

Article

Mixture of Shanker Distributions: Estimation, Simulation, and Application

Tahani A. Abushal ^{1,*} , Tabassum Naz Sindhu ^{2,*} , Showkat Ahmad Lone ³ , Marwa K. H. Hassan ⁴ and Anum Shafiq ^{5,6,*} 

¹ Department of Mathematical Science, Faculty of Applied Science, Umm Al-Qura University, Mecca 24382, Saudi Arabia

² Department of Statistics, Quaid-i-Azam University, Islamabad 44000, Pakistan

³ Department of Basic Science, College of Science and Theoretical Studies, Saudi Electronic University, (Jeddah-M), Riyadh 11673, Saudi Arabia

⁴ Department of Mathematics, Faculty of Education, Ain Shams University, Cairo 11566, Egypt

⁵ School of Mathematics and Statistics, Nanjing University of Information Science and Technology, Nanjing 210044, China

⁶ Jiangsu International Joint Laboratory on System Modeling and Data Analysis, Nanjing University of Information Science and Technology, Nanjing 210044, China

* Correspondence: taabushal@uqu.edu.sa (T.A.A.); sindhuqau@gmail.com (T.N.S.); anumshafiq@gmail.com (A.S.)

Abstract: The Shanker distribution, a one-parameter lifetime distribution with an increasing hazard rate function, is recommended by Shanker for modelling lifespan data. In this study, we examine the theoretical and practical implications of 2-component mixture of Shanker model (2-CMSM). A significant feature of proposed model's hazard rate function is that it has rising, decreasing, and upside-down bathtub forms. We investigate the statistical characteristics of a mixed model, such as the probability-generating function, the factorial-moment-generating function, cumulants, the characteristic function, the Mills ratio, the mean residual life, and the mean time to failure. There is a graphic representation of density, mean, hazard rate functions, coefficient of variation, skewness, and kurtosis. Our final approach is to estimate the parameters of the mixture model using appropriate approaches such as maximum likelihood, least squares, and weighted least squares. Using a simulation analysis, we examined how the estimates behaved graphically. The simulation results demonstrated that, in the majority of cases, the maximum likelihood estimates have the smallest mean square errors among all other estimates. Finally, we observed that when the sample size rises, the precision measures decrease for all of the estimation techniques, indicating that all of the estimation approaches are consistent. Through two real data analyses, the suggested model's validity and adaptability are contrasted with those of other models, including the mixture of the exponential distributions and the Lindley distributions.

Keywords: least square estimation (LSE); mixture distributions; Mills ratio; weighted least square estimation

MSC: 60E05; 62E15; 62E05; 62F10



Citation: Abushal, T.A.; Sindhu, T.N.; Lone, S.A.; Hassan, M.K.H.; Shafiq, A. Mixture of Shanker Distributions: Estimation, Simulation, and Application. *Axioms* **2023**, *12*, 231. <https://doi.org/10.3390/axioms12030231>

Academic Editors: Jiajuan Liang and Kaitai Fang

Received: 24 December 2022

Revised: 13 February 2023

Accepted: 16 February 2023

Published: 22 February 2023



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1. Introduction

Mixture models, more specifically finite-mixture models, have been used more frequently over time to simulate a wider range of events than they were in the early days of statistics. Available data can often be thought of as a mixture of two or more models. Utilizing this concept, we can combine statistical distributions to produce a novel one. The fields of biology, genetics, engineering, medicine, marketing, business, real-world applications, and social sciences are just a few that can benefit from the use of finite-mixture distributions. In mixture distributions, two or more distributions are combined by adjusting the proportions to form a novel model. Therefore, it is essential to look into the statistical characteristics

of the suggested mixture distribution and apply the right techniques to estimate its unknown parameters. Several authors have examined mixing distributions, including [1–5]. Muhammad and Muhammad [6] examined the traditional features of mixture of the Burr XII and Weibull distribution. A two-component mixture of inverse Weibull distribution (2-CMIWD) was proposed by Sultan et al. [7], and some of its characteristics were investigated by utilizing density and hazard function graphs. Jiang et al. [8] focused on the graphical approaches as well as the PDF and hrfs formats to analyze the hybrid of two inverse Weibull distributions. Several authors who address mixture modelling in various real-world issues include those listed below: Mohamed et al. [9], Mohammadi et al. [10], Ateya [11], and Sindhu et al. [12] are a few examples. The generalized method of moments and ML were employed by Al-Moisheer et al. [13] to assess the mixture model's unknown parameters. Some other relevant studies are [14–20].

Shanker [21] proposed the Shanker distribution as a lifetime distribution and estimated its parameter by employing maximum likelihood estimation and moments methodology, as well as by employing those models on how we can model lifetime data from the field of medical science and engineering. The discrete Poisson-Shanker model was developed by Shanker [22], who also studied its mathematical and statistical properties as well as potential uses for count data from many fields of study. In [23], the authors conducted a comparison analysis on modelling lifespan data utilizing one parameter Akash, exponential, and Lindley models, finding that the Akash model of Shanker has flexibility over both exponential and Lindley models. Shanker et al. [24] conducted a thorough comparison of exponential and Lindley models for modelling diverse real-life data sets and discovered that in certain cases, Lindley is superior to exponential, while in others, exponential is superior to Lindley.

The maximal likelihood estimation (MLE) is a well-known estimation approach. Despite the fact that MLE is efficient and has good theoretical features, there is evidence that it does not perform effectively, particularly with small samples. As a result, different estimation approaches have been offered in the literature as alternatives to the conventional method. The weighted least-squares estimation (WLSE), least squares estimator (LSE), L-moments estimator (LME), and percentile estimator (PCE) are among the most frequently recommended. These approaches, in general, do not have good theoretical properties, but they can offer better estimates of unknown parameters in specific instances compared to the MLE. The literature has investigated a variety of estimating techniques for different distributions, such as [25–30]. The aim of the current study is to provide professional statisticians with a framework for choosing the best estimation methodology for 2-component Mixtures of Shanker Models (2-CMSM). This study uses weighted least square estimation (WLSE), least square estimation (LSE) in addition to MLE for estimating the 2-component mixture of the Shanker model (2-CMSM).

Our goal in this study is to develop a distribution for modelling real lifespan data sets from various disciplines of knowledge that are better fitting than both exponential and Lindley distributions. The mixture of the Shanker distribution has one advantage over the one-component Shanker and the exponential distribution: that the Shanker distribution has an increasing, while exponential distribution has constant, hazard rate, whereas the mixture of the Shanker model has decreasing, increasing, rising-decreasing, constant, unimodal, and “upside-down” bathtub curve failure rates.

The 2-CMSM is being developed for analyzing complex data arising from reliability research, survival analysis, statistical mechanics, quality control, economics, biological investigations, and other fields. Our aim is to develop this distribution because of the different shapes of hazard function, including the increasing, decreasing, unimodal, and upside-down bathtub curve, as well as various close-form features of 2-CMSM with simple physical explanations. The originality of this study stems from a comprehensive explanation of mathematical and statistical features of this model, which will hopefully attract more applications in lifespan analysis. Additionally, to our knowledge, no attempt has been made to compare any

of these estimating approaches to estimate parameters of 2-CMSM. For a set of parametric values and sample sizes, we illustrate the efficiency of these estimation techniques.

The rest of article is formatted as displayed. We present 2-CMSM in Section 2. In Section 3, we endeavor to derive the 2-CMSM model's key mathematical and statistical characteristics. In Section 4, we extract some of its general reliability characteristics, such as the cumulative-hazard-rate function, the Mills ratio, the mean time to failure, and the mean residual life. In Section 5, the model pertinent parameters are estimated with LSE, MLE, and WLSE and present simulation outcomes to assess the performance of these estimators in section 6. In Section 7, we show an application to demonstrate the mixing model's applicability. The conclusion can be found in Section 8.

2. 2-Component Mixture of Shanker Model (2-CMSM)

A *r.v.* T is considered to have a finite mixture of 2-component Shanker model (2-CMSM) as its CDF and PDF can be written as:

$$f(t|\check{\Delta}) = \pi f_1(t|\vartheta_1) + \tilde{\pi} f_2(t|\vartheta_2), \quad \tilde{\pi} = 1 - \pi \quad (1)$$

$$f(t|\check{\Delta}) = \pi \frac{\vartheta_1^2}{\vartheta_1^2 + 1} (\vartheta_1 + t) \exp(-\vartheta_1 t) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2 + 1} (\vartheta_2 + t) \exp(-\vartheta_2 t), \quad (2)$$

and

$$F(t|\check{\Delta}) = \pi F_1(t|\vartheta_1) + \tilde{\pi} F_2(t|\vartheta_2), \quad (3)$$

$$F(t|\check{\Delta}) = \pi \left\{ 1 - \frac{\vartheta_1^2 + 1 + \vartheta_1 t}{\vartheta_1^2 + 1} \exp(-\vartheta_1 t) \right\} + \tilde{\pi} \left\{ 1 - \frac{\vartheta_2^2 + 1 + \vartheta_2 t}{\vartheta_2^2 + 1} \exp(-\vartheta_2 t) \right\}, \quad (4)$$

where $\check{\Delta} = (\vartheta_1, \vartheta_2, \pi)$. $(\vartheta_1, \vartheta_2)$ are the scale, whereas π is the mixing parameter.

Figure 1a–h displays a number of graphs of $f(t|\check{\Delta})$ and both component densities for different parameter values. The PDF mentioned above shows how the parametric vector $\check{\Delta}$ affects the density of 2-CMSM($\check{\Delta}$). The PDF curves of 2-CMSM($\check{\Delta}$) shows that it takes several shapes, such as monotonically decreasing, positively skewed, and inverted U with platykurtic, mesokurtic, and leptokurtic curves. This indicates that it can be used to model data of diverse types.

Identifiability

Identifiability is a condition that a statistical model must satisfy for precise inference to be possible. A model is considered identifiable if the true values of its underlying parameters can be learned after an infinite number of observations. It can be seen mathematically as an assertion that different values of the parameters will generate different probabilities of observable variables. Models are usually only identifiable under certain technical conditions, referred to as identification conditions.

Definition 1. Let $f = \{f_{\vartheta_i} : \vartheta_1, \vartheta_2 \in \vartheta_i\}$ be a statistical Shanker model with parameter space ϑ_i . We state that f is said to be identifiable if the mapping $\vartheta_i \rightarrow f_{\vartheta_i}$ is one-to-one:

$$f_{\vartheta_1} = f_{\vartheta_2} \Rightarrow \vartheta_1 = \vartheta_2, \quad \forall \vartheta_1, \vartheta_2 \in \vartheta_i. \quad (5)$$

This means that distinct values of ϑ_i should correspond to distinct probability models: if $\vartheta_1 \neq \vartheta_2$, then also $f_{\vartheta_1} \neq f_{\vartheta_2}$.

By using this approach, we prove the following proposition.

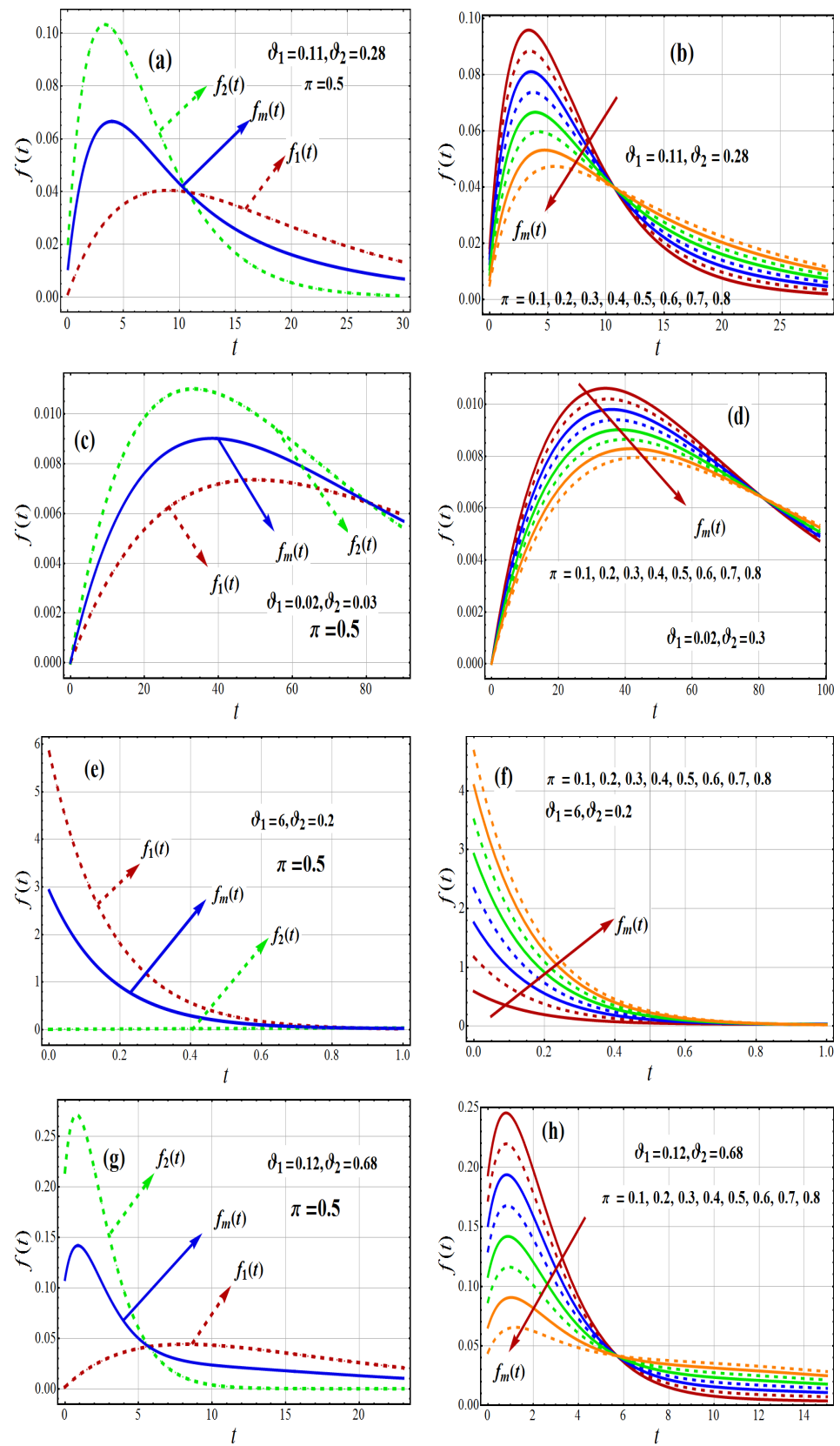


Figure 1. Variations of first and second component of density ($f_1(t), f_2(t)$), respectively, and density of 2-CMSM $f_m(t)$.

Proposition 1. *The class of all finite mixing models relative to Shanker distribution is identifiable. Let f be the pdf of statistical Shanker model:*

$$f = \left\{ f_{\vartheta_i} = \frac{\vartheta_i^2}{\vartheta_i^2 + 1} (\vartheta_i + t) \exp(-\vartheta_i t) : \vartheta_i, i = 1, 2, t > 0 \right\}. \quad (6)$$

$$\begin{aligned} f_{\vartheta_1} &= f_{\vartheta_2}, \\ \Leftrightarrow \frac{\vartheta_1^2}{\vartheta_1^2 + 1} (\vartheta_1 + t) \exp(-\vartheta_1 t) &= \frac{\vartheta_2^2}{\vartheta_2^2 + 1} (\vartheta_2 + t) \exp(-\vartheta_2 t), \end{aligned} \quad (7)$$

Taking log on both sides

$$\begin{aligned} \Leftrightarrow \log \left[\frac{\vartheta_1^2}{\vartheta_1^2 + 1} \right] + \log(\vartheta_1 + t) - \vartheta_1 t &= \log \left[\frac{\vartheta_2^2}{\vartheta_2^2 + 1} \right] + \log(\vartheta_2 + t) - \vartheta_2 t, \\ \Leftrightarrow 2 \log \vartheta_1 - \log(\vartheta_1^2 + 1) + \log(\vartheta_1 + t) - \vartheta_1 t &= 2 \log \vartheta_2 - \log(\vartheta_2^2 + 1) + \log(\vartheta_2 + t) - \vartheta_2 t, \\ \Leftrightarrow 2(\log \vartheta_1 - \log \vartheta_2) - \left[\log(\vartheta_1^2 + 1) - \log(\vartheta_2^2 + 1) \right] + \log(\vartheta_1 + t) - \log(\vartheta_2 + t) - (\vartheta_1 - \vartheta_2)t &= 0, \\ \Leftrightarrow \vartheta_1 &= \vartheta_2 \end{aligned}$$

This expression is equal to zero for almost all t only when all its coefficients are equal to zero.

$$f_{\vartheta_1} = f_{\vartheta_2} \Leftrightarrow \vartheta_1 = \vartheta_2, \quad (8)$$

hence the identifiability is proved.

3. Statistical and Mathematical Characteristics

3.1. Mode

The nonlinear equation w.r.t. t is solved to provide the mode of the 2-CMSM($\check{\Delta}$):

$$\pi \frac{\vartheta_1^2}{\vartheta_1^2 + 1} \{ \exp(-\vartheta_1 t) - (\vartheta_1 + t) \vartheta_1 \exp(-\vartheta_1 t) \} + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2 + 1} \{ \exp(-\vartheta_2 t) - (\vartheta_2 + t) \vartheta_2 \exp(-\vartheta_2 t) \} = 0.$$

3.2. Median

Here, we study about median of 2-CMSM($\check{\Delta}$). Assume that $F(t|\check{\Delta})$ is the CDF of 2-CMSM($\check{\Delta}$) model at 0.5th quantiles $Q_{0.5}$. The median (t^*) can then be obtained by resolving the following non-linear equation for t .

$$\begin{aligned} \pi \left\{ 1 - \frac{\vartheta_1^2 + \vartheta_1 t + 1}{\vartheta_1^2 + 1} \exp(-\vartheta_1 t) \right\} + \tilde{\pi} \left\{ 1 - \frac{\vartheta_2^2 + \vartheta_2 t + 1}{\vartheta_2^2 + 1} \exp(-\vartheta_2 t) \right\} &= 0.5, \\ \pi \left\{ \frac{\vartheta_1^2 + \vartheta_1 t + 1}{\vartheta_1^2 + 1} \exp(-\vartheta_1 t) \right\} + \tilde{\pi} \left\{ \frac{\vartheta_2^2 + \vartheta_2 t + 1}{\vartheta_2^2 + 1} \exp(-\vartheta_2 t) \right\} &= 0.5. \end{aligned} \quad (9)$$

Computing techniques such as the Newton-Raphson methods can be used to obtain the median, t^* , from Equation (9).

Figure 2a–h shows various graphs of hrf of one and 2-CMSM $h(t|\check{\Delta})$ for various parameter values. It should be observed that the parameter values were chosen at random until different shapes could be obtained. An increasing trend can be seen in the hrf of each component distribution, while the hrf of 2-CMSM($\check{\Delta}$) shows a monotonically increasing, decreasing, upside-down bathtub, increasing decreasing-constant (IDC) behavior as shown in the figure. The hazard rate function of the 2-CMSM($\check{\Delta}$) highlights its versatility over one component distribution of Shanker, exponential, and Lindley, and two components of the mixture of exponential and Lindley distributions. These Figures indicate the flexibility of

the 2-CMSM($\check{\Delta}$) distribution to model right skewed data as well as the data with decreasing and upside-down bathtub shapes.

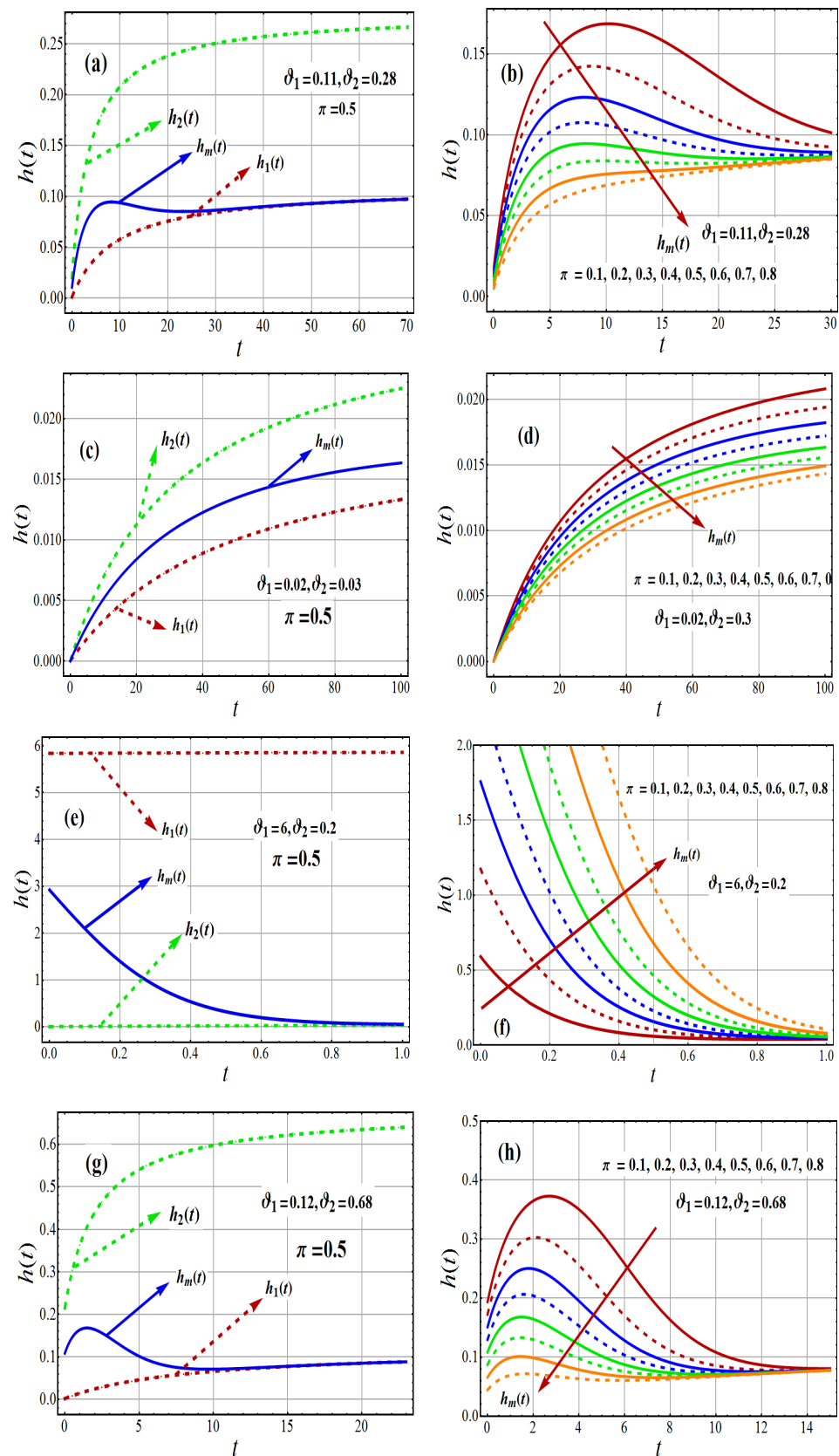


Figure 2. Influences of first and second components hrf ($h_1(t), h_2(t)$) and hrf of 2-CMSM ($h_m(t)$).

Figure 3a–c shows the curves of the mean of 2-CMSM ($\check{\Delta}$) for different parametric values. The mean of 2-CMSM ($\check{\Delta}$) shows a monotonically increasing and decreasing-constant (DC) pattern. For the fixed value of ϑ_2 and varying values of π and ϑ_1 , the mean decreases (see Figure 3a,c), whereas for the fixed value of ϑ_1 , the mean is the increasing function of π and ϑ_2 (see Figure 3b).

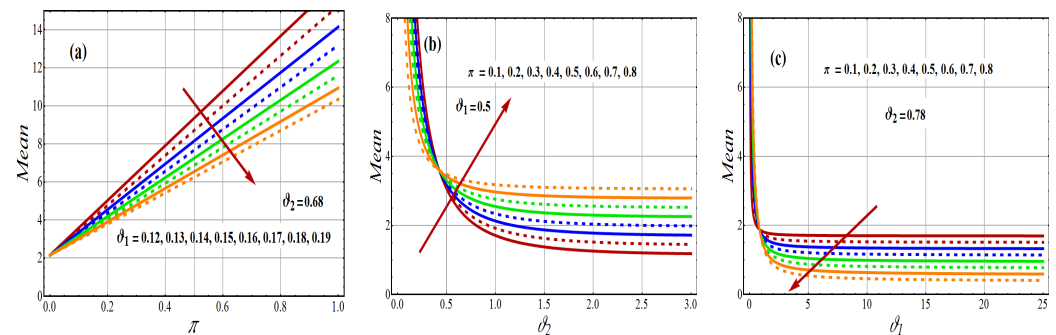


Figure 3. Variations of Mean of 2-CMSM($\check{\Delta}$).

3.3. m th Moments about Origin

The m th moments about the origin of a 2-CMSM($\check{\Delta}$) for the random variable T are as follows:

$$\check{\mu}_m = E(T^m) = \int_0^{\infty} t^m f(t | \check{\Delta}) dt = \int_0^{\infty} t^m \left\{ \pi \frac{\vartheta_1^2}{\vartheta_1^2 + 1} (\vartheta_1 + t) \exp(-\vartheta_1 t) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2 + 1} (\vartheta_2 + t) \exp(-\vartheta_2 t) \right\} dt,$$

$$E(T^m) = \pi \frac{m! (\vartheta_1^2 + m + 1)}{\vartheta_1^m (\vartheta_1^2 + 1)} + \tilde{\pi} \frac{m! (\vartheta_2^2 + m + 1)}{\vartheta_2^m (\vartheta_2^2 + 1)}, m = 1, 2, \dots \quad (10)$$

The mean of the PDF of the 2-CMSM($\check{\Delta}$) is:

$$\check{\mu}_1 = \pi \frac{(\vartheta_1^2 + 2)}{\vartheta_1 (\vartheta_1^2 + 1)} + \tilde{\pi} \frac{(\vartheta_2^2 + 2)}{\vartheta_2 (\vartheta_2^2 + 1)} = \mu, \quad (11)$$

while the variance is given by

$$\check{\sigma} = \pi \frac{(\vartheta_1^4 + 4\vartheta_1^2 + 2)}{\vartheta_1^2 (\vartheta_1^2 + 1)^2} + \tilde{\pi} \frac{(\vartheta_2^4 + 4\vartheta_2^2 + 2)}{\vartheta_2^2 (\vartheta_2^2 + 1)^2}. \quad (12)$$

Particularly, in the first four moments of origin

$$\check{\mu}_1 = \pi \frac{(\vartheta_1^2 + 2)}{\vartheta_1 (\vartheta_1^2 + 1)} + \tilde{\pi} \frac{(\vartheta_2^2 + 2)}{\vartheta_2 (\vartheta_2^2 + 1)}, \quad (13)$$

$$\check{\mu}_2 = \pi \frac{2(\vartheta_1^2 + 3)}{\vartheta_1^2 (\vartheta_1^2 + 1)} + \tilde{\pi} \frac{2(\vartheta_2^2 + 3)}{\vartheta_2^2 (\vartheta_2^2 + 1)}, \quad (14)$$

$$\check{\mu}_3 = \pi \frac{6(\vartheta_1^2 + 4)}{\vartheta_1^3 (\vartheta_1^2 + 1)} + \tilde{\pi} \frac{6(\vartheta_2^2 + 4)}{\vartheta_2^3 (\vartheta_2^2 + 1)}, \quad (15)$$

$$\check{\mu}_4 = \pi \frac{24(\vartheta_1^2 + 5)}{\vartheta_1^4 (\vartheta_1^2 + 1)} + \tilde{\pi} \frac{24(\vartheta_2^2 + 5)}{\vartheta_2^4 (\vartheta_2^2 + 1)}, \quad (16)$$

and the moments about mean of the 2-CMSM($\check{\Delta}$) are:

$$\mu_2 = \pi \frac{(\vartheta_1^4 + 4\vartheta_1^2 + 2)}{\vartheta_1^2(\vartheta_1^2 + 1)^2} + \tilde{\pi} \frac{(\vartheta_2^4 + 4\vartheta_2^2 + 2)}{\vartheta_2^2(\vartheta_2^2 + 1)^2}, \quad (17)$$

$$\mu_3 = \pi \frac{2(\vartheta_1^6 + 6\vartheta_1^4 + 6\vartheta_1^2 + 2)}{\vartheta_1^3(\vartheta_1^2 + 1)^3} + \tilde{\pi} \frac{2(\vartheta_2^6 + 6\vartheta_2^4 + 6\vartheta_2^2 + 2)}{\vartheta_2^3(\vartheta_2^2 + 1)^3}, \quad (18)$$

$$\mu_4 = \pi \frac{3(3\vartheta_1^8 + 24\vartheta_1^6 + 44\vartheta_1^4 + 32\vartheta_1^2 + 8)}{\vartheta_1^4(\vartheta_1^2 + 1)^4} + \tilde{\pi} \frac{3(3\vartheta_2^8 + 24\vartheta_2^6 + 44\vartheta_2^4 + 32\vartheta_2^2 + 8)}{\vartheta_2^4(\vartheta_2^2 + 1)^4}. \quad (19)$$

The coefficient of variation ($\check{\varphi}_{CV}$), the skewness (Ψ_{sk}), and the Kurtosis ($\check{\psi}_K$) of the 2-CMSM($\check{\Delta}$) are:

$$\check{\varphi}_{CV} = \frac{\sqrt{\pi \frac{(\vartheta_1^4 + 4\vartheta_1^2 + 2)}{\vartheta_1^2(\vartheta_1^2 + 1)^2} + \tilde{\pi} \frac{(\vartheta_2^4 + 4\vartheta_2^2 + 2)}{\vartheta_2^2(\vartheta_2^2 + 1)^2}}}{\pi \frac{(\vartheta_1^2 + 2)}{\vartheta_1(\vartheta_1^2 + 1)} + \tilde{\pi} \frac{(\vartheta_2^2 + 2)}{\vartheta_2(\vartheta_2^2 + 1)}}, \quad \check{\psi}_K = \frac{\left(\pi \frac{3(3\vartheta_1^8 + 24\vartheta_1^6 + 44\vartheta_1^4 + 32\vartheta_1^2 + 8)}{\vartheta_1^4(\vartheta_1^2 + 1)^4} + \tilde{\pi} \frac{3(3\vartheta_2^8 + 24\vartheta_2^6 + 44\vartheta_2^4 + 32\vartheta_2^2 + 8)}{\vartheta_2^4(\vartheta_2^2 + 1)^4} \right)}{\left(\pi \frac{(\vartheta_1^4 + 4\vartheta_1^2 + 2)}{\vartheta_1^2(\vartheta_1^2 + 1)^2} + \tilde{\pi} \frac{(\vartheta_2^4 + 4\vartheta_2^2 + 2)}{\vartheta_2^2(\vartheta_2^2 + 1)^2} \right)^2}, \quad (20)$$

and the index of dispersion ($\check{\imath}$) is

$$\check{\imath} = \frac{\pi \frac{(\vartheta_1^4 + 4\vartheta_1^2 + 2)}{\vartheta_1^2(\vartheta_1^2 + 1)^2} + \tilde{\pi} \frac{(\vartheta_2^4 + 4\vartheta_2^2 + 2)}{\vartheta_2^2(\vartheta_2^2 + 1)^2}}{\pi \frac{(\vartheta_1^2 + 2)}{\vartheta_1(\vartheta_1^2 + 1)} + \tilde{\pi} \frac{(\vartheta_2^2 + 2)}{\vartheta_2(\vartheta_2^2 + 1)}}. \quad (21)$$

The 2-CMSM($\check{\Delta}$) distribution is simply demonstrated to be over-dispersed when $\mu_2 > \mu$, equi-dispersed $\mu_2 = \mu$, as well as under-dispersed $\mu_2 < \mu$.

Figure 4 displays the coefficient of variation of 2-CMSM($\check{\Delta}$) graphs for different parametric values. Every component distribution's coefficient of variation rises and is constant as the coefficient of variation of 2-CMSM($\check{\Delta}$) shows monotonically decreasing and increasing behavior. For the fixed value of ϑ_2 and the varying values of π and ϑ_1 , the mean increases (see Figure 4b,d), whereas for the fixed value of ϑ_1 , the mean is the decreasing function of π and ϑ_2 (see Figure 4f).

Figure 5 depicts the graphs of the skewness of one component distribution and the 2-CMSM($\check{\Delta}$) for various parametric values. As noted in Figure 5, the skewness of each component model grows, decreases, and is constant, whereas the skewness for 2-CMSM($\check{\Delta}$) increases the function under different scenarios. The behavior of Kurtosis under different scenarios is observed in Figure 6, which shows, for the fixed value of ϑ_2 and the varying values of π and ϑ_1 , that Kurtosis increases (see Figure 6a,b), whereas for the fixed value of ϑ_1 , Kurtosis is a decreasing function of π and ϑ_2 (see Figure 6c).

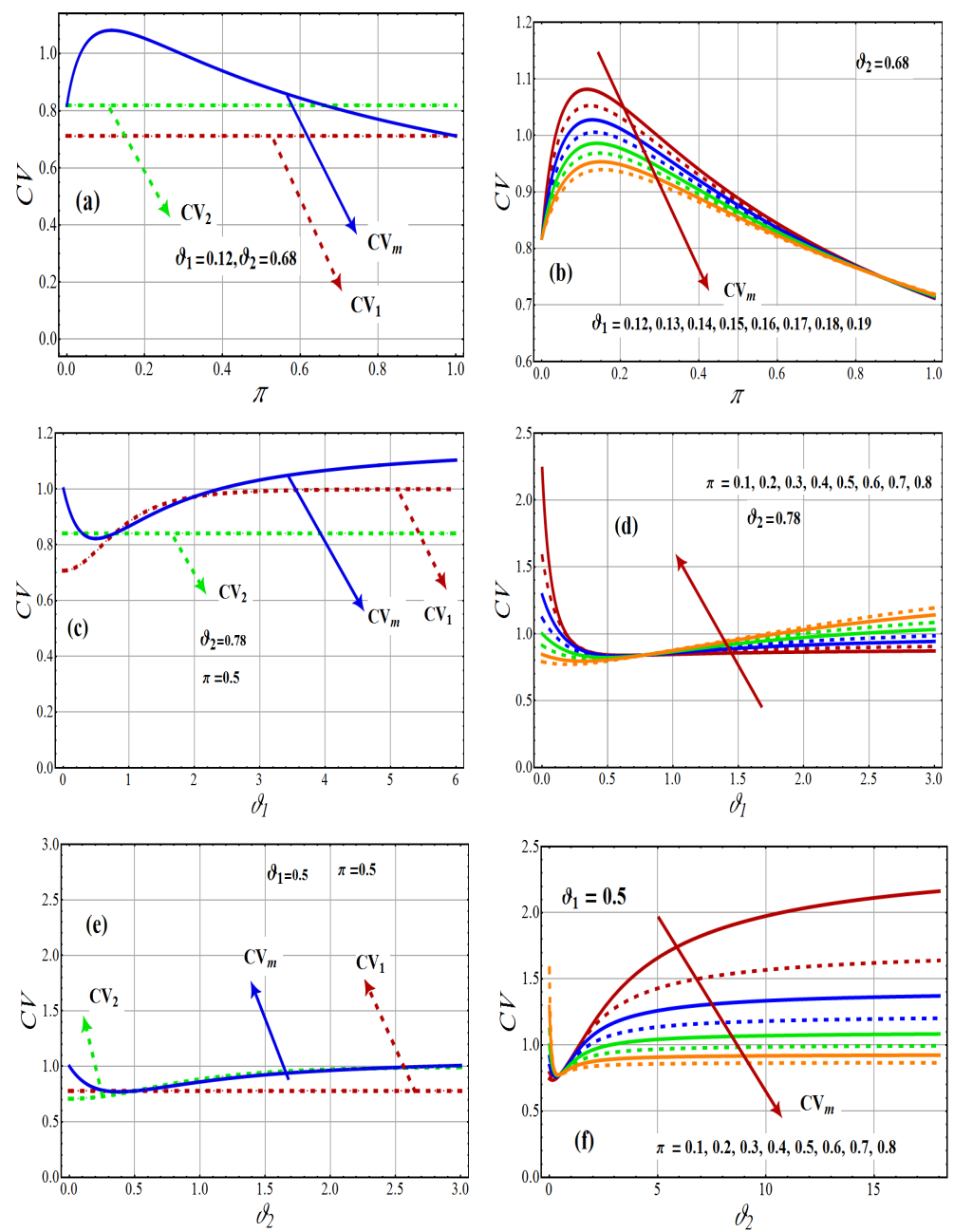


Figure 4. Variations of coefficient of variation, for first component (CV_1), for second component (CV_2), and for 2-CMSM (CV_m).

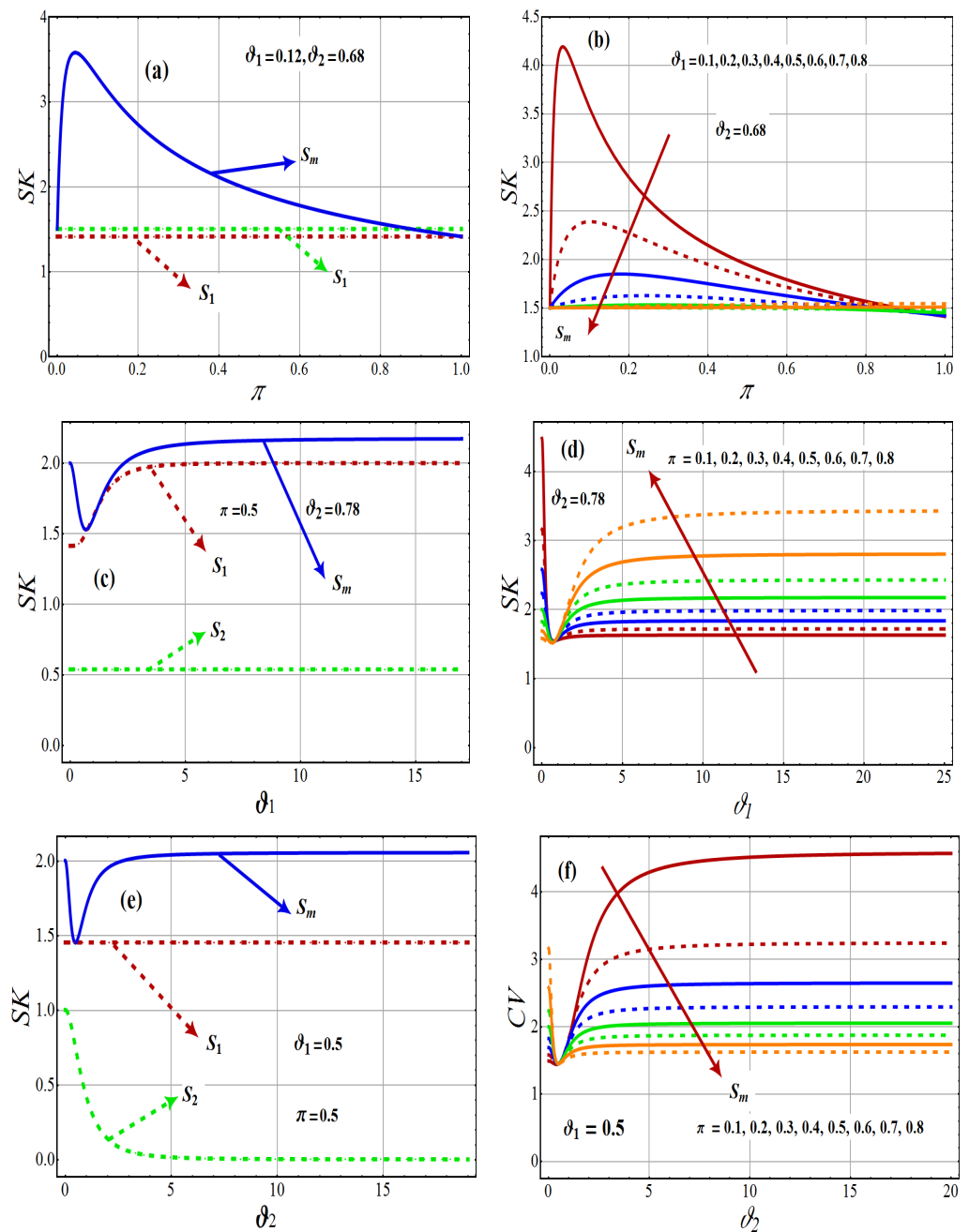


Figure 5. Variations of coefficient of skewness, for first component (S_1), for second component (S_2), and for 2-CMSM (S_m).

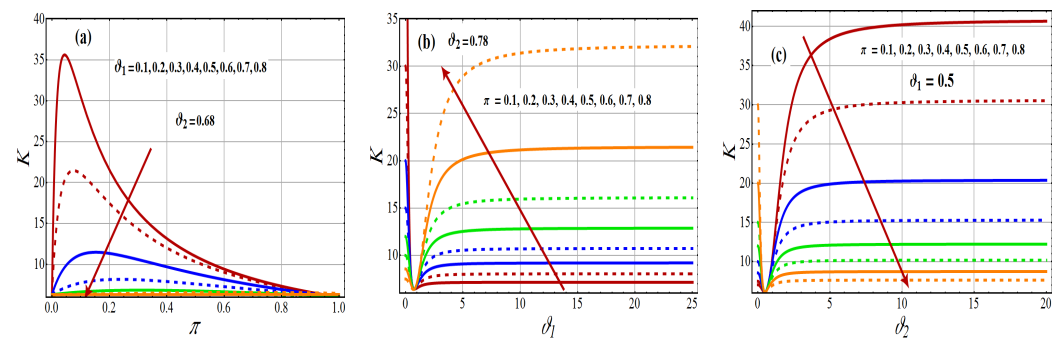


Figure 6. Variations of coefficient of kurtosis (ψ_K), for 2-CMSM.

3.4. Moment Generating Function

The 2-CMSM($\check{\Delta}$)'s MGF is specified as:

$$\tilde{M}_t(v) = E(e^{tv}) = \int_0^\infty e^{tv} \left\{ \pi \frac{\vartheta_1^2}{\vartheta_1^2 + 1} (\vartheta_1 + t) \exp(-\vartheta_1 t) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2 + 1} (\vartheta_2 + t) \exp(-\vartheta_2 t) \right\} dt, \quad (22)$$

$$\tilde{M}_t(v) = \pi \frac{\vartheta_1^2 (\vartheta_1^2 - v\vartheta_1 + 1)}{(\vartheta_1^2 + 1)(\vartheta_1 - v)^2} + \tilde{\pi} \frac{\vartheta_2^2 (\vartheta_2^2 - v\vartheta_2 + 1)}{(\vartheta_2^2 + 1)(\vartheta_2 - v)^2}. \quad (23)$$

3.5. Cumulants

The characteristic function (CF), $\check{\xi}(v) = E[e^{(iv)t}]$ of 2-CMSM($\check{\Delta}$) is obtained by putting v with ' iv ' in Equation (22); CF is obtained as

$$\check{\xi}(v) = \pi \frac{\vartheta_1^2 (\vartheta_1^2 - iv\vartheta_1 + 1)}{(\vartheta_1^2 + 1)(\vartheta_1 - iv)^2} + \tilde{\pi} \frac{\vartheta_2^2 (\vartheta_2^2 - iv\vartheta_2 + 1)}{(\vartheta_2^2 + 1)(\vartheta_2 - iv)^2}, \quad (24)$$

where $i = \sqrt{-1}$ is the complex unit.

3.6. Cumulant Generating Function

The CGF is

$$\check{K}(v) = \pi \left\{ \log \left(1 - \frac{iv\vartheta_1}{\vartheta_1^2 + 1} \right) - 2 \log \left(1 - \frac{iv\vartheta_1}{\vartheta_1} \right) \right\} + \tilde{\pi} \left\{ \log \left(1 - \frac{iv\vartheta_2}{\vartheta_2^2 + 1} \right) - 2 \log \left(1 - \frac{iv\vartheta_2}{\vartheta_2} \right) \right\}. \quad (25)$$

3.7. Probability-Generating Function

In Equation (22), we may obtain the PGF by substituting v with " $\ln(\omega)$ " as shown below:

$$P_t(\omega) = E(\omega^t) = \pi \frac{\vartheta_1^2 (\vartheta_1^2 - \vartheta_1 \ln(\omega) + 1)}{(\vartheta_1^2 + 1)(\vartheta_1 - \ln(\omega))^2} + \tilde{\pi} \frac{\vartheta_2^2 (\vartheta_2^2 - \vartheta_2 \ln(\omega) + 1)}{(\vartheta_2^2 + 1)(\vartheta_2 - \ln(\omega))^2}. \quad (26)$$

3.8. Factorial-Moment-Generating Function

The FMGF can be calculated by substituting v with ' $\ln(1 + \phi)$ ' in Equation (22)

$$\check{F}_t(\omega) = E(e^{t \ln(1 + \phi)}) = \pi \frac{\vartheta_1^2 (\vartheta_1^2 - \vartheta_1 \ln(1 + \phi) + 1)}{(\vartheta_1^2 + 1)(\vartheta_1 - \ln(1 + \phi))^2} + \tilde{\pi} \frac{\vartheta_2^2 (\vartheta_2^2 - \vartheta_2 \ln(1 + \phi) + 1)}{(\vartheta_2^2 + 1)(\vartheta_2 - \ln(1 + \phi))^2}. \quad (27)$$

4. Reliability Measures

In reliability theory, lifespan models are categorized using the reliability function and failure rate. A ratio of the lifespan model to the reliability function is the hazard rate function. An item or component with a lower dependability value will have a shorter lifespan, which means a higher hazard rate and a greater likelihood of failure. In contrast, a lower hazard rate indicates a better reliability function value, which reduces the danger of failure. The reliability characteristics of 2-CMSM($\check{\Delta}$) are currently being explored.

4.1. Reliability Function

The reliability function of 2-CMSM($\check{\Delta}$) is

$$R(t | \check{\Delta}) = \pi \frac{\vartheta_1^2 + 1 + \vartheta_1 t}{\vartheta_1^2 + 1} \exp(-\vartheta_1 t) + \tilde{\pi} \frac{\vartheta_2^2 + 1 + \vartheta_2 t}{\vartheta_2^2 + 1} \exp(-\vartheta_2 t). \quad (28)$$

4.2. Hazard Function

The hazard rate function $h(t|\check{\Delta})$ of 2-CMSM($\check{\Delta}$) is described as follows:

$$h(t|\check{\Delta}) = \frac{\pi \frac{\theta_1^2}{\theta_1^2+1} (\theta_1 + t) \exp(-\theta_1 t) + \tilde{\pi} \frac{\theta_2^2}{\theta_2^2+1} (\theta_2 + t) \exp(-\theta_2 t)}{\pi \frac{\theta_1^2+1+\theta_1 t}{\theta_1^2+1} \exp(-\theta_1 t) + \tilde{\pi} \frac{\theta_2^2+1+\theta_2 t}{\theta_2^2+1} \exp(-\theta_2 t)}. \quad (29)$$

4.3. Mills Ratio

The Mills ratio approaches reliability in a unique way because of its relationship to failure rates.

$$Y(t|\check{\Delta}) = \frac{R(t|\check{\Delta})}{f(t|\check{\Delta})} = \frac{\pi \frac{\theta_1^2+1+\theta_1 t}{\theta_1^2+1} \exp(-\theta_1 t) + \tilde{\pi} \frac{\theta_2^2+1+\theta_2 t}{\theta_2^2+1} \exp(-\theta_2 t)}{\pi \frac{\theta_1^2}{\theta_1^2+1} (\theta_1 + t) \exp(-\theta_1 t) + \tilde{\pi} \frac{\theta_2^2}{\theta_2^2+1} (\theta_2 + t) \exp(-\theta_2 t)}. \quad (30)$$

4.4. Cumulative Hazard Rate Function

The CHRF of 2-CMSM($\check{\Delta}$) is

$$H(t|\check{\Delta}) = \int_0^t h(y|\check{\Delta}) dy = -\log[R(t|\check{\Delta})]. \quad (31)$$

It serves as a risk indicator, with a larger $H(t|\check{\Delta})$ value indicating a greater chance of failure by t -time. It is observed that

$$R(t|\check{\Delta}) = e^{-H(t|\check{\Delta})} \text{ and } f(t|\check{\Delta}) = h(t|\check{\Delta}) e^{-H(t|\check{\Delta})}. \quad (32)$$

Therefore,

$$H(t|\check{\Delta}) = -\log \left[\pi \frac{\theta_1^2+1+\theta_1 t}{\theta_1^2+1} \exp(-\theta_1 t) + \tilde{\pi} \frac{\theta_2^2+1+\theta_2 t}{\theta_2^2+1} \exp(-\theta_2 t) \right]. \quad (33)$$

4.5. Reversed-Hazard-Rate Function

The ratio of life likelihood function to its distribution function represents the reversed hazard rate for a random life.

$$\check{h}(t|\check{\Delta}) = \frac{f(t|\check{\Delta})}{F(t|\check{\Delta})} = \frac{\pi \frac{\theta_1^2}{\theta_1^2+1} (\theta_1 + t) \exp(-\theta_1 t) + \tilde{\pi} \frac{\theta_2^2}{\theta_2^2+1} (\theta_2 + t) \exp(-\theta_2 t)}{1 - \pi \frac{\theta_1^2+1+\theta_1 t}{\theta_1^2+1} \exp(-\theta_1 t) - \tilde{\pi} \frac{\theta_2^2+1+\theta_2 t}{\theta_2^2+1} \exp(-\theta_2 t)}. \quad (34)$$

4.6. Mean Time to Failure

The MTTF provides information about the anticipated (or average) period of time during which a device performs satisfactorily. If 2-CMSM($\check{\Delta}$), the reliability function is used to express MTTF and looks like this:

$$\check{M}(t|\check{\Delta}) = \int_0^{+\infty} \check{R}(x) dx, \quad (35)$$

where $\check{R}(t)$ is given in Equation (28). Hence,

$$\check{M}(t|\check{\Delta}) = \pi \frac{(\theta_1^2+2)}{\theta_1(\theta_1^2+1)} + \tilde{\pi} \frac{(\theta_2^2+2)}{\theta_2(\theta_2^2+1)}. \quad (36)$$

4.7. Mean Residual Life

The MRL has been investigated by reliability experts, statisticians, and survival analysts, among other disciplines. It has produced a lot of beneficial outcomes. The residual lifetime after t for a system or component of age t is random. It is known as the mean residual lifetime or mean remaining life $\check{M}_R(t|\check{\Delta})$, and it is calculated as follows:

$$\check{M}_R(t|\check{\Delta}) = \frac{1}{R(t|\check{\Delta})} \int_t^{+\infty} R(x|\check{\Delta}) dx, \quad (37)$$

$$\check{M}_R(t|\check{\Delta}) = \frac{\left\{ \frac{\pi(\vartheta_1^2+2+\vartheta_1 t) \exp(-\vartheta_1 t)}{\vartheta_1(\vartheta_1^2+1)} + \frac{\tilde{\pi}(\vartheta_2^2+2+\vartheta_2 t) \exp(-\vartheta_2 t)}{\vartheta_2(\vartheta_2^2+1)} \right\}}{\pi \frac{\vartheta_1^2+1+\vartheta_1 t}{\vartheta_1^2+1} \exp(-\vartheta_1 t) + \tilde{\pi} \frac{\vartheta_2^2+1+\vartheta_2 t}{\vartheta_2^2+1} \exp(-\vartheta_2 t)}, \quad (38)$$

where $R(t|\check{\Delta})$ is given in Equation (28).

5. Estimation Inference via Simulation

This section contributes a number of statistical properties of the 2-CMSM($\check{\Delta}$), taking into account that the parametric vector $\check{\Delta}$ is unknown. Three widely used estimation techniques (maximum likelihood estimation, least squares estimation, and weighted least squares estimation) are used to evaluate parametric vectors $\check{\Delta}$. From now, $t_1, t_2, \dots, t_n$ represent n observed values from T and their ascending ordering values $t_{(1)} \leq t_{(2)} \leq \dots \leq t_{(n)}$.

5.1. Maximum Likelihood Estimation

The most used technique for the parameter estimate is the maximum likelihood technique. The method's widespread use is a result of its many desirable characteristics, including consistency, normalcy, and asymptotic effectiveness. Suppose that n observed values, $t_1, t_2, \dots, t_n$ from the Equation (2) and $\check{\Delta}$ is the vector of unknown parameters. The assessments of MLEs of $\check{\Delta}$ can be provided by optimizing the likelihood function with respect to ϑ_1, ϑ_2 , and π given by $L(\mathbf{t}|\check{\Delta}) = \prod_{i=1}^n f(t_i; \check{\Delta})$ or likewise the log-likelihood function for $\check{\Delta}$ given by

$$l(\mathbf{t}|\check{\Delta}) = \ln \prod_{i=1}^n f(t_i; \check{\Delta}), \quad (39)$$

$$l(\mathbf{t}|\check{\Delta}) = \sum_{i=1}^n \ln \left\{ \pi \frac{\vartheta_1^2}{\vartheta_1^2+1} (\vartheta_1 + t_i) \exp(-\vartheta_1 t_i) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2+1} (\vartheta_2 + t_i) \exp(-\vartheta_2 t_i) \right\}. \quad (40)$$

So, by partially differentiating $l(\mathbf{t}|\check{\Delta})$ with respect to parameters $(\vartheta_1, \vartheta_2, \pi)$ and putting the findings equal to zero, therefore, MLEs of the considered parameters are attained; the likelihood equations are

$$\frac{\partial l(\mathbf{t}|\check{\Delta})}{\partial \vartheta_1} = \sum_{i=1}^n \frac{\exp(-\vartheta_1 t_i) \left[\pi(\vartheta_1 + t_i) \left\{ \frac{2\vartheta_1}{(\vartheta_1^2+1)} - \frac{\vartheta_1^2 t_i}{(\vartheta_1^2+1)} - \frac{2\vartheta_1^3}{(\vartheta_1^2+1)^2} \right\} + \frac{\pi \vartheta_1^2}{(\vartheta_1^2+1)} \right]}{\left\{ \pi \frac{\vartheta_1^2}{\vartheta_1^2+1} (\vartheta_1 + t_i) \exp(-\vartheta_1 t_i) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2+1} (\vartheta_2 + t_i) \exp(-\vartheta_2 t_i) \right\}}, \quad (41)$$

$$\frac{\partial l(\mathbf{t}|\check{\Delta})}{\partial \vartheta_2} = \sum_{i=1}^n \frac{\exp(-\vartheta_2 t_i) \left[\tilde{\pi}(\vartheta_2 + t_i) \left\{ \frac{2\vartheta_2}{(\vartheta_2^2+1)} - \frac{\vartheta_2^2 t_i}{(\vartheta_2^2+1)} - \frac{2\vartheta_2^3}{(\vartheta_2^2+1)^2} \right\} + \frac{\tilde{\pi} \vartheta_2^2}{(\vartheta_2^2+1)} \right]}{\left\{ \pi \frac{\vartheta_1^2}{\vartheta_1^2+1} (\vartheta_1 + t_i) \exp(-\vartheta_1 t_i) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2+1} (\vartheta_2 + t_i) \exp(-\vartheta_2 t_i) \right\}}, \quad (42)$$

$$\frac{\partial l(\mathbf{t}|\check{\Delta})}{\partial \pi} = \sum_{i=1}^n \frac{\frac{\vartheta_1^2}{\vartheta_1^2+1}(\vartheta_1+t_i)\exp(-\vartheta_1 t_i) - \frac{\vartheta_2^2}{\vartheta_2^2+1}(\vartheta_2+t_i)\exp(-\vartheta_2 t_i)}{\left\{ \pi \frac{\vartheta_1^2}{\vartheta_1^2+1}(\vartheta_1+t_i)\exp(-\vartheta_1 t_i) + \tilde{\pi} \frac{\vartheta_2^2}{\vartheta_2^2+1}(\vartheta_2+t_i)\exp(-\vartheta_2 t_i) \right\}}. \quad (43)$$

This nonlinear system of equations can therefore be solved to obtain the MLE. Although we cannot solve these equations analytically, we can solve them using statistical software using an iterative approach like the Newton method or fixed-point iteration approaches.

5.2. Least Square Estimators

The ordinary least squares technique is well known for estimating unknown parameters [31]. The LSEs of ϑ_1 , ϑ_2 , and π , denoted by $\tilde{\vartheta}_{1LSE}$, $\tilde{\vartheta}_{2LSE}$, and $\tilde{\pi}_{LSE}$ can be attained by minimizing the function

$$LS(\check{\Delta}) = \sum_{i=1}^n \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right]^2, \quad (44)$$

with respect to ϑ_1 , ϑ_2 , and π , where $F(\cdot)$ is given by Equation (4). The following nonlinear equations can be resolved in the same manner to generate them:

$$\frac{\partial LS(\check{\Delta})}{\partial \vartheta_1} = \sum_{i=1}^n \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right] \Psi_1(t_{(i)}|\vartheta_1) = 0, \quad (45)$$

$$\frac{\partial LS(\check{\Delta})}{\partial \vartheta_2} = \sum_{i=1}^n \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right] \Psi_2(t_{(i)}|\vartheta_2) = 0, \quad (46)$$

and

$$\frac{\partial LS(\check{\Delta})}{\partial \pi} = \sum_{i=1}^n \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right] \Psi_3(t_{(i)}|\pi) = 0, \quad (47)$$

where

$$\Psi_1(t_{(i)}|\vartheta_1) = \pi \frac{t_{(i)}\vartheta_1 \exp(-\vartheta_1 t_{(i)}) (3\vartheta_1 + t_{(i)} + \vartheta_1^3 + t_{(i)}\vartheta_1^2)}{(\vartheta_1^2 + 1)^2}, \quad (48)$$

$$\Psi_2(t_{(i)}|\vartheta_2) = \tilde{\pi} \frac{t_{(i)}\vartheta_2 \exp(-\vartheta_2 t_{(i)}) (3\vartheta_2 + t_{(i)} + \vartheta_2^3 + t_{(i)}\vartheta_2^2)}{(\vartheta_2 + 1)^2}, \quad (49)$$

$$\Psi_3(t_{(i)}|\pi) = \frac{\exp(-\vartheta_2 t_{(i)}) (1 + \vartheta_2 (1 + t_{(i)}))}{(\vartheta_2^2 + 1)} - \frac{\exp(-\vartheta_1 t_{(i)}) (1 + \vartheta_1 (1 + t_{(i)}))}{(\vartheta_1^2 + 1)}. \quad (50)$$

5.3. Weighted Least Squares Estimators

Consider the weighted function below (see [32]).

$$\kappa_i = \frac{(n+1)^2(n+2)}{i(n-i+1)}. \quad (51)$$

The WLSEs $\tilde{\vartheta}_{1WLSE}$, $\tilde{\vartheta}_{2WLSE}$, and $\tilde{\pi}_{WLSE}$, can be attained via minimizing the function

$$WLS(\check{\Delta}) = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right]^2, \quad (52)$$

Additionally, one can obtain these estimators by solving:

$$\frac{\partial WLS(\check{\Delta})}{\partial \theta_1} = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right] \Psi_1(t_{(i)}|\theta_1) = 0, \quad (53)$$

$$\frac{\partial WLS(\check{\Delta})}{\partial \theta_2} = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right] \Psi_2(t_{(i)}|\theta_2) = 0, \quad (54)$$

and

$$\frac{\partial WLS(\check{\Delta})}{\partial \pi} = \sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(t_{(i)}|\check{\Delta}) - \frac{i}{n+1} \right] \Psi_3(t_{(i)}|\pi) = 0, \quad (55)$$

where $\Psi_1(t_{(i)}|\theta_1)$, $\Psi_2(t_{(i)}|\theta_2)$ and $\Psi_3(t_{(i)}|\pi)$ are given in Equations (48)–(50).

6. Simulation Study

The various estimation approaches described in the preceding subsection are evaluated using the results of the simulation research. Monte Carlo simulations are run using different model parameters and mixing proportions π . The efficiency of the LSEs, MLE, and WLSEs of 2-CMSM($\check{\Delta}$) parameters are calculated using four simulation experiments. The bias and MSE indicators are used to discuss the ability of the MLEs, LSEs, and WLSEs. The efficiency of each parameter estimation approach for the 2-CMSM($\check{\Delta}$) model in terms of n is considered. The simulation algorithm's steps are as follows:

1. By varying the mixing proportion π and the model parameters $(\theta_1, \theta_2, \pi) = (0.25, 0.30, 0.56)$, $(0.5, 0.3, 0.4)$, $(0.15, 0.03, 0.6)$, and $(0.35, 0.45, 0.65)$, generate random samples of sizes 10, 25, ..., 300 from the 2-CMSM($\check{\Delta}$). The simulation's random samples are generated as described in the next stage.
2. Using the R uniform generator (runif), create one variate u from the uniform distribution $U(0, 1)$.
3. If $u \leq \pi$, then generate a random variate from the first component, which is a Shanker distribution (θ_1). If $u > \pi$, the second component, the Shanker distribution (θ_2), is used to generate a random variate.
4. Follow (2) until you have the required sample size n .
5. Using 1000 replications, repeat steps 1–4 again. Compute the MLEs, LSEs, and WLSEs for the 1000 samples, say $\check{\theta}_j$ for $j = 1, 2, \dots, 1000$, using the optima function and the Nelder-Mead technique in R to calculate the estimator values.
6. Determine biases and MSEs. These goals are accomplished using the following formulas:

$$Bias_{\theta}(n) = \frac{1}{1000} \sum_{j=1}^{1000} (\check{\theta}_j - \theta), \quad (56)$$

$$MSE_{\theta}(n) = \frac{1}{1000} \sum_{j=1}^{1000} (\check{\theta}_j - \theta)^2, \quad (57)$$

where $\theta = (\theta_1, \theta_2, \pi)$.

Figures 7–14 display the empirical findings. These empirical results demonstrate that the 2-CMSM($\check{\Delta}$) parameters may be accurately estimated using the suggested estimate methods. We can infer that the estimators display asymptotic unbiasedness because the bias goes to zero as n grows. On the other hand, the mean squared error behaviour suggests consistency because the mistakes trend to zero as n grows. The following observations can be drawn from Figures 7–14.

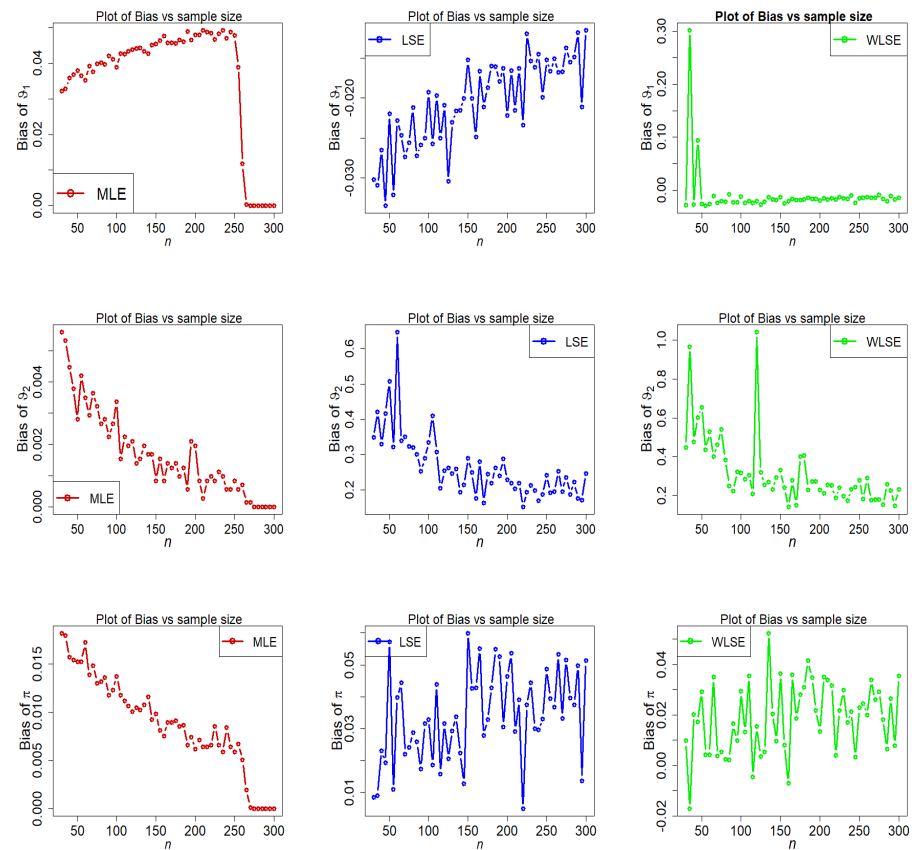


Figure 7. Fluctuations of bias of estimators under different methods for parametric set I.

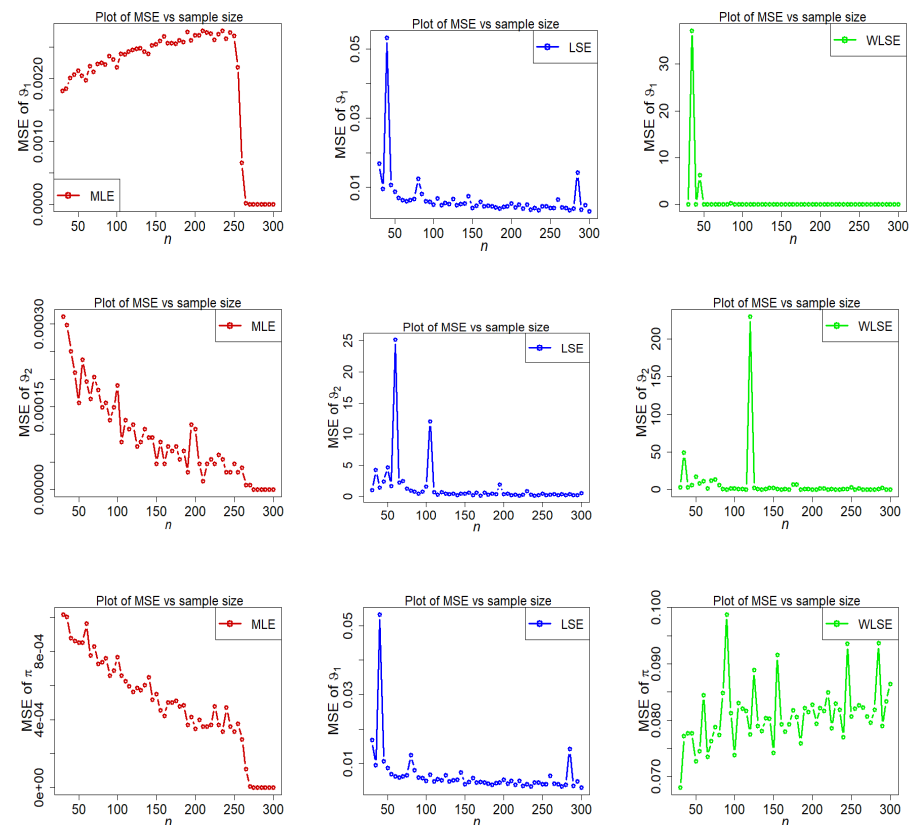


Figure 8. Fluctuations of MSE of estimators under different methods for parametric set I.

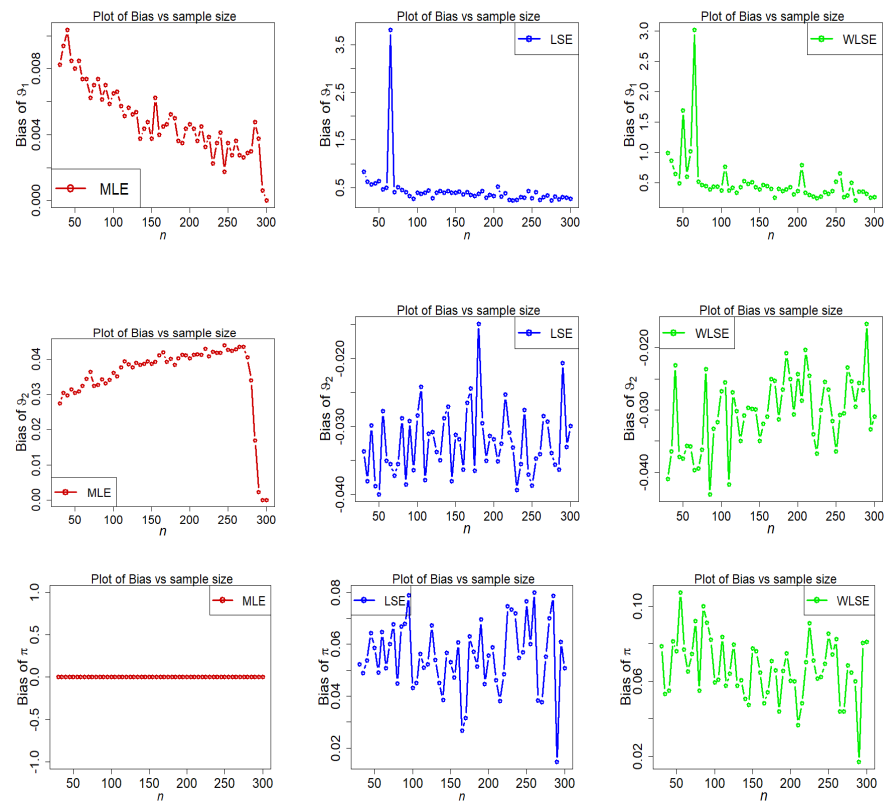


Figure 9. Fluctuations of bias of estimators under different methods for parametric set II.

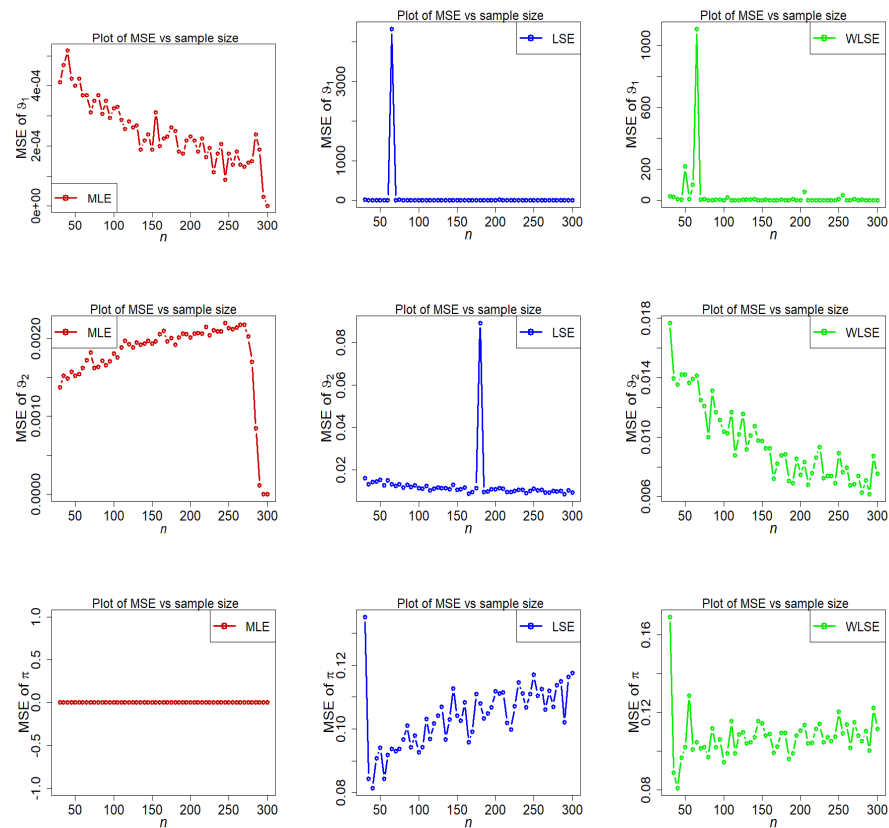


Figure 10. Fluctuations of MSE of estimators under different methods for parametric set II.

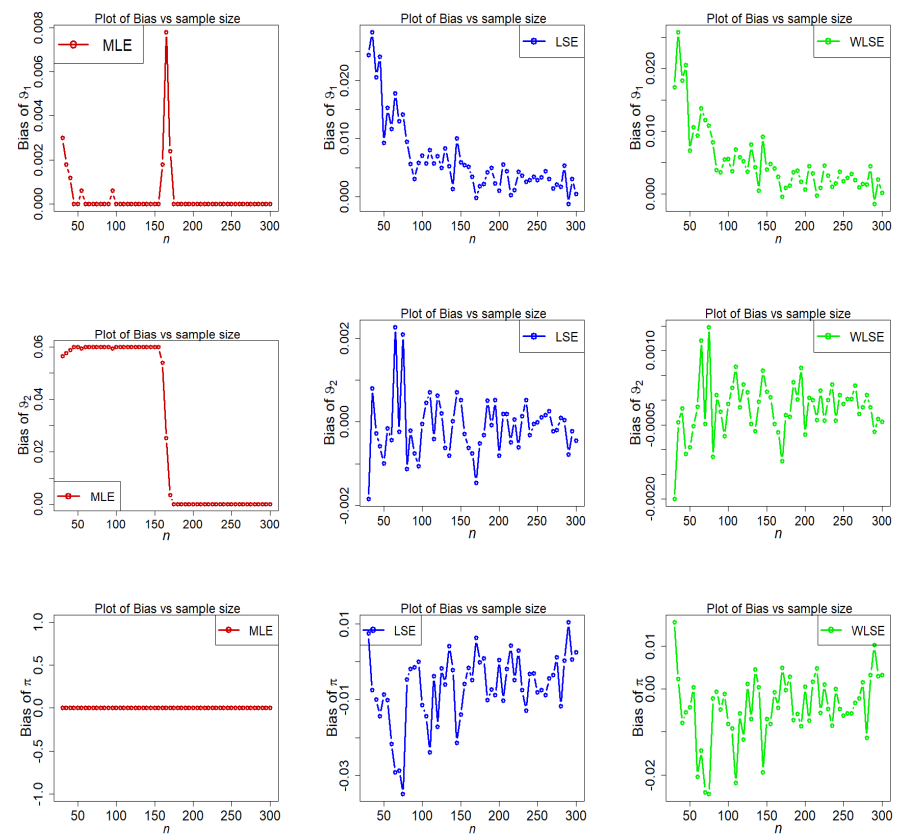


Figure 11. Fluctuations of bias of estimators under different methods for parametric set III.

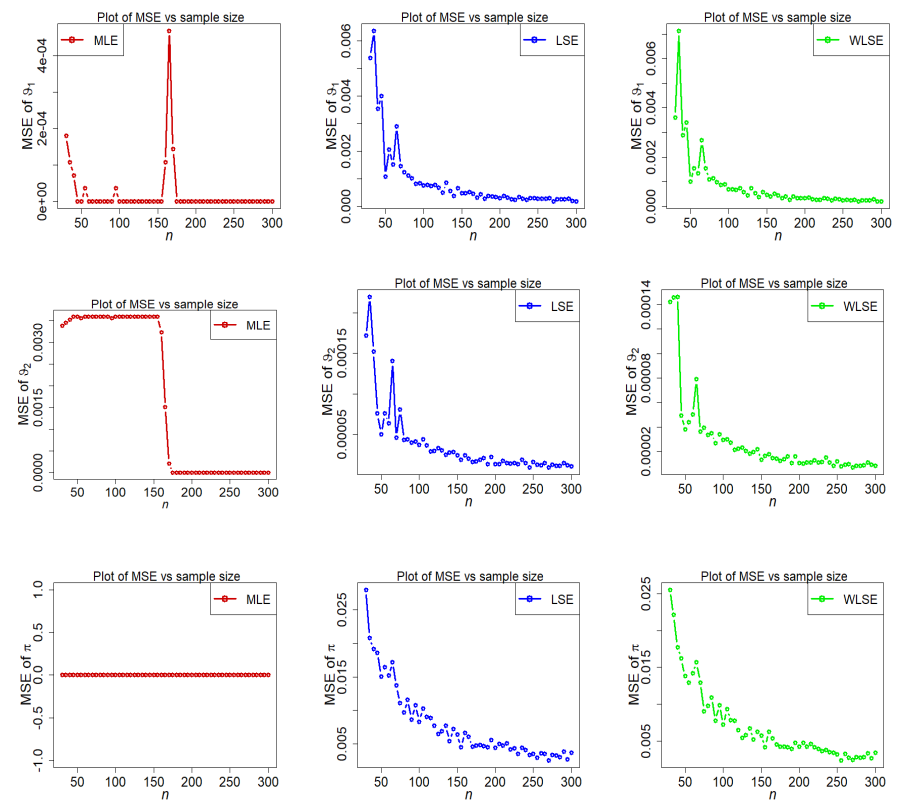


Figure 12. Fluctuations of MSE of estimators under different methods for parametric set III.

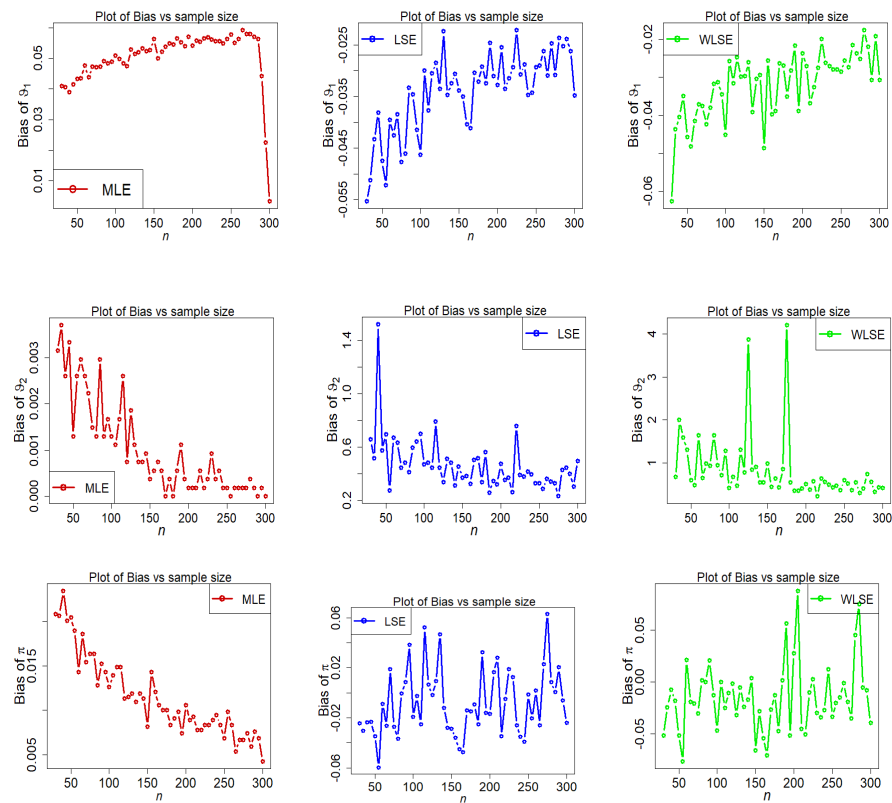


Figure 13. Fluctuations of bias of estimators under different methods for parametric set IV.

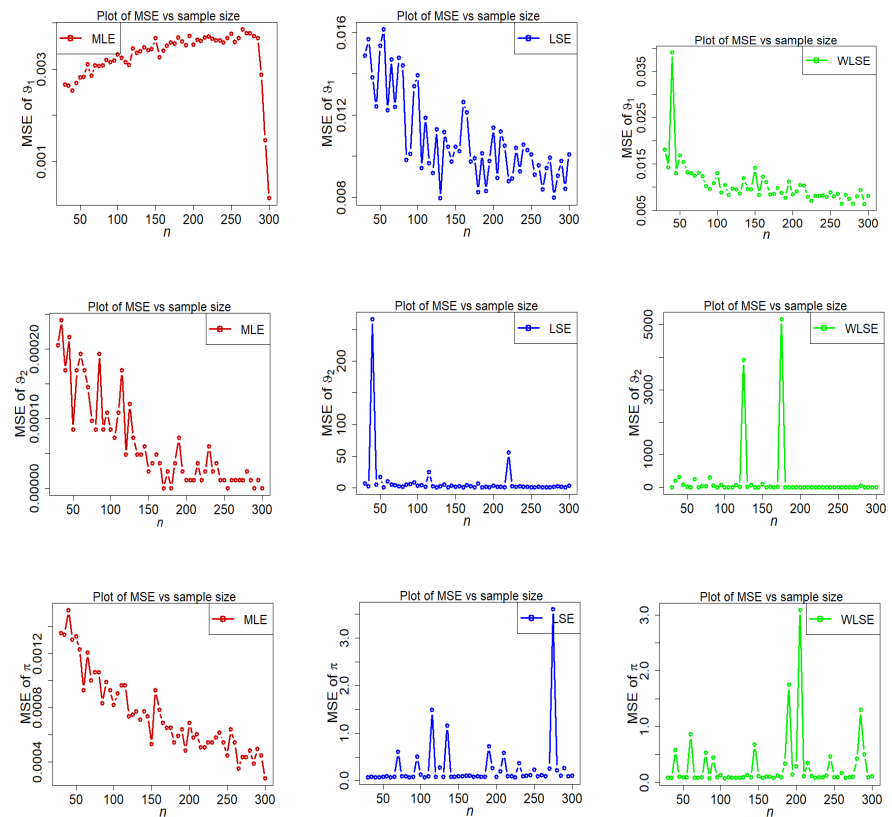


Figure 14. Fluctuations of MSE of estimators under different methods for parametric set IV.

- The estimated bias of parameters θ_1, θ_2, π , decreases as n increases under all three estimation approaches.
- For parametric Set-I, we can see that the estimated bias of parameters θ_1 under LSE is negative, whereas π and θ_2 is over-estimated while MSE of θ_2 is higher under WLSE (see Figures 7 and 8).
- For parametric Set-II, we can see that the estimated bias of parameters θ_1 and π is over-estimated in all three estimation methods, while θ_2 is under-estimated in the LSE and WLSE estimation method (see Figure 9).
- The MSE of θ_1 is strongly stimulated and higher under the LSE and WLSE estimation methods when $n < 100$ (see Figure 10).
- Figures 10 and 11 demonstrate the influence of choice of the parameters on the estimation approaches; here, bias and MSEs are comparatively low than selected set of parameters (see Figures 11 and 12).
- Some big shifts in MSEs of θ_1 under LSE and WLSE are observed when $n < 30$ and $n > 150$.
- In terms of bias, the MLE's performance is relatively favorable (see Figures 7, 9, 11 and 13).
- Moreover, when n grows, the MSE reduces for all three estimating techniques, satisfying the consistency criteria (Figures 8, 10, 12, and 14).
- The discrepancy between estimates and assumed parameters goes to zero as the sample size grows in all estimating approaches.
- When compared to alternative estimate procedures for all specified parameter values, the MLE estimation is frequently stronger in terms of bias and MSE as the sample size approaches infinity (see Figures 7–14).

The final finding from the above figures is that as the sample size grows, under all estimation methods, the estimated bias and MSE graphs for parameters θ_1, θ_2 , and π finally approach zero. This confirms the reliability of these estimating methods as well as the numerical computations for the 2-CMSM($\check{\Delta}$) parameters.

7. Applications

We demonstrate the flexibility of the 2-CMSM($\check{\Delta}$) model in this section by examining two real-world datasets. The fits of the 2-CMSM($\check{\Delta}$) model are compared to the competitive models such as the two component mixture of exponential model (2-CMEM($\check{\Delta}$)) and the two component mixture of Lindley (2-CMLM($\check{\Delta}$)) using the function `maxLik()` in R. The following excellently statistical benchmarks have been used to compare these models: -Log-likelihood (-LL), the AIC (Akaike information criterion), the BIC (Bayesian information criterion), and the AICC (Akaike information criterion corrected). The best model for the real data set might be the one with the lowest values of the above-mentioned goodness-of-fit (GOF) measures.

DataSet-1: Ghitany et al. [33] evaluated and analyzed the first data set, which represents the waiting times (in minutes) before the service of 100 Bank clients, in order to fit the Lindley distribution.

DataSet-2: The first set of data is from [34]; it is also analyzed by [35]. It comprises 40 measurements of active repair times for airborne communication transceivers. The hour is the unit of measurement.

Tables 1 and 2 show the MLEs for the 2-CMSM($\check{\Delta}$) and goodness-of-fit (GoF) metrics. Table 1 clearly demonstrates that the 2-CMSM($\check{\Delta}$) is superior to the 2-CMEM($\check{\Delta}$) and 2-CMLM($\check{\Delta}$). For further visual detailed comparison of the estimated densities, the estimated PDFs and CDFs for the considered models for Datasets I and II have been graphed (Figures 15 and 16). The CDF of the 2-CMSM($\check{\Delta}$), yet again, is clearly closer to the empirical distribution than the 2-CMEM($\check{\Delta}$) and 2-CMLM($\check{\Delta}$). To provide a different perspective, we employ probability–probability (PP) graphs in Figure 17 for Datasets I and II, respectively, to demonstrate the models' adequacy. As seen in the Figures, the 2-CMSM($\check{\Delta}$) gives a very strong fit for these datasets when compared to 2-CMEM($\check{\Delta}$) and 2-CMLM($\check{\Delta}$). Specifically, for Dataset II, it is obvious that the 2-CMSM($\check{\Delta}$) model provides a superior fit than the other models due to the exact adjustment of the scatter plot by the

PP line. In summary, the 2-CMSM($\check{\Delta}$) model emerges as the more appropriate model for the two datasets, demonstrating its applicability in a real life situation.

Table 1. MLEs, and GOF statistics for Dataset I.

Distributions		MLEs	$-LL$	AIC	BIC	AICC
2-CMSM	$\check{\theta}_1$	0.134313	−317.6296	641.2591	646.0098	641.5092
	$\check{\theta}_2$	0.198818				
	$\check{\tau}$	0.005594				
2-CMLM	$\check{\theta}_1$	0.18601	−319.0374	644.0748	648.8254	644.3248
	$\check{\theta}_2$	0.18658				
	$\check{\tau}$	0.01055				
2-CMEM	$\check{\theta}_1$	0.10124	−329.0209	664.0418	668.7924	664.2918
	$\check{\theta}_2$	0.10125				
	$\check{\tau}$	0.10561				

Table 2. MLEs, and GOF statistics for Dataset II.

Distributions		MLEs	$-LL$	AIC	BIC	CAIC
2-CMSM	$\check{\theta}_1$	0.15920	−93.12697	192.2539	197.0045	192.9206
	$\check{\theta}_2$	0.65810				
	$\check{\tau}$	0.14734				
2-CMLM	$\check{\theta}_1$	0.15046	−93.31165	192.6233	197.3739	193.2900
	$\check{\theta}_2$	0.61692				
	$\check{\tau}$	0.14163				
2-CMEM	$\check{\theta}_1$	0.10171	−94.13504	194.2701	199.0206	194.9367
	$\check{\theta}_2$	0.36262				
	$\check{\tau}$	0.17721				

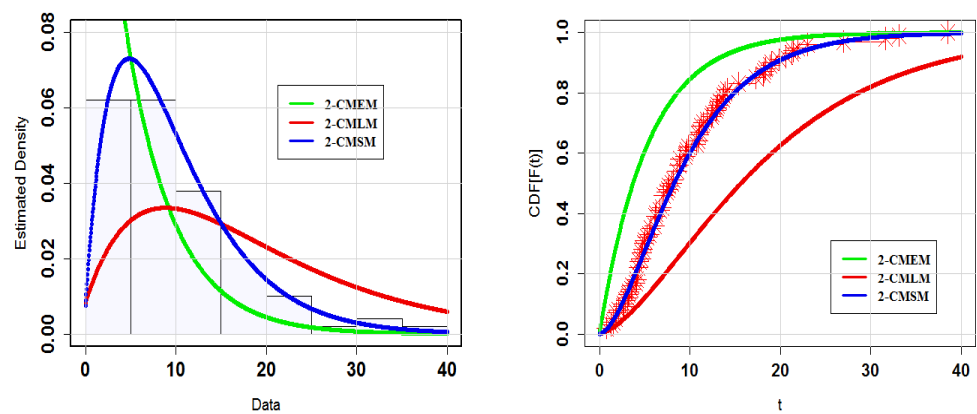


Figure 15. Plots for the estimated PDFs and CDFs for Dataset I.

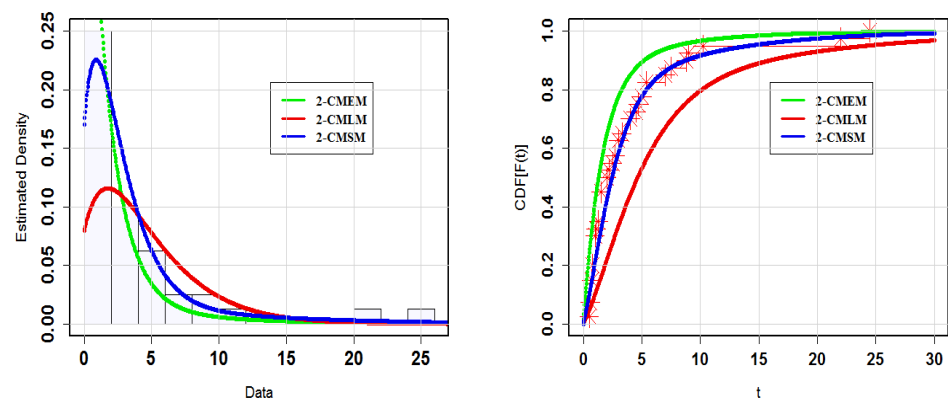


Figure 16. Plots for the estimated PDFs and CDFs for Dataset II.

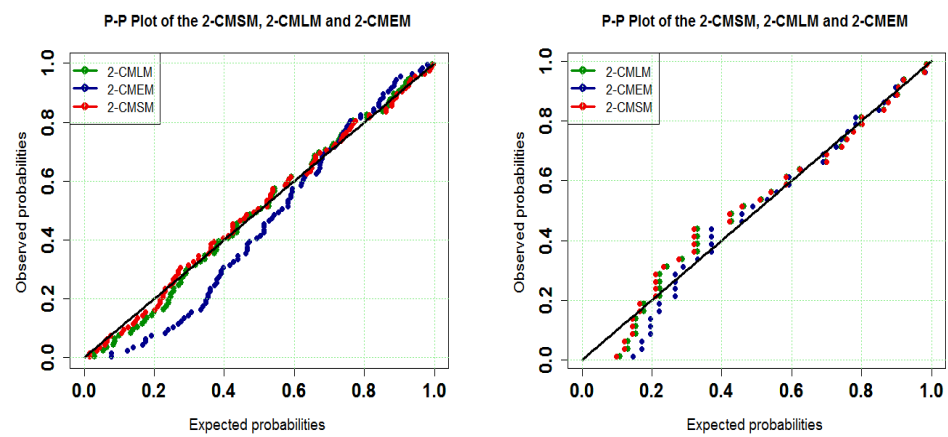


Figure 17. The probability–probability (P-P) plots for Dataset I and II.

Figures 18 and 19 show the profiles of the log-likelihood function (PLL) based on datasets that support the findings of Tables 1 and 2. Based on such graphical methods, we can suggest that the 2-CMSM ($\tilde{\Delta}$) model is a better model for the data sets under consideration. Figures 18 and 19 illustrate a graphical representation of the existence and uniqueness of MLEs, respectively.

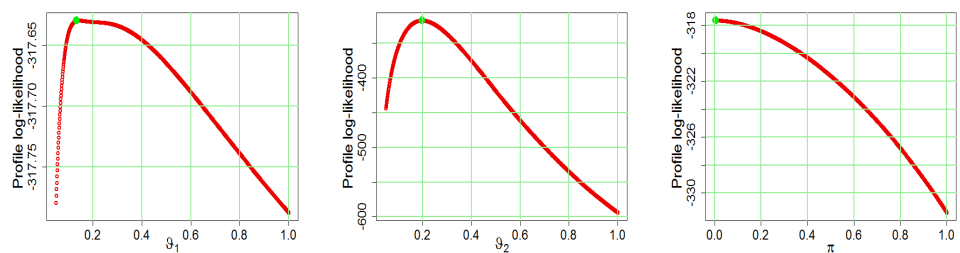


Figure 18. The PLLF for Dataset I.

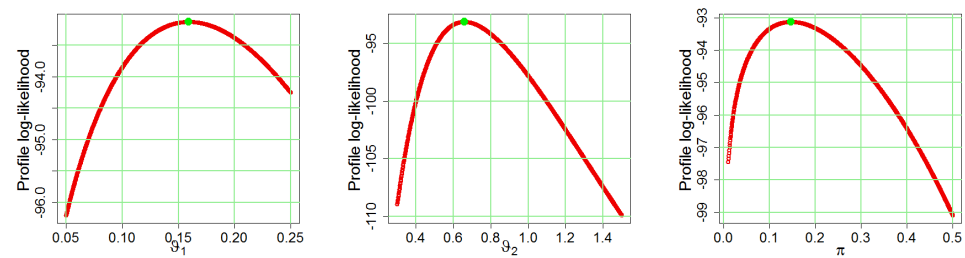


Figure 19. The PLL for Dataset II.

8. Conclusions

In this paper, we worked on two component mixtures of Shanker models using three estimate techniques: MLE, LSE, and WLSE. Additionally, the Shanker mixture model's additional statistical and reliability characteristics, including central moments, cumulants, the cumulant generating function, the factorial-moment-generating function, the probability-generating function, skewness and kurtosis, the coefficient of variation, the Mills ratio, the mean time to failure, the reversed-hazard-rate function, and the mean residual life were obtained. To investigate and compare the performance of the estimation methodologies, a simulation study with 1000 iterations was carried out. As a consequence, we discovered that the ML technique performed better than the others in terms of accuracy and consistency while predicting model unknown parameters. Furthermore, we used real datasets to demonstrate the utility of the underlying mixture model. The current study can be expanded in the future by using a mixture model with more than two components.

Author Contributions: Conceptualization, T.N.S. and T.A.A.; Methodology, S.A.L., T.N.S. and M.K.H.H.; Software, T.A.A., A.S., T.N.S.; Validation, M.K.H.H., T.N.S.; Formal analysis, A.S., S.A.L., M.K.H.H. and T.A.A.; Investigation, A.S., S.A.L. and T.N.S.; Data curation, A.S., T.N.S.; Writing—original draft, A.S., T.N.S., M.K.H.H. and T.A.A.; Writing—review & editing, A.S. and T.A.A.; Supervision, A.S.; Funding acquisition, T.A.A. All authors have read and agreed to the published version of the manuscript.

Funding: The authors would like to thank the Deanship of Scientific Research at Umm Al-Qura University for supporting this work by Grant Code 22UQU4310063DSR13

Data Availability Statement: The data used to support the findings of this study are included within the article.

Conflicts of Interest: The authors declare that have no conflict of interest.

Nomenclature

Symbols

$f(t \check{\Delta})$	PDF
$R(t \check{\Delta})$	RF
$H(t \check{\Delta})$	CHRF
$Y(t \check{\Delta})$	Mills ratio
$\tilde{M}_t(v)$	MGF
$P_t(\omega)$	PGF
$\check{K}(v)$	CGF
$\tilde{M}(t \check{\Delta})$	MTTF
$F(t \check{\Delta})$	CDF
$h(t \check{\Delta})$	HRF
$Q(q;\check{\Delta})$	QF
$R(t \check{\Delta})$	RF
$\check{\xi}_t(v)$	CF
$\tilde{F}_t(\omega)$	FMGF
$h(t \check{\Delta})$	RHRF
$\tilde{M}_R(t \check{\Delta})$	MRL

Abbreviations

The following abbreviations are used in this manuscript:

PDF	Probability density function
PGF	Probability-generating function
CDF	Cumulative distribution function
FMGF	Factorial-moment-generating function
MLE	Maximum likelihood estimator
MGF	Moment-generating function
RHRF	Reversed-hazard-rate function
WLSE	Weighted least square estimator
CGF	Cumulant generating function
CHRF	Cumulative-hazard-rate function
TTF	Time-to-failure
QF	Quantile function
MTTF	Mean time to failure
HRF	Hazard rate function
MSE	Mean square error
CF	Characteristic function
RF	Reliability function
MRL	Mean residual life
MGF	Moment-generating function
LSE	Least square estimator

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