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A Study of the Monotonic Properties of Solutions of Neutral Differential Equations and Their Applications

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Abstract: In this paper, we aim to study the monotonic properties of the solutions of a class of neutral delay differential equations. The importance of this study lies in the fact that the monotonic properties largely control the study of the oscillation and asymptotic behaviour of the solutions to delay differential equations. Then, by using the new properties, we create improved criteria for testing the oscillation of solutions to the studied equation. We also find new criteria that can be applied more than once. Moreover, we discuss the importance and novelty of the results through the application to a special case of the studied equation.

Keywords: delay differential equation; neutral delay; monotonic properties; oscillation

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1. Introduction

Differential equations are the most important link between mathematics and applied sciences, biology, engineering and others. Differential equation models that describe different phenomena enable us to study, analyse and understand these phenomena. However, this requires either solving these models or studying the properties of their solutions. The first aspect is covered by analytical or numerical methods by finding exact or approximate solutions to these models. As for the other side, it is covered by the qualitative theory, which is concerned with investigating the qualitative characteristics of solutions such as oscillation, periodicity, stability, and others.

Oscillation theory is the theory concerned with the investigation of the asymptotic and oscillatory behaviour of solutions to differential equations. This theory is concerned with finding conditions that confirm that all solutions of the equation are oscillatory, guarantee the existence of an oscillatory solution, provide an asymptotic property for non-oscillatory solutions, or study the distance between the zeros of oscillatory solutions.

Neutral differential equations (NDEs) are one type of delay differential equation (DDEs) in which the highest derivative appears on the solution with and without delay. In electrical circuits containing lossless transmission lines and in the study of vibrating masses, models of NDEs appear, see [1]. With the development of new models and the significant technical and scientific advancement that the world is currently experiencing in engineering, biology, and physics, interest in understanding the qualitative properties of DDEs is growing, see [2–5].

In this work, we investigate the asymptotic behaviour of solutions to the even-order NDEs of the form

$$\frac{\mathrm{d}^n}{\mathrm{d}s^n}\mathcal{U}(s) + \phi(s)x(\delta(s)) = 0, \tag{1}$$

where $s \geq s_0$, $n \geq 4$ is even, and $\mathcal{U} = x + \varphi \cdot (x \circ \beta)$. We also assume the following conditions:

(C1) φ and ϕ are continuous on $[s_0,\infty)$ and satisfy the conditions: $0 \le \varphi(s) \le \varphi_0$, $\phi(s) > 0$, and ϕ does not vanish identically on any half-line $[s_*,\infty)$, for $s_* \ge s_0$.

(C2) β and δ are continuous delay functions on $[s_0,\infty)$ and fulfil the conditions: $\beta(s) \le s$, $\delta(s) \le s$, $\delta'(s) \ge 0$ and $\lim_{s\to\infty} \beta(s) = \lim_{s\to\infty} \delta(s) = \infty$.

For a solution of (1), we mean a real function $x \in \mathbf{C}([s_*,\infty))$ for $s_* \ge s_0$, which has the property $\mathcal{U} \in \mathbf{C}^n([s_*,\infty))$ and x satisfies (1) on $[s_*,\infty)$. We take into account these solutions x of (1) such that $\sup\{|x(s)| : s \ge s_1\} > 0$ for $s_1 \ge s_*$. A solution x of (1) is said to be non-oscillatoryif it is eventually positive or negative; otherwise, it is said to be oscillatory.

The last decade has witnessed a great development in the study of the oscillatory behaviour of different-order DDEs. Monographs [6–10] have collected the most important results in the oscillation theory of DDEs up to the decade before last.

It is easy to notice the great development in the study of oscillations of second-order DDEs. For example, Bohner et al. [11] and Džurina et al. [12] developed an improved approach to study the oscillation of NDE

$$\left(r(s)\left(\mathcal{U}'(s)\right)^{\alpha}\right)' + \phi(s)x^{\alpha}(\delta(s)) = 0, \tag{2}$$

in the non-canonical case. Later, Grace et al. [13] extended the approach in [11] to the canonical case of NDE (2). Moaaz et al. [14] presented more efficient criteria for testing the oscillation of NDE (2) in the canonical case based on the definition of two Riccati substitutions. Whereas more recently, Bohner et al. [15] and Jadlovská [16] obtained sharp criteria to ensure the oscillation of NDE (2).

On the other hand, the study of oscillation of higher-order DDEs has also received great attention recently. Agarwal et al. [17] and Li and Rogovchenko [18] introduced criteria for the oscillation of NDE (1). Therefore, from [18], we mention the following result:

Theorem 1. *Assume that $\beta'(s) \ge 0$ and there are functions $\varkappa \in \mathbf{C}([s_0,\infty))$ and $\theta \in \mathbf{C}^1([s_0,\infty))$ such that $\theta'(s) \ge 0$, $\varkappa(s) \to \infty$ and $\theta(s) \to \infty$ as $s \to \infty$,*

$$\max\{\varkappa(s),\ \theta(s)\} \le \delta(s) \text{ and } \max\{\varkappa(s),\ \theta(s)\} < \beta(s).$$

If

$$\liminf_{s\to\infty} \int_{\beta^{-1}(\delta(s))}^{s} \phi(l)K_1(\delta(l))\left(\beta^{-1}(\varkappa(l))\right)^{n-1} \mathrm{d}l > \frac{(n-1)!}{\mathrm{e}} \tag{3}$$

and

$$\liminf_{s\to\infty} \int_{\beta^{-1}(\delta(s))}^{s} \left(\int_{l}^{\infty} \phi(\nu)(l-\nu)^{n-3}K_2(\delta(l))\mathrm{d}\nu\right)\beta^{-1}(\delta(l))\mathrm{d}l > \frac{(n-3)!}{\mathrm{e}}, \tag{4}$$

then all solutions of (1) oscillate, where

$$K_1(s) := \frac{1}{\varphi(\beta^{-1}(s))}\left[1 - \frac{\left(\beta^{-1}(\beta^{-1}(s))\right)^{n-1}}{(\beta^{-1}(s))^{n-1}\varphi(\beta^{-1}(\beta^{-1}(s)))}\right],$$

and

$$K_2(s) := \frac{1}{\varphi(\beta^{-1}(s))}\left[1 - \frac{\beta^{-1}(\beta^{-1}(s))}{\beta^{-1}(s)\varphi(\beta^{-1}(\beta^{-1}(s)))}\right].$$

The oscillatory behaviour of solutions of the DDE

$$\left(r(s)\left(x^{(n-1)}(s)\right)^{\alpha}\right)' + \phi(s)f(x(\delta(s))) = 0 \tag{5}$$

has been studied by several techniques. In 2012, Baculikova et al. [19] derived criteria for oscillation using comparative principles by comparing DDE (5) with three first-order equations, whereas Zhang et al. [20] and Li and Rogovchenko [21] used the Riccati substitution to obtain criteria for the oscillation of DDE (5) when $f(x) = x^{\beta}$. Moaaz and Muhib [22] used general Riccati substitution to improve the results in [19,20] when $n = 4$. Moaaz et al. [23] improved and simplified the oscillation criteria for (5).

In [24–27], the oscillation of NDE

$$\left(r(\mathrm{s})\left(\mathcal{U}^{(n-1)}(\mathrm{s})\right)^{\alpha}\right)' + \phi(\mathrm{s})f(x(\delta(\mathrm{s}))) = 0, \tag{6}$$

or special cases of it, has been studied. Zhang et al. [24] considered DDE (6) when $r(\mathrm{s}) = 1$ and $\alpha = 1$, and obtained conditions for oscillation of all solutions. By using the Riccati transformation technique, Baculikova and Dzurina [25] studied the oscillatory behaviour of (6), whereas Baculikova and Dzurina [26] were interested in studying the linear case of (6) by using the comparison technique. Very recently, Salah et al. [27] presented a comparison between the different approaches that relied on the comparison technique to study the oscillation of solutions to (6).

In this article, we find new monotonic properties of a class of positive solutions to DDE (1). Using these properties, we improve the relationship between the solution x and its corresponding function $\mathcal{U}$. To increase positive solutions, the traditional relation $x > (1-\varphi)\mathcal{U}$ is usually used which requires that $\varphi < 1$ be specified. Furthermore, the works that studied the case $\varphi \geq 1$ imposed restrictions on the delay functions in the form $\beta \circ \delta = \delta \circ \beta$. Our results consider the case $\varphi \geq 1$ but do not require the condition $\beta \circ \delta = \delta \circ \beta$. We use the comparison technique to obtain the oscillation theorems that provide criteria ensuring that all solutions of DDE (1) oscillate.

2. Monotonic Properties

Before looking at the oscillation of the DDE, it is known that determining the signs of the derivatives of the solution is necessary. Establishing relationships between derivatives of various orders is also crucial, although doing so may impose further limitations on the study. The most influential factor in the relationships between derivatives is the monotonic properties of the solutions of these equations. Therefore, improving these properties or finding new properties of an iterative nature greatly affects the qualitative study of solutions to these equations.

While presenting the results, we will need the following notations:

$$F_{[1]} := F\,,\, F_{[i+1]} = F \circ F_{[i]}, \text{ for } i = 1,2,3,....$$

The following lemma can be directly obtained from applying Lemma 2.2.1 in [28].

Lemma 1. *Assume that x is one of the eventually positive solutions of (1). Then $\mathcal{U}(\mathrm{s}) > 0$, $\mathcal{U}^{(n-1)}(\mathrm{s}) > 0$, $\mathcal{U}^{(n)}(\mathrm{s}) \leq 0$, and one of the following possibilities is satisfied, eventually:*
(D_1) $\mathcal{U}^{(i)}(\mathrm{s}) > 0$ *for* $i = 1,2,...,n-1$;
(D_2) $(-1)^{i+1}\mathcal{U}^{(i)}(\mathrm{s}) > 0$ *for* $i = 1,...,n-2$.

Notation 1. *Solutions x whose corresponding function $\mathcal{U}$ satisfy case (D_1) are indicated by class $\mathcal{F}_*$. Moreover, we will use the following condition to prove the main results:*
(C) *there is a $\kappa > 0$ such that $(1-\varphi(\mathrm{s}))\mathrm{s}\,\delta^{n-1}(\mathrm{s})\,\phi(\mathrm{s}) \geq (n-1)!\,\kappa$.*

Lemma 2. *Assume that $x \in \mathcal{F}_*$. Then, eventually,*

$$\mathcal{U}(\mathrm{s}) \geq \frac{\epsilon_1 \mathrm{s}}{(n-1)}\frac{\mathrm{d}}{\mathrm{ds}}\mathcal{U}(\mathrm{s}), \tag{7}$$

and

$$\mathcal{U}(\mathrm{s}) \geq \frac{\epsilon_2 \mathrm{s}^{n-1}}{(n-1)!}\frac{\mathrm{d}^{n-1}}{\mathrm{ds}^{n-1}}\mathcal{U}(\mathrm{s}), \tag{8}$$

for all $\epsilon_i \in (0,1)$, $i = 1,2$.

Proof. By using Lemma 1 in [29] and Lemma 2.2.3 in [28], we directly obtain the proof of this lemma. Therefore, it has been left out. □

Lemma 3. *Assume that* $x \in \mathcal{F}_*$ *and* (C) *holds. Then,*

(a) $\lim \frac{\mathcal{U}^{(n-r)}}{\mathrm{s}^{r-1}} = 0$,
(b) $\frac{\mathrm{d}}{\mathrm{ds}} \frac{\mathcal{U}^{(n-r)}}{\mathrm{s}^{r-1}} < 0$,
for $r = 1,2,...,n$, *eventually.*

Proof. Using the fact that $\mathcal{U}^{(n-1)}$ is a non-increasing positive function, we obtain $\lim_{\mathrm{s}\to\infty} \mathcal{U}^{(n-1)} = k \geq 0$. Suppose that $k > 0$. Then, $\mathcal{U}^{(n-1)} \geq k$, for $\mathrm{s} \geq \mathrm{s}_1$. From Lemma 2, we arrive at

$$x(\mathrm{s}) \geq (1 - \varphi(\mathrm{s}))\mathcal{U}(\mathrm{s}) \geq \frac{k\epsilon_2(1-\varphi(\mathrm{s}))}{(n-1)!}\mathrm{s}^{n-1},$$

which with (1) and (C) gives

$$\begin{aligned} \frac{\mathrm{d}^n}{\mathrm{ds}^n}\mathcal{U}(\mathrm{s}) &\leq -\frac{k\epsilon_2(1-\varphi(\mathrm{s}))}{(n-1)!}\delta^{n-1}(\mathrm{s})\phi(\mathrm{s}) \\ &\leq -\frac{k\epsilon_2}{(n-1)!}\frac{1}{\mathrm{s}}. \end{aligned} \tag{9}$$

Integrating (9) from s_1 to s gives

$$\begin{aligned} \mathcal{U}^{(n-1)}(\mathrm{s}_1) &\geq \mathcal{U}^{(n-1)}(\mathrm{s}) + \frac{k\epsilon_2}{(n-1)!}\ln\frac{\mathrm{s}}{\mathrm{s}_1} \\ &\geq k + \frac{k\epsilon_2}{(n-1)!}\ln\frac{\mathrm{s}}{\mathrm{s}_1} \to \infty \text{ as } \mathrm{s} \to \infty, \end{aligned}$$

which is a contradiction. Thus, $\lim_{\mathrm{s}\to\infty} \mathcal{U}^{(n-1)} = 0$. Now, by applying l'Hôpital's rule, we obtain that (a) holds.

Next, we have

$$\begin{aligned} \mathcal{U}^{(n-2)} &= \mathcal{U}^{(n-2)}(\mathrm{s}_1) + \int_{\mathrm{s}_1}^{\mathrm{s}} \mathcal{U}^{(n-1)}(l)\mathrm{d}l \\ &\geq \mathcal{U}^{(n-2)}(\mathrm{s}_1) + (\mathrm{s} - \mathrm{s}_1)\mathcal{U}^{(n-1)}(\mathrm{s}). \end{aligned} \tag{10}$$

Since $\lim_{\mathrm{s}\to\infty} \mathcal{U}^{(n-1)} = 0$, there is an $\mathrm{s}_2 \geq \mathrm{s}_1$ such that $\mathcal{U}^{(n-2)}(\mathrm{s}_1) - \mathrm{s}_1\mathcal{U}^{(n-1)}(\mathrm{s}) \geq 0$ for $\mathrm{s} \geq \mathrm{s}_2$. Thus, (10) becomes $\mathcal{U}^{(n-2)} \geq \mathrm{s}\mathcal{U}^{(n-1)}$, and so

$$\frac{\mathrm{d}}{\mathrm{ds}}\frac{\mathcal{U}^{(n-2)}}{\mathrm{s}} < 0.$$

Using the fact that $\mathcal{U}^{(n-2)}/\mathrm{s}$ is positive and decreasing, we obtain

$$\begin{aligned} \mathcal{U}^{(n-3)}(\mathrm{s}) &= \mathcal{U}^{(n-3)}(\mathrm{s}_2) + \int_{\mathrm{s}_2}^{\mathrm{s}} \mathcal{U}^{(n-2)}(l)\mathrm{d}l \\ &\geq \mathcal{U}^{(n-3)}(\mathrm{s}_2) + \frac{\mathcal{U}^{(n-2)}(\mathrm{s})}{\mathrm{s}}\int_{\mathrm{s}_2}^{\mathrm{s}} l\mathrm{d}l \\ &= \mathcal{U}^{(n-3)}(\mathrm{s}_2) + \frac{1}{2}\left(\mathrm{s}^2 - \mathrm{s}_2^2\right)\frac{\mathcal{U}^{(n-2)}(\mathrm{s})}{\mathrm{s}}. \end{aligned} \tag{11}$$

Since $\lim_{s\to\infty}\mathcal{U}^{(n-2)}/s = 0$, there is an $s_3 \geq s_2$ such that $\mathcal{U}^{(n-3)}(s_2) - \frac{s_2^2}{2s}\mathcal{U}^{(n-2)}(s) \geq 0$ for $s \geq s_3$. Thus, (11) becomes $\mathcal{U}^{(n-3)} \geq \frac{1}{2}s\mathcal{U}^{(n-2)}$, and hence

$$\frac{d}{ds}\frac{\mathcal{U}^{(n-3)}}{s^2} < 0.$$

By repeating the same approach, we obtain (b). The proof is complete. □

Lemma 4. *Assume that $x \in \mathcal{F}_*$ and (C) holds. Then,*

$$x(s) \geq \sum_{k=1}^{m}\left(\prod_{i=1}^{2k-1}\frac{1}{\varphi\left(\beta_{[i]}^{-1}(s)\right)}\right)\left[1 - \frac{1}{\varphi\left(\beta_{[2k]}^{-1}(s)\right)}\left(\frac{\beta_{[2k]}^{-1}(s)}{\beta_{[2k-1]}^{-1}(s)}\right)^{(n-1)}\right]\mathcal{U}\left(\beta_{[2k-1]}^{-1}(s)\right),$$

for all $\epsilon \in (0,1)$.

Proof. Let $x \in \mathcal{F}_*$. From the definition of $\mathcal{U}$, we arrive at

$$\begin{aligned} x(s) &= \frac{\mathcal{U}(\beta^{-1}(s)) - x(\beta^{-1}(s))}{\varphi(\beta^{-1}(s))} \\ &= \frac{\mathcal{U}(\beta^{-1}(s))}{\varphi(\beta^{-1}(s))} - \frac{\mathcal{U}\left(\beta_{[2]}^{-1}(s)\right) - x\left(\beta_{[2]}^{-1}(s)\right)}{\varphi(\beta^{-1}(s))\varphi\left(\beta_{[2]}^{-1}(s)\right)} \\ &= \frac{\mathcal{U}(\beta^{-1}(s))}{\varphi(\beta^{-1}(s))} - \frac{\mathcal{U}\left(\beta_{[2]}^{-1}(s)\right)}{\varphi\left(\beta_{[1]}^{-1}(s)\right)\varphi\left(\beta_{[2]}^{-1}(s)\right)} + \frac{\mathcal{U}\left(\beta_{[3]}^{-1}(s)\right) - x\left(\beta_{[3]}^{-1}(s)\right)}{\varphi\left(\beta_{[1]}^{-1}(s)\right)\varphi\left(\beta_{[2]}^{-1}(s)\right)\varphi\left(\beta_{[3]}^{-1}(s)\right)}, \end{aligned}$$

and so

$$\begin{aligned} x(s) &= \sum_{k=1}^{2m}\left(\prod_{i=1}^{k}\frac{1}{\varphi\left(\beta_{[i]}^{-1}(s)\right)}\right)(-1)^{k+1}\mathcal{U}\left(\beta_{[k]}^{-1}(s)\right) + x\left(\beta_{[2m]}^{-1}(s)\right)\prod_{i=1}^{2m}\frac{1}{\varphi\left(\beta_{[i]}^{-1}(s)\right)} \\ &\geq \sum_{k=1}^{m}\left(\prod_{i=1}^{2k-1}\frac{1}{\varphi\left(\beta_{[i]}^{-1}(s)\right)}\right)\left[\mathcal{U}\left(\beta_{[2k-1]}^{-1}(s)\right) - \frac{1}{\varphi\left(\beta_{[2k]}^{-1}(s)\right)}\mathcal{U}\left(\beta_{[2k]}^{-1}(s)\right)\right]. \end{aligned} \tag{12}$$

From Lemma 3 and the fact that $\beta(s) \leq s$, we obtain

$$\mathcal{U}\left(\beta_{[2k]}^{-1}(s)\right) \leq \left(\frac{\beta_{[2k]}^{-1}(s)}{\beta_{[2k-1]}^{-1}(s)}\right)^{(n-1)}\mathcal{U}\left(\beta_{[2k-1]}^{-1}(s)\right),$$

which in (12) gives

$$x(s) \geq \sum_{k=1}^{m}\left(\prod_{i=1}^{2k-1}\frac{1}{\varphi\left(\beta_{[i]}^{-1}(s)\right)}\right)\left[1 - \frac{1}{\varphi\left(\beta_{[2k]}^{-1}(s)\right)}\left(\frac{\beta_{[2k]}^{-1}(s)}{\beta_{[2k-1]}^{-1}(s)}\right)^{(n-1)}\right]\mathcal{U}\left(\beta_{[2k-1]}^{-1}(s)\right).$$

The proof is complete. □

3. Oscillation Results

Lemma 5. *Assume that $\delta(\mathrm{s}) \leq \beta(\mathrm{s})$, β^{-1} is non-decreasing, and (C) holds. If*

$$\limsup_{\mathrm{s}\to\infty}\left(\beta^{-1}(\delta(\mathrm{s}))\right)^{n-1}\sum_{k=1}^{m}\left[\int_{\beta^{-1}(\delta(\mathrm{s}))}^{\mathrm{s}}\phi(l)\beta_k(\delta(l))\left(\frac{\beta_{[2k-1]}^{-1}(\delta(l))}{\beta_{[2k-1]}^{-1}(\delta(\mathrm{s}))}\right)^{(n-1)}\mathrm{d}l+\int_{\mathrm{s}}^{\infty}\phi(l)\beta_k(\delta(l))\mathrm{d}l\right] > (n-1)!, \tag{13}$$

for any $m \in \mathbb{N}$, then $\mathcal{F}_* = \emptyset$, where

$$\beta_k(\mathrm{s}) := \left(\prod_{i=1}^{2k-1}\frac{1}{\varphi\left(\beta_{[i]}^{-1}(\mathrm{s})\right)}\right)\left[1-\frac{1}{\varphi\left(\beta_{[2k]}^{-1}(\mathrm{s})\right)}\left(\frac{\beta_{[2k]}^{-1}(\mathrm{s})}{\beta_{[2k-1]}^{-1}(\mathrm{s})}\right)^{(n-1)}\right]. \tag{14}$$

Proof. Let $x \in \mathcal{F}_*$. From Lemma 3, we have (a) and (b) hold. From Lemma 4, Equation (1) becomes

$$\mathcal{U}^{(n)}(\mathrm{s})+\phi(\mathrm{s})\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s}))\mathcal{U}\left(\beta_{[2k-1]}^{-1}(\delta(\mathrm{s}))\right) \leq 0, \tag{15}$$

An integration of (15) yields

$$\mathcal{U}^{(n-1)}(\mathrm{s}) \geq \int_{\mathrm{s}}^{\infty}\left(\phi(l)\sum_{k=1}^{m}\beta_k(\delta(l))\mathcal{U}\left(\beta_{[2k-1]}^{-1}(\delta(l))\right)\right)\mathrm{d}l. \tag{16}$$

If $\delta(\mathrm{s}) \leq \beta(\mathrm{s})$, then we obtain

$$\begin{aligned}\mathcal{U}^{(n-1)}\left(\beta^{-1}(\delta(\mathrm{s}))\right) &\geq \int_{\beta^{-1}(\delta(\mathrm{s}))}^{\infty}\left(\phi(l)\sum_{k=1}^{m}\beta_k(\delta(l))\mathcal{U}\left(\beta_{[2k-1]}^{-1}(\delta(l))\right)\right)\mathrm{d}l\\ &= \int_{\beta^{-1}(\delta(\mathrm{s}))}^{\mathrm{s}}\left(\phi(l)\sum_{k=1}^{m}\beta_k(\delta(l))\mathcal{U}\left(\beta_{[2k-1]}^{-1}(\delta(l))\right)\right)\mathrm{d}l\\ &\quad+\int_{\mathrm{s}}^{\infty}\left(\phi(l)\sum_{k=1}^{m}\beta_k(\delta(l))\mathcal{U}\left(\beta_{[2k-1]}^{-1}(\delta(l))\right)\right)\mathrm{d}l.\end{aligned}$$

Using (b) and the fact that $\mathcal{U}'(\mathrm{s}) \geq 0$, we find

$$\begin{aligned}&\mathcal{U}^{(n-1)}\left(\beta^{-1}(\delta(\mathrm{s}))\right)\\ \geq\ &\sum_{k=1}^{m}\mathcal{U}\left(\beta_{[2k-1]}^{-1}(\delta(\mathrm{s}))\right)\left[\int_{\beta^{-1}(\delta(\mathrm{s}))}^{\mathrm{s}}\phi(l)\beta_k(\delta(l))\left(\frac{\beta_{[2k-1]}^{-1}(\delta(l))}{\beta_{[2k-1]}^{-1}(\delta(\mathrm{s}))}\right)^{(n-1)}\mathrm{d}l+\int_{\mathrm{s}}^{\infty}\phi(l)\beta_k(\delta(l))\mathrm{d}l\right]\\ \geq\ &\mathcal{U}\left(\beta^{-1}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\left[\int_{\beta^{-1}(\delta(\mathrm{s}))}^{\mathrm{s}}\phi(l)\beta_k(\delta(l))\left(\frac{\beta_{[2k-1]}^{-1}(\delta(l))}{\beta_{[2k-1]}^{-1}(\delta(\mathrm{s}))}\right)^{(n-1)}\mathrm{d}l+\int_{\mathrm{s}}^{\infty}\phi(l)\beta_k(\delta(l))\mathrm{d}l\right].\end{aligned}$$

From (b), we arrive at

$$1 \geq \frac{\left(\beta^{-1}(\delta(\mathrm{s}))\right)^{n-1}}{(n-1)!}\sum_{k=1}^{m}\left[\int_{\beta^{-1}(\delta(\mathrm{s}))}^{\mathrm{s}}\phi(l)\beta_k(\delta(l))\left(\frac{\beta_{[2k-1]}^{-1}(\delta(l))}{\beta_{[2k-1]}^{-1}(\delta(\mathrm{s}))}\right)^{(n-1)}\mathrm{d}l+\int_{\mathrm{s}}^{\infty}\phi(l)\beta_k(\delta(l))\mathrm{d}l\right],$$

which contradicts (13). The proof is complete. □

Lemma 6. *Assume that* (C) *holds and, for any* $m \in \mathbb{N}$, *the DDE*

$$w'(\mathrm{s}) + \frac{\epsilon}{(n-1)!}\phi(\mathrm{s})w\left(\beta^{-1}_{[2m-1]}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s}))\left(\beta^{-1}_{[2k-1]}(\delta(\mathrm{s}))\right)^{n-1} = 0, \text{ if } \delta(\mathrm{s}) \le \beta_{[2m-1]}(\mathrm{s}), \tag{17}$$

or

$$y'(\mathrm{s}) + \phi(\mathrm{s})\frac{\epsilon}{(n-1)!}\left(\beta^{-1}(\delta(\mathrm{s}))\right)^{n-1}y\left(\beta^{-1}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s})) = 0., \text{ if } \delta(\mathrm{s}) \le \beta(\mathrm{s}), \tag{18}$$

is oscillatory for some $\epsilon \in (0,1)$, *then* $\mathcal{F}_* = \emptyset$, *where* β_k *is defined as in (14).*

Proof. Let $x \in \mathcal{F}_*$. From Lemma 2, we have that (8) holds. Using Lemma 4, Equation (1) reduces to (15). Thus, from (8), we obtain

$$\mathcal{U}^{(n)}(\mathrm{s}) + \frac{\epsilon}{(n-1)!}\phi(\mathrm{s})\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s}))\left(\beta^{-1}_{[2k-1]}(\delta(\mathrm{s}))\right)^{n-1}\mathcal{U}^{(n-1)}\left(\beta^{-1}_{[2k-1]}(\delta(\mathrm{s}))\right) \le 0,$$

which, with the facts that $\mathcal{U}^{(n)} \le 0$ and $\beta^{-1}_{[2k-1]}(\mathrm{s}) \le \beta^{-1}_{[2m-1]}$ for $k = 1,2,...,m$, gives

$$\mathcal{U}^{(n)}(\mathrm{s}) + \frac{\epsilon}{(n-1)!}\phi(\mathrm{s})\mathcal{U}^{(n-1)}\left(\beta^{-1}_{[2m-1]}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s}))\left(\beta^{-1}_{[2k-1]}(\delta(\mathrm{s}))\right)^{n-1} \le 0.$$

Suppose that $w := \mathcal{U}^{(n-1)}$. Then $w > 0$ is a solution of

$$w'(\mathrm{s}) + \frac{\epsilon}{(n-1)!}\phi(\mathrm{s})w\left(\beta^{-1}_{[2m-1]}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s}))\left(\beta^{-1}_{[2k-1]}(\delta(\mathrm{s}))\right)^{n-1} \le 0.$$

It follows from Theorem 1 in [30] that Equation (17) also has a positive solution, a contradiction.

On the other hand, using the fact that $\mathcal{U}' > 0$ and $\beta^{-1}(\mathrm{s}) \le \beta^{-1}_{[2k-1]}$ for $k = 1,2,...,m$, the inequality in (15) becomes

$$\mathcal{U}^{(n)}(\mathrm{s}) + \phi(\mathrm{s})\mathcal{U}\left(\beta^{-1}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s})) \le 0.$$

Thus, from (8), we obtain

$$\mathcal{U}^{(n)}(\mathrm{s}) + \phi(\mathrm{s})\frac{\epsilon\left(\beta^{-1}(\delta(\mathrm{s}))\right)^{n-1}}{(n-1)!}\mathcal{U}^{(n-1)}\left(\beta^{-1}(\delta(\mathrm{s}))\right)\sum_{k=1}^{m}\beta_k(\delta(\mathrm{s})) \le 0.$$

Therefore, it follows from Theorem 1 in [30] that Equation (18) has a positive solution, a contradiction. The proof is complete. □

Corollary 1. *Assume that* (C) *holds,*

$$\liminf_{\mathrm{s}\to\infty}\int_{\beta^{-1}_{[2m-1]}(\delta(\mathrm{s}))}^{\mathrm{s}}\left(\phi(l)\sum_{k=1}^{m}\beta_k(\delta(l))\left(\beta^{-1}_{[2k-1]}(\delta(l))\right)^{n-1}\right)\mathrm{d}l > \frac{(n-1)!}{\mathrm{e}}, \text{ if } \delta(\mathrm{s}) \le \beta_{[2m-1]}(\mathrm{s}), \tag{19}$$

or

$$\liminf_{s\to\infty}\int_{\beta^{-1}(\delta(s))}^{s}\left(\phi(l)\left(\beta^{-1}(\delta(l))\right)^{n-1}\sum_{k=1}^{m}\beta_k(\delta(l))\right)\mathrm{d}l > \frac{(n-1)!}{\mathrm{e}},\ \textit{if}\ \delta(s)\le\beta(s), \tag{20}$$

is oscillatory, then $\mathcal{F}_* = \emptyset$*, where* β_k *is defined as in (14).*

Proof. From Theorem 2 in [31], conditions in (19) and (20) imply the oscillation of Equations (17) and (18), respectively. □

Theorem 2. *Assume that* $\delta(s) \le \beta(s)$*,* β^{-1} *is non-decreasing, and* (C) *and (13) hold. Then, Equation (1) is oscillatory if (4) holds.*

Proof. Assume that x is an eventually positive solution of (1). From Lemma 1, one of the possibilities (D_1) or (D_2) is satisfied. Using Lemma 5, we have $\mathcal{F}_* = \emptyset$. Then, case (D_2) holds. In exactly the same way as Theorem 2.1 in [18], we obtain a contradiction with (4). The proof is complete. □

Theorem 3. *Assume that* (C) *holds, and one of the conditions in (19) or (20) is satisfied. Then, Equation (1) is oscillatory if (4) holds.*

4. Application and Discussion

Example 1. *Consider the NDE*

$$(x(s)+\varphi_0 x(\mu s))^{(4)} + \frac{\phi_0}{s^4}x(\lambda s) = 0, \tag{21}$$

where $\varphi_0 > 0$*,* $\lambda < \mu \in (0,1)$*,* $\phi_0 > 0$*, and* $\mu^3\varphi_0 > 1$*. In the following we will apply the conditions of the theorems in the previous section to check the oscillation of this equation. Conditions in (13), (19) and (20) reduce to*

$$\phi_0\left(\frac{\lambda}{\mu}\right)^3\left[\ln\frac{\mu}{\lambda}+\frac{1}{3}\right]\left[1-\frac{1}{\mu^3\varphi_0}\right]\sum_{k=1}^{m}\frac{1}{\varphi_0^{2k-1}} > 3!, \tag{22}$$

$$\phi_0\lambda^3\left[1-\frac{1}{\mu^3\varphi_0}\right]\ln\left(\frac{\mu^{2m-1}}{\lambda}\right)\sum_{k=1}^{m}\left(\frac{1}{\varphi_0^{2k-1}}\left(\frac{1}{\mu^{2k-1}}\right)^3\right) > \frac{3!}{\mathrm{e}},\ \textit{if}\ \lambda < \mu^{2m-1} \tag{23}$$

and

$$\phi_0\left(\frac{\lambda}{\mu}\right)^3\left[1-\frac{1}{\mu^3\varphi_0}\right]\left(\ln\frac{\mu}{\lambda}\right)\sum_{k=1}^{m}\frac{1}{\varphi_0^{2k-1}} > \frac{3!}{\mathrm{e}},, \tag{24}$$

respectively. The condition in (4) becomes

$$\phi_0\frac{1}{3\varphi_0}\frac{\lambda}{\mu}\left[1-\frac{1}{\mu\varphi_0}\right]\ln\frac{\mu}{\lambda} > \frac{1}{\mathrm{e}}. \tag{25}$$

By using Theorems 2 and 3, Equation (21) is oscillatory if (25) and one of the conditions in (22), (23) or (24) are satisfied.

Remark 1. *Applying the results in the previous example to the special case of Equation (21), when* $\varphi_0 = 16$*,* $\mu = 1/2$*, and* $\lambda = 1/6$*, we conclude that Equation (21) is oscillatory if*

$$\phi_0 > \frac{1152}{7e\ln 3},\ [\textit{condition}\ (25)]$$

and one of conditions (22), (23) or (24) is satisfied, see Table 1.

Table 1. Conditions (22), (23) and (24) when $\varphi_0 = 16$, $\mu = 1/2$, and $\lambda = 1/6$.

Condition	(22)	(23)	(24)
	$\phi_0 > 3606.1$	$\phi_0 > 1736$	$\phi_0 > 1729.1$

Therefore, Equation (21) is oscillatory if $\phi_0 > 1729.1$, while the results of [18] state that (21) is oscillatory if $\phi_0 > 1736$. Thus, our results improve upon those in [18].

Remark 2. *In Example 1, we note that criterion (24) often provides the best results. For comparison between the criteria in (3) and (24), we consider the special case when $\varphi_0 = 1/\mu^4$, and $\lambda = \mu^3$. Conditions in (3) and (24) reduce to*

$$\phi_0 > \frac{3!}{e\mu^{10}\left(\ln\frac{1}{\mu^2}\right)(1-\mu)} \tag{26}$$

and

$$\phi_0 > \frac{3!}{e\mu^{6}\left(\ln\frac{1}{\mu^2}\right)(1-\mu)\sum_{k=1}^{50}\mu^{8k-4}}, \tag{27}$$

respectively. Figure 1 shows a comparison of the lower bounds for the values of ϕ_0 for the conditions in (3) and (24) when $\mu \in (0.7, 0.9)$.

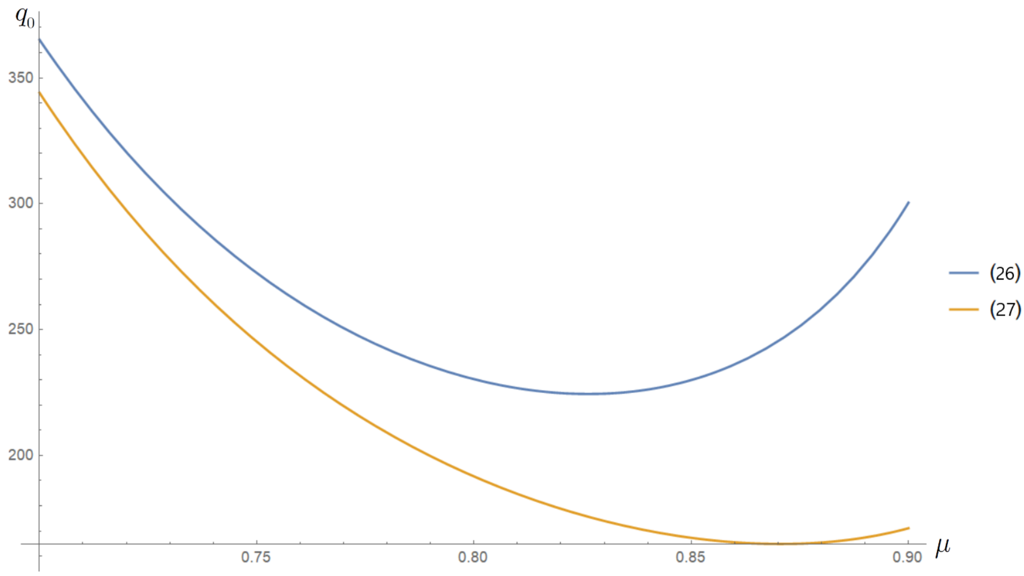

Figure 1. The minimum values of ϕ_0 for which (3) and (24) are satisfied.

5. Conclusions

The study of the oscillatory behaviour of DDEs depends mainly on the monotonic properties of the solutions. These properties control the relationships between the derivatives as well as the relationship between the solution and its corresponding function. Therefore, finding new or improving monotonic properties plays an important role in improving the oscillation parameters.

In this work, we obtained new monotonic properties, through which we were able to obtain a new and improved relationship linking the solution and its corresponding function. Then, we used this relationship to obtain oscillation criteria for the studied equation. Finally, we provided an example and comparisons to illustrate the importance of the results.

Recently, there has been a lot of research activity focused on studying the properties of solutions to fractional differential equations. It would be interesting to extend our results to fractional differential equations.

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