac{h(s)^d}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||}.
\end{aligned}$$

So we have

$$h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq M(t_0)D(t_0)2^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||, \tag{11}$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

Case 1.2′: $h(t) < 2s$ with $Q(t_0,x_0)v_0 \neq 0$.

$$\begin{aligned}
h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| &\leq M(t_0)h(t)^d\left(\frac{h(t)}{h(s)}\right)^{\omega}||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| = \\
&= M(t_0)\left(\frac{h(t)}{h(s)}\right)^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0|| \leq \\
&\leq M(t_0)2^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||.
\end{aligned}$$

So

$$h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq M(t_0)2^{\omega+d}h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||, \tag{12}$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$.

From (11) and (12), we obtain that there exists $N(t_0) = M(t_0)D(t_0)2^{\omega+d}$ such that

$$h(t)^d||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0|| \leq N(t_0)h(s)^d||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||,$$

for any $(t,s,t_0,x_0,v_0) \in T \times X \times V$. □

Corollary 6. *If the pair (Φ,P) has nonuniform exponential growth, with Φ being a strongly measurable skew-evolution cocycle, then (Φ,P) is nonuniformly exponentially dichotomic if and only if there are a nondecreasing function $D : \mathbb{R}_+ \to [1,\infty)$ and a constant $d > 1$ with*

(neD_1'') $\quad \int_s^{\infty} e^{t(d-1)}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt \leq D(t_0)e^{ds}||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$

for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$.

(neD_2'') $\displaystyle\int_s^\infty \frac{e^{t(d-1)}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}dt \leq \frac{D(t_0)e^{ds}}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||},$
for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$ *with* $Q(t_0,x_0)v_0 \neq 0$.

Proof. This follows from Theorem 4 by taking $h(t) = e^t$. □

Corollary 7. *If the pair* (Φ, P) *has nonuniform polynomial growth, with* Φ *being a strongly measurable skew-evolution cocycle, then* (Φ, P) *is nonuniformly polynomially dichotomic if and only if there are a nondecreasing function* $D : \mathbb{R}_+ \to [1,\infty)$ *and a constant* $d > 1$ *with*

(npD_1'') $\displaystyle\int_s^\infty (t+1)^{d-1}||\Phi(t,t_0,x_0)P(t_0,x_0)v_0||dt \leq D(t_0)(s+1)^d||\Phi(s,t_0,x_0)P(t_0,x_0)v_0||,$
for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$.

(npD_2'') $\displaystyle\int_s^\infty \frac{(t+1)^{d-1}}{||\Phi(t,t_0,x_0)Q(t_0,x_0)v_0||}dt \leq \frac{D(t_0)(s+1)^d}{||\Phi(s,t_0,x_0)Q(t_0,x_0)v_0||},$
for any $(s,t_0,x_0,v_0) \in \Delta \times X \times V$ *with* $Q(t_0,x_0)v_0 \neq 0$.

Proof. This results from Theorem 4 if the growth rate equals $t+1$. □

4. Conclusions

The theory of skew-evolution cocycles is used to study the asymptotic properties of time-varying linear systems. In this sense, we can say that it is possible to give generalizations of Datko [6] for the concepts of stability or dichotomy. So, the question is whether we can find a system associated with skew-evolution cocycles in order to use the concepts of nonuniform h-dichotomy, nonuniform exponential dichotomy and nonuniform polynomial dichotomy.

The aim of this paper is to find an answer for this question. Therefore, we succeeded in giving necessary and sufficient conditions for nonuniform h-dichotomy and its correspondents for the particular cases of nonuniform exponential dichotomy and nonuniform polynomial dichotomy of skew-evolution cocycles in Banach spaces.

The article gives characterizations for nonuniform dichotomic behaviors of dynamical systems described via skew-evolution cocycles that contain as special cases evolution equations in abstract spaces and introduces a new concept of nonuniform dichotomy compared to the classical ones considered in Bento, Megan, Lupa and Silva [28], Stoica and Borlea [34], Stoica and Megan [13] and Stoica [22].

The main results of our work are presented in *Theorems 2–4* for the concept of nonuniform h-dichotomy of skew-evolution cocycles in Banach spaces.

The connection between nonuniform h-dichotomy and nonuniform exponential dichotomy is given by *Theorem 1* and between nonuniform polynomial dichotomy and nonuniform exponential dichotomy is given by *Corollary 1*.

As particular cases of *Theorems 2–4* , we obtain *Corollaries 2, 4 and 6* when the growth rate is the exponential function e^t and *Corollaries 3, 5 and 7* when we have $h(t) = t+1$.

In the near future, the goals pursued by the authors are to obtain integral characterizations for these three concepts which extend the results of Barbashin [35] and also for the concept of trichotomy described by skew-evolution cocycles in Banach spaces.

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