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On the Structure of Coisometric Extensions

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Abstract: If T is a bounded linear operator on a Hilbert space $\mathscr{H}$ and V is a given linear isometry on a Hilbert space $\mathscr{K}$, we present necessary and sufficient conditions on T in order to ensure the existence of a linear isometry $\pi : \mathscr{H} \to \mathscr{K}$ such that $\pi T = V^*\pi$ (i.e., (π, V^*) extends T). We parametrize the set of all solutions π of this equation. We show, for example, that for a given unitary operator U on a Hilbert space $\mathscr{E}$ and for the multiplication operator by the independent variable M_z on the Hardy space $H^2_{\mathscr{D}}(\mathbb{D})$, there exists an isometric operator $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ such that $(\pi, (U \oplus M_z)^*)$ extends T if and only if T is a contraction, the defect index $\delta_T \le \dim \mathscr{D}$ and, for some $Y : \mathscr{A}_T \to \mathscr{E}$, (Y, U^*) extends the isometric operator $A_T^{1/2}h \mapsto A_T^{1/2}Th$ on the space $\mathscr{A}_T = \overline{A_T\mathscr{H}}$, where A_T is the asymptotic limit associated with T. We also prove that if T is isometric and V is unitary, there exists an isometric operator $\pi : \mathscr{H} \to \mathscr{K}$ such that (π, V) extends T if and only if (a) the spectral measures of the unitary part of T (in its Wold decomposition) and the restriction of V to one of its reducing subspaces $\mathscr{K}_0$ possess identical multiplicity functions and (b) $\dim(\ker T^*) = \dim(\mathscr{K}_1 \ominus V\mathscr{K}_1)$ for a certain subspace $\mathscr{K}_1$ of $\mathscr{K}$ that contains $\mathscr{K}_0$ and is invariant under V. The precise form of π, in each situation, and characterizations of the minimality conditions are also included. Several examples are given for illustrative purposes.

Keywords: coisometric extension; isometric dilation; contraction; asymptotic limit; unilateral shift

MSC: 47A20; 47A45; 47A15; 47A62; 47A65

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1. Introduction

Throughout this paper, our studies are directed toward the category of Hilbert spaces. Its objects are all Hilbert spaces, i.e., real or complex vector spaces $\mathscr{A}$ endowed with hermitian positive definite sesquilinear forms $\langle \cdot, \cdot \rangle$ on $\mathscr{A}$ (inner products) such that the associated norms $\mathscr{A} \ni a \mapsto \|a\| := \sqrt{\langle a, a \rangle} \in \mathbb{R}_+$ are complete. The morphisms of this category are all the maps C between Hilbert spaces $\mathscr{A}$ and $\mathscr{B}$ that are linear (i.e., $C(\alpha a + \alpha' a') = \alpha Ca + \alpha' Ca'$ for all scalars α and α') and bounded (i.e., $\|C\| := \sup\{\frac{\|Ca\|}{\|a\|} \mid a \neq 0\} < \infty$).

The space $\mathcal{B}(\mathscr{A}, \mathscr{B})$ of all bounded linear maps between Hilbert spaces $\mathscr{A}$ and $\mathscr{B}$ is actually a Banach space with the pointwise operations and the operator norm defined above. The adjoint of an operator $T \in \mathcal{B}(\mathscr{A}, \mathscr{B})$ is the operator $T^* \in \mathcal{B}(\mathscr{B}, \mathscr{A})$, defined uniquely by the formula $\langle Ta, b \rangle = \langle a, T^*b \rangle, a \in \mathscr{A}, b \in \mathscr{B}$. The space $\mathcal{B}(\mathscr{A}) := \mathcal{B}(\mathscr{A}, \mathscr{A})$ has the structure of a C^* algebra with the usual composition of operators and with the involution $T \mapsto T^*$. Certain special operators $A \in \mathcal{B}(\mathscr{A})$ are involved in our research: positive operators (i.e., $\langle Aa, a \rangle \ge 0, a \in \mathscr{A}$), contractions (i.e., $\|A\| \le 1$), (linear) isometries (i.e., $\|Aa\| = \|a\|, a \in \mathscr{A}$; equivalently, $A^*A = I_{\mathscr{A}}$, where $I_{\mathscr{A}}$ denotes the identity operator on $\mathscr{A}$), coisometries (i.e., A^* is isometric), and unitary (i.e., A is both isometric and coisometric) and normal operators (i.e., $AA^* = A^*A$). $A \in \mathcal{B}(\mathscr{A})$ and $B \in \mathcal{B}(\mathscr{B})$ are said to be unitarily equivalent if there exists a unitary operator $U : \mathscr{A} \to \mathscr{B}$ such that $BU = UA$.

If $\mathscr{A}$ and $\mathscr{B}$ are two Hilbert spaces, then the Cartesian product $\mathscr{A} \times \mathscr{B}$ can be naturally endowed with a Hilbert space structure using the addition and multiplication by components and the inner product $\langle (a, b), (a', b') \rangle = \langle a, a' \rangle + \langle b, b' \rangle, (a, b), (a', b') \in \mathscr{A} \times \mathscr{B}$.

The new structure is usually denoted by $\mathscr{A} \oplus \mathscr{B}$ and called the direct sum between Hilbert spaces $\mathscr{A}$ and $\mathscr{B}$. If $A \in \mathcal{B}(\mathscr{A})$ and $B \in \mathcal{B}(\mathscr{B})$, then their operator direct sum is the map $A \oplus B \in \mathcal{B}(\mathscr{A} \oplus \mathscr{B})$ given by $(A \oplus B)(a, b) := (Aa, Bb), (a, b) \in \mathscr{A} \times \mathscr{B}$.

A subspace $\mathscr{A}_0$ of a Hilbert space $\mathscr{A}$ is a closed linear manifold of $\mathscr{A}$. Its orthogonal complement $\mathscr{A}_0^{\perp} = \mathscr{A} \ominus \mathscr{A}_0$ is the subspace of all vectors $a \in \mathscr{A}$, which are orthogonal to $\mathscr{A}_0$ (i.e., $\langle a, a_0 \rangle = 0$ for every $a_0 \in \mathscr{A}_0$; the usual notation for this orthogonality is $a \perp a_0$). The orthogonal complement $\mathscr{A}_0^{\perp}$ is also a direct complement, i.e., $\mathscr{A} = \mathscr{A}_0 + \mathscr{A}_0^{\perp}$ and a representation of an element $a \in \mathscr{A}$ into the form $a = a_0 + a_0'$ is uniquely determined by the conditions $a_0 \in \mathscr{A}_0$ and $a_0' \in \mathscr{A}_0^{\perp}$. Thus, we are entitled to define the orthogonal projection $P_{\mathscr{A}_0} = P_{\mathscr{A}_0}^{\mathscr{A}}$ of $\mathscr{A}$ onto $\mathscr{A}_0$, which is a bounded linear operator between $\mathscr{A}$ and $\mathscr{A}_0$ (or $\mathscr{A}$), by $\mathscr{A} \ni a \mapsto a_0 \in \mathscr{A}_0$. The (Hilbert) dimension of $\mathscr{A}$, denoted by $\dim \mathscr{A}$, is the number of elements in an orthonormal basis (i.e., a maximal set of unit norm pairwise orthogonal vectors in $\mathscr{A}$). For two subspaces $\mathscr{A}_0$ and $\mathscr{A}_1$ of $\mathscr{A}$, we use the notation $\mathscr{A}_0 \oplus \mathscr{A}_1$ for the usual sum $\mathscr{A}_0 + \mathscr{A}_1$ indicating the fact that $\mathscr{A}_0$ and $\mathscr{A}_1$ are orthogonal (orthogonal sum). For a family $\{\mathscr{A}_i\}_{i \in I}$ of subspaces, their intersection $\bigcap_{i \in I} \mathscr{A}_i$ is also a subspace. This property is not inherited by their union. In this case, one can consider their closed linear span $\bigvee_{i \in I} \mathscr{A}_i$, i.e., the smallest subspace of $\mathscr{A}$, which contains all the subspaces $\mathscr{A}_i, i \in I$. If $\mathscr{A}_i, i \in I$ are pairwise orthogonal, we use the notation $\bigoplus_{i \in I} \mathscr{A}_i$ for the closed span $\bigvee_{i \in I} \mathscr{A}_i$. If $T \in \mathcal{B}(\mathscr{A}, \mathscr{B})$, then its kernel $\ker T := T^{-1}(\{0\})$ is a subspace of $\mathscr{A}$, while its range $\operatorname{ran} T := T\mathscr{A}$ is generally only a linear manifold of $\mathscr{B}$. A subspace $\mathscr{A}_0$ of $\mathscr{A}$ is invariant under $A \in \mathcal{B}(\mathscr{A})$ if $A\mathscr{A}_0 \subseteq \mathscr{A}_0$. $\mathscr{A}_0$ is reducing for A if it is invariant under both A and A^*.

These basic facts on the category of Hilbert spaces can be found in any introductory book on functional analysis or linear operator theory. The author's recommendations are [1–3].

The famous dilation theorem of Béla Sz.-Nagy [4] shows that every contraction T that acts on a Hilbert space $\mathscr{H}$ can be (power) dilated by an isometric (equivalently, a unitary) operator V on a bigger space $\mathscr{K} \supseteq \mathscr{H}$:

$$T^n = P_{\mathscr{H}} V^n|_{\mathscr{H}}, \quad n \geq 0.$$

Nowadays, dilations represent an important instrument of study in many disciplines such as invariant subspace theory, interpolation theory, operator algebras, dynamical systems, control, prediction theory, and so on (see the excellent survey article of Orr Moshe Shalit [5] or the books by Sz.-Nagy, Foiaș, Bercovici and Kérchy [6], Foiaș and Frazho [7], Foiaș, Frazho, Gohberg and Kaashoeck [8], Rosenblum and Rovnyak [9], Kakihara [10], and so on).

If V^* is a coisometric extension of T, i.e., $\mathscr{H}$ is invariant under V^* and $V^*|_{\mathscr{H}} = T$, then V is an isometric dilation of T^*. In minimality conditions, i.e., $\mathscr{K} = \bigvee_{n \geq 0} V^n \mathscr{H}$, the two concepts are actually equivalent. In addition to the numerous applications in dilation theory, the theory of extensions was proved by P.S. Muhly and B. Solel to be useful in the representation theory of tensor algebras associated with C^*correspondences or in the study of C^* dynamical systems [11,12]. A good reference in this context is the Ph.D. thesis of T. Wolf [13].

In our approach, a coisometric extension on a Hilbert space $\mathscr{K}$ of the contraction T is a pair (π, V^*), where $V \in \mathcal{B}(\mathscr{K})$ and $\pi \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ are isometric operators such that $\pi T = V^* \pi$ (T is a quasi-affine transform of V^*, with quasi-affinity π, according to the terminology introduced in [14]). In fact, in this setting, $\pi \mathscr{H}$ is invariant under V^* and $V^*|_{\pi \mathscr{H}}$ is unitarily equivalent with T. Given the operator V, it is our aim in this paper to characterize T such that there exists π, which intertwines T and V^*. We also describe and classify the isometric operators π. This type of extension has been considered recently, e.g., in [15,16].

By the Wold–Halmos decomposition theorem (Theorem 1), any isometric operator on $\mathscr{H}$ can be represented as the direct sum between a unitary operator and a unilateral shift. Unitary operators are well understood. They have a spectral representation and an associated functional calculus. On the other hand, unilateral shifts possess a simple geometrical structure. In fact, a *unilateral shift* (or, simply, a shift) is just an operator that is unitarily equivalent to the operator M_z of multiplication by the independent variable, $(M_z f)(\lambda) := \lambda f(\lambda), \lambda \in \mathbb{D}$, on the $\mathscr{D}$-valued Hardy space

$$H^2_{\mathscr{D}}(\mathbb{D}) := \{f : \mathbb{D} \to \mathscr{D} \mid f(\lambda) = \sum_{n=0}^{\infty} \lambda^n d_n, \lambda \in \mathbb{D}, \{d_n\}_{n\geq 0} \subseteq \mathscr{D} \\ \text{and } \|f\|^2 = \sum_{n=0}^{\infty} \|d_n\|^2 < \infty\}$$

for a certain complex Hilbert space $\mathscr{D}$. In fact, an isometric operator S on $\mathscr{H}$ is a shift if and only if there exists a subspace $\mathscr{L}$ of $\mathscr{H}$, which is *wandering*, i.e.,

$$S^n\mathscr{L} \perp S^m\mathscr{L} \text{ for } m, n \geq 0,\ m \neq n$$

and *generating*, i.e.,

$$\mathscr{H} = \bigoplus_{n\geq 0} S^n\mathscr{L}.$$

The subspace $\mathscr{L}$ is called the *defect* of S and can be computed in terms of S (in fact, $\mathscr{L} = \ker S^*$). The dimension of $\mathscr{L}$, i.e., the *multiplicity* of S determines S up to a unitary equivalence. Let us also note that S is a shift if and only if it is completely non-unitary (i.e., there are no non-null subspaces reducing S to a unitary operator) if and only if S is of class $\mathcal{C}_{\cdot 0}$ (or, S^* is strongly stable). As introduced in [6] (Chapter II.4), a contraction T on $\mathscr{H}$ is of *class* $\mathcal{C}_{0\cdot}$ if T^n tends strongly to 0 as $n \to \infty$ (i.e., $\|T^n h\| \xrightarrow{n\to\infty} 0$ for every $h \in \mathscr{H}$). T is of *class* $\mathcal{C}_{\cdot 0}$ if T^* belongs to the class $\mathcal{C}_{0\cdot}$. A very good introductory material on shifts is [17]. The included terminology has been extracted from [6] (Chapter I).

Let A and B be bounded linear operators acting on complex Hilbert spaces $\mathscr{A}$ and $\mathscr{B}$, respectively. Let also $\pi : \mathscr{A} \to \mathscr{B}$ be a linear isometry.

Definition 1. *The pair (π, B) is said to:*

- extend A *if* $\pi A = B\pi$;
- (power) dilate A *if* $\pi^* B^n \pi = A^n$ *for every* $n \geq 0$.

The paper starts with a general discussion on the link between extensions and dilations. Their main properties are emphasized here. We show that an extension is always a dilation. However, if (π, B) is a dilation of A, then (π, B) is also an extension of A if and only if $\pi\mathscr{A}$ is invariant under B. We also prove that for the property of a pair (π, B) for a dilation, respectively, an extension is preserved when we restrict B to one of its invariant subspaces, which contains $\pi\mathscr{A}$.

Definition 2. *A dilation (π, B) of A is said to be* minimal *if $\mathscr{B}$ is the smallest subspace that is invariant under B and contains $\pi\mathscr{A}$, i.e.,*

$$\mathscr{B} = \bigvee_{n\geq 0} B^n \pi\mathscr{A}. \tag{1}$$

We observe that, for a given minimal dilation (π, B) of A, its adjoint (π, B^*) extends A^*. We also indicate necessary and sufficient conditions for two minimal dilations of A in order to be equivalent. The novelty of the results presented in this section is that they are formulated in full generality, while in books on dilation theory, the study is restricted to the case when A is a contraction, B is isometric or unitary, $\mathscr{A}$ is a subspace of $\mathscr{B}$, while π is just the inclusion map (see, e.g., [6]).

In Section 3, we present necessary and sufficient conditions on an isometric operator V on $\mathscr{H}$ in order to be π-extended by a given unitary operator U on $\mathscr{K}$. This description is given only by numerical invariants, i.e., multiplicities of certain isometric operators associated with U. In order to formulate this result precisely, we include some definitions for increased readability. A complex spectral measure is a map E on the Borel σ-algebra $\mathcal{B}$ on $\mathbb{C}$ into $\mathcal{B}(\mathscr{H})$ such that $E(\mathbb{C}) = I_{\mathscr{H}}$, $E(\mathcal{M}) = E(\mathcal{M})^2 = E(\mathcal{M})^*$ for every Borel set $\mathcal{M}$ and $E(\bigcup_{n\geq 0}\mathcal{M}_n) = \sum_{n\geq 0} E(\mathcal{M}_n)$ for every pairwise disjoint family $\{\mathcal{M}_n\}_{n\geq 0}$ of Borel sets. We sketch the construction of Halmos [18] (Chapter III) for the multiplicity function u attached to the spectral measure E. For a given nonzero finite measure μ on $\mathcal{B}$, its multiplicity $u(\mu)$ is the minimum value between the powers of maximal orthogonal systems of type $P_{\{h\in\mathscr{H}|\rho(h)\ll\mu_0\}}$ for nonzero measures μ_0, which are absolutely continuous with respect to μ, where, for $h \in \mathscr{H}$, $\rho(h)$ is the measure $\mathcal{B} \ni \mathcal{M} \mapsto \langle E(\mathcal{M})h, h\rangle \in \mathbb{C}$. We show that (π, U) extends V if and only if (a) the spectral measures of the unitary part of V and the restriction of U to one of its reducing subspaces $\mathscr{K}_0$ possess identical multiplicity functions and (b) the shift part of V and the shift part of $U|_{\bigvee_{n\geq 0}(\mathscr{K}_1\ominus U^n\mathscr{K}_1)}$ have identical multiplicities, where $\mathscr{K}_1$ is a subspace of $\mathscr{K}$, which contains $\mathscr{K}_0$ and it is invariant under U.

An important result of Ciprian Foiaș [19] describes the structure of contractions of class $\mathcal{C}_{0\cdot}$: they can be represented as restrictions of backward shifts (i.e., adjoints of shifts). We prove, in the following section, that there exists an isometric operator $\pi : \mathscr{H} \to H^2_{\mathscr{D}}(\mathbb{D})$ such that (π, M_z^*) extends T if and only if T is a contraction of class $\mathcal{C}_{0\cdot}$ and $\delta_T \leq \dim \mathscr{D}$. Any solution π of the equation $\pi T = M_z^*\pi$ has the form $\pi = \pi_V$, where:

$$(\pi_V h)(\lambda) = V D_T (I_{\mathscr{H}} - \lambda T)^{-1} h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

where $V : \mathscr{D}_T \to \mathscr{D}$ is an arbitrary isometric operator. Here, according to the B. Sz.-Nagy and C. Foiaș terminology [6] (Chapter I.3), $D_T := (I_{\mathscr{H}} - T^*T)^{1/2}$ is the *defect operator* of T, $\mathscr{D}_T := \overline{D_T\mathscr{H}}$ is the corresponding *defect space*, and $\delta_T = \dim \mathscr{D}_T$ is the *defect index* of T.

A model for the minimal isometric dilation V of T (hence, the model also describes the coisometric extension V^* of T^*) has been proposed by Schäffer in [20]. More precisely, the minimal isometric dilation of T is given on the Hilbert space $\mathscr{H} \oplus H^2_{\mathscr{D}_T}(\mathbb{D})$ by the formula:

$$V(h, f) = (Th, D_T h + M_z f), \quad h \in \mathscr{H}, f \in H^2_{\mathscr{D}_T}(\mathbb{D}).$$

We deduce that the operator V^* is a coisometric extension of T^*. This result was the starting point for developing a precise geometrical structure for such dilations, respectively extensions (cf. [21–23]). In the model above, in the terminology introduced in Definition 1, the pair $(i_{\mathscr{H}}, V^*)$ extends T^*, where $i_{\mathscr{H}}$ is the embedding $\mathscr{H} \hookrightarrow \mathscr{H} \oplus H^2_{\mathscr{D}_T}(\mathbb{D})$. In our approach, our job is, given V, to find necessary and sufficient conditions on T in order to ensure the existence of an isometric operator π such that $\pi T = V^*\pi$. Therefore, we look for operators π that might be different from just an embedding. In order to solve this problem, we need to introduce a special positive contraction. Since $(T^{*n}T^n)_{n\geq 0}$ is a decreasing sequence of positive operators, it has a strong limit, which will be denoted by A_T and called the *asymptotic limit* associated with T. It has been used as a tool in the construction of the isometric dilation in [6] (Chapter I.10), in various invariant subspace problems [24], in similarity problems [25], for Putnam–Fuglede-type results [26], and so on (see, also, [27–31]). It seems that this limit appeared for the first time in 1967, in the french edition of the book [6]. Associated notions are the *asymptotic space* of T, i.e., the closure $\mathscr{A}_T$ of the range of A_T and the *asymptotic index* a_T of T, i.e., the Hilbert dimension of $\mathscr{A}_T$.

In the last section of the paper, we move on to the general case in which V is written, by the Wold–Halmos decomposition theorem (Theorem 1), as the direct sum between a unitary operator U and the shift M_z. We prove that, given complex Hilbert spaces $\mathscr{D}, \mathscr{E}$ and a unitary operator U on $\mathscr{E}$, there exists an isometric operator $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ such that $(\pi, (U \oplus M_z)^*)$ extends T if and only if T is a contraction, $\delta_T \leq \dim \mathscr{D}$ and (Y, U^*) extends the isometric operator

$$\mathscr{A}_T \ni A_T^{1/2} h \mapsto V_T A_T^{1/2} h := A_T^{1/2} T h \in \mathscr{A}_T,$$

for a certain isometric operator Y that acts from $\mathscr{A}_T := \overline{A_T\mathscr{H}}$ into $\mathscr{E}$. Any solution π to this problem has the form:

$$\pi h = (YA_T^{1/2}h, W_X h), \quad h \in \mathscr{H},$$

where $Y : \mathscr{A}_T \to \mathscr{E}$ is an isometric operator that intertwines V_T and U^*, while W_X has the form:

$$(W_X h)(\lambda) = XD_T(I_{\mathscr{H}} - \lambda T)^{-1}h, \quad \lambda \in \mathbb{D}, h \in \mathscr{H},$$

X being any isometric operator from $\mathscr{D}_T$ into $\mathscr{D}$. We also characterize π-extensions in which the corresponding adjoint dilations are minimal. Several examples are given for illustrative purposes.

2. Extensions and Dilations

Throughout this section, the symbols $\mathscr{A}$ and $\mathscr{B}$ denote complex Hilbert spaces with $\dim \mathscr{A} \leq \dim \mathscr{B}$, $\pi : \mathscr{A} \to \mathscr{B}$ is a linear isometry, while A and B are bounded linear operators acting on $\mathscr{A}$ and $\mathscr{B}$, respectively.

When $\mathscr{A}$ is a closed subspace of $\mathscr{B}$, one can take π as the inclusion map of $\mathscr{A}$ into $\mathscr{B}$. Then π^* is the orthogonal projection $P_{\mathscr{A}}$ of $\mathscr{B}$ onto $\mathscr{A}$. Hence, $(i_{\mathscr{A}}, B)$ is the usual extension of A: $\mathscr{A}$ is invariant under B and $B|_{\mathscr{A}} = A$. At the same time, the pair $(i_{\mathscr{A}}, B)$ dilates A if

$$P_{\mathscr{A}} B^n a = A^n a, \quad a \in \mathscr{A}, n \geq 0,$$

which is exactly the concept introduced by Béla Sz.-Nagy in [4].

Even under minimality conditions, a dilation (π, B) might not be unique, up to a unitary equivalence. One may note that, in general, the structures of π and B could become quite complicated.

It is an immediate consequence of Definition 1 that (π, B) is a dilation of A if and only if (π, B^*) is a dilation of A^*. Such a property is not valid, in general, for extensions. Another easy consequence of the corresponding definitions is their transitive properties: if (π, B) extends (dilates) A and (ρ, C) extends (dilates) B, then $(\rho\pi, C)$ extends (dilates) A. Here, C is a bounded linear operator that acts on the complex Hilbert space $\mathscr{C}$, and $\rho : \mathscr{B} \to \mathscr{C}$ is a linear isometry.

Remark 1. *(a) If (π, B) extends A, i.e., $\pi A = B\pi$, then $\pi A^2 = B\pi A = B^2\pi$ and, inductively, $\pi A^n = B\pi A^{n-1} = B^n\pi$ for every $n \geq 1$. This shows that (π, B) dilates A.*

The converse is, in general, false. However,

(b) If (π, B) dilates A, then (π, B) also extends A if and only if $\pi\mathscr{A}$ is invariant under B. Indeed, for each $a \in \mathscr{A}$, it holds:

$$\langle \pi Aa, B\pi a - \pi Aa\rangle = \langle Aa, \pi^* B\pi a\rangle - \langle \pi Aa, \pi Aa\rangle = 0,$$

so

$$\|B\pi a\|^2 = \|Aa\|^2 + \|B\pi a - \pi Aa\|^2. \tag{2}$$

Hence, (π, B) extends A if and only if $\|B\pi a\| = \|Aa\|$, $a \in \mathscr{A}$. Since $A = \pi^* B\pi$, we deduce that, for every $a \in \mathscr{A}$, $B\pi a$ must be in the range of π. It follows that $B\pi\mathscr{A} \subseteq \pi\mathscr{A}$, as required.

(c) In particular, if (π, B) dilates A, A is an isometry and B is a contraction, then (π, B) extends A. Indeed, for $a \in \mathscr{A}$, $\|B\pi a\| \leq \|\pi a\| = \|a\|$ and $\|Aa\| = \|a\|$. This forces $\|B\pi a - \pi Aa\| = 0$ in (2).

(d) If $\pi\mathscr{A}$ is invariant under B, then $A = \pi^* B\pi$ is a restriction of (π, B), i.e., (π, B) extends A. Any restriction of (π, B) has this form. We use the fact that $\pi\mathscr{A}$ is invariant under B if and only if $B\pi = \pi\pi^* B\pi$, that is $B\pi = \pi A$. Hence, (π, B) is an extension of A. If $A' \in \mathcal{B}(\mathscr{A})$ is a restriction of (π, B), then

$$A' = \pi^*\pi A' = \pi^* B\pi.$$

If $\mathscr{B}_0$ is a subspace of $\mathscr{B}$, which is invariant under B and contains $\pi\mathscr{A}$, then one can consider the operators $B_0 = B|_{\mathscr{B}_0}$ and $\pi_0 : \mathscr{A} \to \mathscr{B}_0$ defined by $\pi_0 a = \pi a,\ a \in \mathscr{A}$. Since $\pi_0^* = \pi^*|_{\mathscr{B}_0}$, we immediately deduce that (π, B) dilates (extends) A if and only if (π_0, B_0) dilates (extends) A. This property can be extended to the case when $\mathscr{B}_0$ is not necessarily a subspace of $\mathscr{B}$.

Proposition 1. *Let $\mathscr{B}_0$ be a complex Hilbert space and $\rho : \mathscr{B}_0 \to \mathscr{B}$ an isometric operator such that $\rho\mathscr{B}_0$ contains $\pi\mathscr{A}$, and it is invariant under B. Then (π, B) dilates (extends) A if and only if $(\rho^*\pi, B_0 := \rho^* B\rho)$ dilates (extends) A. In this case, (ρ, B) extends B_0, which is unitarily equivalent to $B|_{\rho\mathscr{B}_0}$, i.e., $B_0 = U^* B|_{\rho\mathscr{B}_0} U$, where $U : \mathscr{B}_0 \to \rho\mathscr{B}_0$ is the unitary operator*

$$\mathscr{B}_0 \ni b_0 \mapsto \rho b_0 \in \rho\mathscr{B}_0.$$

Proof. As seen before, $\rho\mathscr{B}_0$ is invariant under B if and only if $\rho\rho^* B\rho = B\rho$. Observe, also, that $\rho^*\pi$ is an isometric operator:

$$\|\rho^*\pi a\| = \|U^*\pi a\| = \|a\|, \quad a \in \mathscr{A}.$$

We used, for the first equality above, the fact that $\rho\mathscr{B}_0$ contains $\pi\mathscr{A}$ and $U^* = \rho^*|_{\rho\mathscr{B}_0}$. We note that, for $a \in \mathscr{A}$ and $n \geq 0$, the following set of inequalities holds true:

$$\begin{aligned}
\pi^*\rho B_0^n \rho^*\pi a &= \pi^*\rho \underbrace{(\rho^* B\rho)\dots(\rho^* B\rho)}_{n \text{ times}} \rho^*\pi a \\
&= \pi^* B\rho \underbrace{(\rho^* B\rho)\dots(\rho^* B\rho)}_{n-1 \text{ times}} \rho^*\pi a \\
&\dots\dots\dots\dots\dots\dots\dots\dots \\
&= \pi^* B^n \rho\rho^*\pi a \\
&= \pi^* B^n P_{\rho\mathscr{B}_0}\pi a \\
&= \pi^* B^n \pi a.
\end{aligned} \tag{3}$$

Similarly,

$$B_0\rho^*\pi a = \rho^* B\rho\rho^*\pi a = \rho^* B\pi a, \quad a \in \mathscr{A}. \tag{4}$$

According to (3), (π, B) is a dilation of A if and only if $(\rho^*\pi, B_0)$ is a dilation of A. In view of (4), if A is extended by (π, B), then $B_0\rho^*\pi = \rho^* B\pi = \rho^*\pi A$. Conversely, if A is extended by $(\rho^*\pi, B_0)$, then it follows by (4) that $\rho^* B\pi a = \rho^*\pi A a,\ a \in \mathscr{A}$. Since both $B\pi a$ and $\pi A a$ are elements of $\rho\mathscr{B}_0$ and $U^* = \rho^*|_{\rho\mathscr{B}_0}$, we deduce that $U^* B\pi a = U^*\pi A a,\ a \in \mathscr{A}$. Therefore, $B\pi = \pi A$, as required.

(ρ, B) extends B_0 followed by Remark 1(d). The unitary equivalence between B_0 and $B|_{\rho\mathscr{B}_0}$ is a consequence of the definition of B_0. □

We now start with a bounded linear operator A on $\mathscr{A}$, with a Hilbert space $\mathscr{B}$ such that $\dim \mathscr{B} \geq \dim \mathscr{A}$ and with a linear isometry $\pi : \mathscr{A} \to \mathscr{B}$. Our next aim is to build and parametrize the class of all operators $B \in \mathcal{B}(\mathscr{A})$ such that (π, B) is an extension of A.

In order to obtain the equality, $\pi A = B\pi$ is enough to define B on $\pi\mathscr{A}$ as follows:

$$Bb := \pi A\pi^* b, \quad b \in \pi\mathscr{A}.$$

One can take for $B|_{\mathscr{B}\ominus\pi\mathscr{A}}$ any bounded linear operator from $\mathscr{B} \ominus \pi\mathscr{A}$ into $\mathscr{B}$. Thus, according to the decomposition $\mathscr{B} = \pi\mathscr{A} \oplus (\mathscr{B} \ominus \pi\mathscr{A})$, B has the matrix representation:

$$B = \begin{pmatrix} \pi A\pi^*|_{\pi\mathscr{A}} & X \\ 0 & Y \end{pmatrix}.$$

where $X \in \mathcal{B}(\mathscr{B} \ominus \pi\mathscr{A}, \pi\mathscr{A})$ and $Y \in \mathcal{B}(\mathscr{B} \ominus \pi\mathscr{A})$.

We observe that, by an easy computation with matrices, B is isometric if and only if A is isometric, $\pi A\mathscr{A}$ is orthogonal to ran X, and (X, Y) is a row isometry (i.e., $X^*X + Y^*Y = I_{\mathscr{B}\ominus\pi\mathscr{A}}$). One can take $X = 0$ and A, Y isometric operators in order to build an isometric extension (π, B) of A.

Similar computations show that B is coisometric if and only if Y is coisometric, ran X^* and ran Y^* are orthogonal and $XX^* = \pi(I_{\mathscr{A}} - AA^*)\pi^*|_{\pi\mathscr{A}}$. Taking again $X = 0$, the extension (π, B) is coisometric if and only if A and Y are coisometric operators.

We deduce that B is unitary if and only if A is isometric, Y is coisometric, ran $X^* \perp$ ran Y^*, ran X $\perp \pi A\mathscr{A}$ and $X^*X + Y^*Y = I_{\mathscr{B}\ominus\pi\mathscr{A}}$.

As observed in Proposition 1, the dilation extension properties, respectively, are preserved if we restrict B to one of its invariant subspaces, which contains $\pi\mathscr{A}$.

As we will see later in this section, the minimality condition does not always ensure the uniqueness, up to a unitary equivalence, of such a dilation.

As noted earlier, (π, B) is a dilation of A if and only if (π, B^*) is a dilation of A^*. However, there is no reason to assume that the minimality of (π, B) implies the minimality of (π, B^*). If (π, B) extends A, then the smallest subspace of $\mathscr{B}$ that is invariant under B and contains $\pi\mathscr{A}$ is

$$\bigvee_{n\geq 0} B^n\pi\mathscr{A} = \bigvee_{n\geq 0} \pi A^n\mathscr{A} = \pi\mathscr{A}.$$

In other words, if π_0 is the unitary operator $\mathscr{A} \ni a \mapsto \pi_0 a := \pi a \in \pi\mathscr{A}$ and $B_0 := B|_{\pi\mathscr{A}} \in \mathcal{B}(\pi\mathscr{A})$, then (π_0, B_0) is an extension of A, which is unitarily equivalent to A:

$$A = \pi_0^* B_0 \pi_0.$$

Hence, this situation does not present any real interest.

A dilation is not, in general, an extension. However, under the minimality assumption, the dilation adjoint is actually an extension.

Remark 2. If (π, B) is a minimal dilation of A, then (π, B^*) is an extension of A^*. *Let $a, a' \in \mathscr{A}$ and $n \geq 0$. Then*

$$\langle B^*\pi a - \pi A^* a, B^n\pi a'\rangle = \langle a, \pi^* B^{n+1}\pi a'\rangle - \langle A^* a, \pi^* B^n \pi a'\rangle = 0,$$

since $\pi^* B^m \pi a = A^m a$ for $m \in \{n, n+1\}$. In view of the minimality condition (1), this implies that $B^*\pi = \pi A^*$.

Our final aim in this section is to express the necessary and sufficient conditions in order to ensure that a minimal dilation is unique.

Definition 3. *Two minimal dilations (π_1, B_1) and (π_2, B_2) of A are said to be* equivalent *if there exists a unitary operator $U : \mathscr{B}_1 \to \mathscr{B}_2$ (here, $B_1 \in \mathcal{B}(\mathscr{B}_1)$ and $B_2 \in \mathcal{B}(\mathscr{B}_2)$) such that $U\pi_1 = \pi_2$ and $UB_1 = B_2U$.*

If (π_1, B_1) and (π_2, B_2) are equivalent, then, according to Definition 1, (U, B_2) extends B_1 and (U^*, B_1) extends B_2.

We are now in a position to present the conditions that describe the equivalence between two minimal dilations.

Proposition 2. *Two minimal dilations (π_1, B_1) and (π_2, B_2) of A are equivalent if and only if*

$$\pi_1^* B_1^{*m} B_1^n \pi_1 = \pi_2^* B_2^{*m} B_2^n \pi_2 \text{ for every } m, n \geq 0. \tag{5}$$

Proof. The operator U must be defined on the generators of $\mathscr{B}_1$, which are, by minimality, $B_1^n\pi_1 a$, where $n \geq 0$ and $a \in \mathscr{A}$. More precisely, in view of the axioms of Definition 3, we must have:

$$UB_1^n\pi_1 a = B_2 UB_1^{n-1}\pi_1 a = \cdots = B_2^n U\pi_1 a = B_2^n \pi_2 a. \tag{6}$$

Formula (6) correctly defines an isometric operator on $\mathscr{B}_1$ if and only if

$$\langle \sum_{m\geq 0} B_1^m \pi_1 a_m, \sum_{n\geq 0} B_1^n \pi_1 a'_n \rangle = \langle \sum_{m\geq 0} B_2^m \pi_2 a_m, \sum_{n\geq 0} B_2^n \pi_2 a'_n \rangle \tag{7}$$

for all sequences $(a_m)_{m\geq 0}$ and $(a'_n)_{n\geq 0}$ of elements in $\mathscr{A}$ with finite support. We can write (7) in equivalent form as:

$$\sum_{m,n\geq 0} \langle a_m, \pi_1^* B_1^{*m} B_1^n \pi_1 a'_n \rangle = \sum_{m,n\geq 0} \langle a_m, \pi_2^* B_2^{*m} B_2^n \pi_2 a'_n \rangle.$$

We finally deduce (5).

By minimality, the operator U, as defined in (6), is also surjective. Hence, U is a unitary operator. Moreover, by the same Formula (6), $U\pi_1 = \pi_2$ $(n = 0)$ and

$$UB_1(B_1^n\pi_1 a) = UB_1^{n+1}\pi_1 a = B_2^{n+1}\pi_2 a = B_2 U(B_1^n \pi_1 a), \quad n \geq 0, a \in \mathscr{A},$$

that is, $UB_1 = B_2 U$, according to the minimality of B_1. □

Minimal extensions are indeed equivalent: if (π, B) extends A, then

$$\pi^* B^{*m} B^n \pi = A^{*m} A^n, \quad m, n \geq 0$$

it is independent of the choices of π and B.

Examples of equivalent classes of dilations are the ones of selfadjoint operators and isometric operators, respectively.

3. Unitary Extensions

In this section, we provide necessary and sufficient conditions on a given unitary operator U in order to extend, in the sense of Definition 1, a given isometric operator V. We also discuss the structure of U.

The Norwegian statistician Herman Wold discovered in [32] a remarkable decomposition for every stationary stochastic process. More precisely, his decomposition separates the deterministic part from the part corrupted by noises (in fact, the moving average of white noise). It was the cornerstone of prediction theory for such processes and has nowadays applications in many domains such as machine learning [33], traffic flow prediction [34], modeling of helicopter rotor aerodynamic noise [35], or image processing [36]. The Wold decomposition theorem has been formulated for the general case of isometric operators on Hilbert spaces (cf., e.g., [17,37,38]):

Theorem 1. *Let V be an isometric operator on the Hilbert space $\mathscr{H}$. Then there exists an orthogonal decomposition of the form*

$$\mathscr{H} = \mathscr{H}_u^V \oplus \mathscr{H}_s^V,$$

uniquely determined by the conditions:

(a) $\mathscr{H}_u^V$ *reduces V to a unitary operator V_u;*

(b) $\mathscr{H}_s^V$ *reduces V to a shift V_s.*

More precisely, $\mathscr{H}_u^V = \bigcap_{n\geq 0} V^n \mathscr{H}$ *and* $\mathscr{H}_u^s = \bigoplus_{n\geq 0} V^n \ker V^*$.

According to the classical spectral theorem, to every normal operator N on a complex Hilbert space $\mathscr{H}$, it corresponds a unique compact, complex spectral measure E such that $N = \int \lambda dE(\lambda)$ (cf., e.g., [18], §44, Theorem 1). It follows immediately that $U^*NU =$

$\int \lambda d(U^* E(\lambda) U)$ for any given unitary operator U from a complex Hilbert space $\mathscr{K}$ onto $\mathscr{H}$. In other words, two normal operators N_1 and N_2 are unitarily equivalent if and only if their spectral measures E_1 and E_2 are unitarily equivalent:

$$U^* E_1(\mathcal{M}) U = E_2(\mathcal{M})$$

for a certain unitary operator U and for all complex Borel sets $\mathcal{M}$. According to the theory developed for nonseparable Hilbert spaces by Paul R. Halmos [18] (Chapter III) and Arlen Brown [39], the equivalence of two spectral measures is characterized by the equality of their multiplicity functions. Therefore, it holds the following description of the unitary equivalence between two normal operators.

Theorem 2. *Two normal operators are unitarily equivalent if and only if their spectral measures possess identical multiplicity functions.*

We now have all the ingredients to prove the main result of this section.

Theorem 3. *Let V be an isometric operator on a complex Hilbert space $\mathscr{H}$ with the Wold decomposition $\mathscr{H} = \mathscr{H}_u \oplus \mathscr{H}_s$. Furthermore, let U be a unitary operator on another complex Hilbert space $\mathscr{K}$. Then, there exists an isometric operator $\pi : \mathscr{H} \to \mathscr{K}$ such that (π, U) extends V if and only if:*

- *(a) There exists a subspace $\mathscr{K}_0$ of $\mathscr{K}$, which reduces U such that the spectral measures of the unitary operators V_u and $U|_{\mathscr{K}_0}$ possess identical multiplicity functions;*
- *(b) There exists a subspace $\mathscr{K}_1$ of $\mathscr{K}$, which contains $\mathscr{K}_0$, and it is invariant under U such that the shifts V_s and $(U|_{\widetilde{\mathscr{K}_1}})_s$ possess identical multiplicities (here, $\widetilde{\mathscr{K}_1} := \bigvee_{n\geq 0}(\mathscr{K}_1 \ominus U^n \mathscr{K}_1)$); equivalently, $\dim(\ker V^*) = \dim(\mathscr{K}_1 \ominus U\mathscr{K}_1)$.*

The operator π has the form:

$$\pi h := U_1 h_u + U_2 h_s, \quad h = h_u + h_s, h_u \in \mathscr{H}_u^V, h_s \in \mathscr{H}_s^V. \tag{8}$$

Here $U_1 : \mathscr{H}_u^V \to \mathscr{K}_0$ is a unitary operator that satisfies $U_1^* U|_{\mathscr{K}_0} U_1 = V|_{\mathscr{H}_u^V}$, while U_2 is the unitary operator defined for every $h_s = \sum_{n\geq 0} V^n h_n \in \mathscr{H}_s^V$ ($\{h_n\}_{n\geq 0} \subseteq \ker V^*$), by

$$U_2 h_s := \sum_{n\geq 0} U^n \varphi h_n \in \widetilde{\mathscr{K}_1}, \tag{9}$$

where φ is a unitary operator between $\ker V^*$ and $\widetilde{\mathscr{K}_1} \ominus U\widetilde{\mathscr{K}_1} = \mathscr{K}_1 \ominus U\mathscr{K}_1$.

Proof. We firstly show that, for a subspace $\mathscr{K}_1$ of $\mathscr{K}$ that is invariant under U, $\mathscr{K}_1 \ominus U\mathscr{K}_1$ is wandering for U and

$$\widetilde{\mathscr{K}_1} = \bigoplus_{n\geq 0} U^n(\mathscr{K}_1 \ominus U\mathscr{K}_1). \tag{10}$$

Indeed, for $k, k' \in \mathscr{K}_1 \ominus U\mathscr{K}_1$ and $n \geq 1$, $U^{n-1}k \in \mathscr{K}_1$ and $U^* k' \perp \mathscr{K}_1$, that is

$$\langle U^n k, k' \rangle = \langle U^{n-1}k, U^* k' \rangle = 0.$$

Furthermore, for any $n \geq 0$,

$$\mathscr{K}_1 \ominus U^n \mathscr{K}_1 = \bigoplus_{k=0}^{n} U^k(\mathscr{K}_1 \ominus U\mathscr{K}_1).$$

The equality $\widetilde{\mathscr{K}_1} \ominus U\widetilde{\mathscr{K}_1} = \mathscr{K}_1 \ominus U\mathscr{K}_1$ is a consequence of (10).

Let us now suppose that, for a certain isometric operator $\pi : \mathscr{H} \to \mathscr{K}$, (π, U) extends V. Then $\mathscr{K}_0 := \pi\mathscr{H}_u^V$ is a closed subspace of $\mathscr{K}$ (as π is isometric), which reduces U:

$$U\mathscr{K}_0 = U\pi\mathscr{H}_u^V = \pi V\mathscr{H}_u^V = \pi\mathscr{H}_u^V = \mathscr{K}_0.$$

The unitary operator $\pi_u := \pi|_{\mathscr{H}_u^V} : \mathscr{H}_u^V \to \mathscr{K}_0$ intertwines $U|_{\mathscr{K}_0}$ and $V|_{\mathscr{H}_u^V}$:

$$U|_{\mathscr{K}_0}\pi_u h_u = U\pi h_u = \pi V h_u = \pi_u V|_{\mathscr{H}_u^V} h_u, \quad h_u \in \mathscr{H}_u^V.$$

In view of Theorem 2, the spectral measures of $U|_{\mathscr{K}_0}$ and $V|_{\mathscr{H}_u^V}$ possess identical multiplicity functions. Finally, the subspace $\mathscr{K}_1 := \pi\mathscr{H}$ is invariant under U (indeed, $U\mathscr{K}_1 = \pi \operatorname{ran} V$), it contains $\mathscr{K}_0$ and

$$\mathscr{K}_1 \ominus U\mathscr{K}_1 = \pi(\mathscr{H} \ominus \operatorname{ran} V) = \pi \ker V^*.$$

Hence, $\mathscr{K}_1 \ominus U\mathscr{K}_1$ and $\ker V^*$ have identical Hilbert dimensions.

Conversely, let $\mathscr{K}_0$ be a reducing subspace for U, $U_1 : \mathscr{H}_u^V \to \mathscr{K}_0$, a unitary operator that intertwines V_u and $U|_{\mathscr{K}_0}$, $\mathscr{K}_1$, a subspace of $\mathscr{K}$ that is invariant under U, contains $\mathscr{K}_0$ and such that there exists a unitary operator $\varphi : \ker V^* \to \mathscr{K}_1 \ominus U\mathscr{K}_1$ (according to (a) and (b)). Let U_2 be the operator defined as in (9). Then, for every $hs = \sum_{n\geq 0} V^n h_n \in \mathscr{H}_s^V$ ($\{h_n\}_{n\geq 0} \subseteq \ker V^*$),

$$\|U_2 h_s\|^2 = \sum_{n\geq 0} \|U^n \varphi h_n\|^2 = \|h_s\|^2.$$

This shows that U_2 is an isometric operator. It is also surjective since for every $k = \sum_{n\geq 0} U^n k_n \in \widetilde{\mathscr{K}_1}$ ($k_n \in \mathscr{K}_1 \ominus U\mathscr{K}_1, n \geq 0$), $h_s = \sum_{n\geq 0} V^n \varphi^* k_n \in \mathscr{H}_s^V$ and $U_2 h_s = k$. In addition,

$$UU_2(V^n h) = U(U^n \varphi h) = U^{n+1}\varphi h = U_2(V^{n+1}h) = U_2 V(V^n h),\ n \geq 0, h \in \ker V^*.$$

We deduce that $U|_{\widetilde{\mathscr{K}_1}} U_2 = U_2 V_s$.

The operator π, defined by (8), is isometric:

$$\|\pi h\|^2 = \|U_1 h_u\|^2 + \|U_2 h_s\|^2 = \|h\|^2, \quad h = h_u + h_s, h_u \in \mathscr{H}_u^V, h_s \in \mathscr{H}_s^V,$$

since $U_1\mathscr{H}_u^V = \mathscr{K}_0$ and $U_2\mathscr{H}_s^V = \widetilde{\mathscr{K}_1}$ are orthogonal subspaces:

$$\langle k_0, U^n k_1\rangle = \langle UU^{*n+1}k_0, k_1\rangle = 0, \quad k_0 \in \mathscr{K}_0, k_1 \in \mathscr{K}_1 \ominus U\mathscr{K}_1,$$

due to the observation that $UU^{*n+1}k_0 \in UU^{*n+1}\mathscr{K}_0 = U\mathscr{K}_0 \subseteq U\mathscr{K}_1$ and $k_1 \perp U\mathscr{K}_1$.

Finally,

$$\begin{aligned}\pi V h &= \pi(Vh_u + Vh_s) = U_1 V_u h_u + U_2 V_s h_s \\ &= UU_1 h_u + UU_2 h_s = U\pi h, \quad h = h_u + h_s, h_u \in \mathscr{H}_u^V, h_s \in \mathscr{H}_s^V.\end{aligned}$$

This proves that (π, U) extends V. $\square$

As observed in the first section, (π, U^*) dilates V^*. We can thus consider the smallest subspace of $\mathscr{K}$, which is invariant under U^* and contains $\pi\mathscr{H}$:

$$\mathscr{K}_0 := \bigvee_{n\geq 0} U^{*n}\pi\mathscr{H}.$$

We observe that

$$U\mathscr{K}_0 = U\pi\mathscr{H} \vee \mathscr{K}_0 = \pi V\mathscr{H} \vee \mathscr{K}_0 = \mathscr{K}_0,$$

that is, $\mathscr{K}_0$ reduces U. We next show that:

$$U^*\pi\mathscr{H} = \pi\mathscr{H} \oplus U^*\pi \ker V^*. \tag{11}$$

The sum in the right side of this equation is orthogonal:

$$\langle \pi h, U^*\pi h' \rangle = \langle U\pi h, \pi h' \rangle = \langle \pi V h, \pi h' \rangle = \langle h, V^* h' \rangle = 0, \quad h \in \mathscr{H}, h' \in \ker V^*.$$

If we apply U to equation (11), we obtain the relation:

$$\pi\mathscr{H} = U\pi\mathscr{H} \oplus \pi \ker V^* = \pi(V\mathscr{H} \oplus \ker V^*),$$

which is obviously true. We proceed inductively to prove that:

$$\bigvee_{k=1}^{n} U^{*k}\pi\mathscr{H} = \pi\mathscr{H} \oplus \bigoplus_{k=1}^{n} U^{*k}\pi \ker V^*, \quad n \geq 1.$$

Consequently,

$$\mathscr{K}_0 = \pi\mathscr{H} \oplus \bigoplus_{n\geq 1} U^{*n}\pi \ker V^*.$$

Let $Z : \mathscr{H} \oplus H^2_{\ker V^*}(\mathbb{D}) \to \mathscr{K}$ be the isometric operator defined by:

$$Z(h, f) := \pi h + \sum_{n\geq 0} U^{*n+1}\pi h_n,$$

where $h \in \mathscr{H}$ and $f \in H^2_{\ker V^*}(\mathbb{D})$,

$$\mathbb{D} \ni \lambda \mapsto f(\lambda) := \sum_{n\geq 0} \lambda^n h_n \in \ker V^*.$$

Then

$$\begin{aligned}
Z^*UZ(h, f) &= Z^*U(\pi h + \sum_{n\geq 0} U^{*n+1}\pi h_n) \\
&= Z^*(\pi V h + \sum_{n\geq 0} U^{*n}\pi h_n) \\
&= Z^*(\pi(Vh + h_0) + \sum_{n\geq 0} U^{*n+1}\pi h_{n+1}) \\
&= (Vh + f(0), M_z^* f).
\end{aligned}$$

We deduce, by Proposition 1, that $(Z^*\pi, Z^*UZ)$ is a unitary extension of V, while (Z, U) is a unitary extension of Z^*UZ.

We proved the following theorem.

Theorem 4. *Let V be a linear isometry on the complex Hilbert space $\mathscr{H}$. Then $(i_{\mathscr{H}}, W)$ is a unitary extension of V, where $i_{\mathscr{H}}$ is the inclusion map of $\mathscr{H}$ into $\mathscr{H} \oplus H^2_{\ker V^*}(\mathbb{D})$, while W, which acts on $\mathscr{H} \oplus H^2_{\ker V^*}(\mathbb{D})$, has the matrix representation:*

$$W = \begin{pmatrix} V & E_0 \\ 0 & M_z^* \end{pmatrix},$$

E_0 being the map of evaluation in 0:

$$H^2_{\ker V^*}(\mathbb{D}) \ni f \mapsto f(0) \in \ker V^*.$$

In addition, $(i_{\mathscr{H}}, W^*)$ is a minimal unitary dilation of V^*.

4. Restrictions of Backward Shifts

Our next aim is to study the case of extensions (π, B) of a given operator T on a complex Hilbert space $\mathscr{H}$, where B is a backward shift. In this situation or, more generally, when B is just contractive, T should necessarily be a contraction:

$$\|Th\| = \|\pi Th\| = \|B\pi h\| \leq \|\pi h\| = \|h\|, \quad h \in \mathscr{H}.$$

The following norm formula, which relates the contraction T with its defect operator D_T, will be frequently used in our discussion:

$$\|D_T h\|^2 = \|h\|^2 - \|Th\|^2, \quad h \in \mathscr{H}. \tag{12}$$

Let us also note that, for every $h \in \mathscr{H}$, the sequence $(\|T^n h\|)_{n\geq 0}$ is decreasing and bounded below and, hence, convergent.

As mentioned in Section 1, shift operators S are unitarily equivalent with the operators M_z of multiplication by the independent variable on the Hardy space $H^2_{\ker S^*}(\mathbb{D})$. This is the reason why in our description of backward shift extensions of T, we replace B by M_z^*.

Theorem 5. *Let $\mathscr{H}, \mathscr{D}$ be complex Hilbert spaces and $T \in \mathcal{B}(\mathscr{H})$. Then there exists a linear isometry $\pi : \mathscr{H} \to H^2_{\mathscr{D}}(\mathbb{D})$ such that (π, M_z^*) extends T if and only if $\|T\| \leq 1, T \in \mathcal{C}_{0\cdot}$ and $\delta_T \leq \dim \mathscr{D}$.*

Any solution π to the equation $\pi T = M_z^* \pi$ has the form $\pi = \pi_V$, where

$$(\pi_V h)(\lambda) := V D_T (I_{\mathscr{H}} - \lambda T)^{-1} h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D} \tag{13}$$

and $V : \mathscr{D}_T \to \mathscr{D}$ is an arbitrary isometric operator.

Proof. It T is extended by (π, M_z^*), then, according to the discussion at the beginning of this section, T must be a contraction.

Any isometric operator π from $\mathscr{H}$ into $H^2_{\mathscr{D}}(\mathbb{D})$ has the general form:

$$(\pi h)(\lambda) = \sum_{n\geq 0} \lambda^n T_n h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

where $T_n \in \mathcal{B}(\mathscr{H}, \mathscr{D}), n \geq 0$ and

$$\sum_{n\geq 0} \|T_n h\|^2 = \|h\|^2, \quad h \in \mathscr{H}. \tag{14}$$

The condition $\pi(Th)(\lambda) = M_z^*(\pi h)(\lambda), h \in \mathscr{H}, \lambda \in \mathbb{D}$ can be written in equivalent form as:

$$T_n T = T_{n+1}, \quad n \geq 0.$$

If we fix T_0, then any contraction operator $T_n, n \geq 0$ can be represented in terms of T_0 and T: $T_n = T_0 T^n$. Thus, Formula (14) becomes:

$$\sum_{n\geq 0} \|T_0 T^n h\|^2 = \|h\|^2, \quad h \in \mathscr{H}. \tag{15}$$

We replace h by Th in (15) and obtain that

$$\|Th\|^2 + \|T_0 h\|^2 = \|h\|^2, \quad h \in \mathscr{H},$$

that is

$$\|T_0 h\| = \|D_T h\|, \quad h \in \mathscr{H} \quad (\text{by } (12)). \tag{16}$$

Consequently, we can eliminate T_0 from (15):

$$\sum_{n\geq 0} \|D_T T^n h\|^2 = \|h\|^2, \quad h \in \mathscr{H}.$$

Another application of (12), i.e., $\|D_T T^n h\|^2 = \|T^n h\|^2 - \|T^{n+1}h\|^2, n \geq 0$, shows that:

$$\|h\|^2 = \lim_{n\to\infty} \sum_{k=0}^{n} (\|T^k h\|^2 - \|T^{k+1}h\|^2), \quad h \in \mathscr{H}.$$

Hence, T is of class $\mathcal{C}_{0\cdot}$, and the only condition imposed on T_0 should be (16). In other words, there exists an isometric operator $V : \mathscr{D}_T \to \mathscr{D}$ such that $T_0 h = V D_T h, h \in \mathscr{H}$. Thus $\delta_T = \dim \mathscr{D}_T \leq \dim \mathscr{D}$ and π has the form given by (13).

Conversely, if T is a contraction of class $\mathcal{C}_{0\cdot}$ and $\delta_T \leq \dim \mathscr{D}$, there exists an isometric operator $V : \mathscr{D}_T \to \mathscr{D}$. The operator π_V, introduced by (13), is well-defined and isometric since

$$\sum_{n\geq 0} \|V D_T T^n h\|^2 = \|h\|^2 - \lim_{n\to\infty} \|T^{n+1}h\|^2 = \|h\|^2.$$

In addition,

$$(\pi_V(Th))(\lambda) = \sum_{n\geq 0} \lambda^n V D_T T^{n+1} h = M_z^*(\pi_V h)(\lambda), \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

which shows that (π_V, M_z^*) extends T. $\square$

We compute the elements involved in this theorem for rank one contractions.

Example 1. *Let $\mathscr{H}, \mathscr{D}$ be infinite dimensional complex Hilbert spaces and $x, y \in \mathscr{H}$ be two linearly independent vectors such that $\|x\|\|y\| \leq 1$. If $T = x \otimes y$ is the tensor product:*

$$\mathscr{H} \ni h \mapsto \langle h, y\rangle x \in \mathscr{H},$$

then T is a contraction of class $\mathcal{C}_{0\cdot}$. In fact, $T^n = \langle x, y\rangle^{n-1} x \otimes y$ $(n \geq 1)$ tends uniformly to 0 as n tends to ∞:

$$\|T^n\| = |\langle x, y\rangle|^{n-1} \|x\|\|y\| \to 0 \text{ as } n \to \infty,$$

since, by the Cauchy–Schwarz inequality, $|\langle x, y\rangle| < \|x\|\|y\| \leq 1$.

Our next aim is to compute the square root D_T of $I_{\mathscr{H}} - T^*T = I_{\mathscr{H}} - \|x\|^2 y \otimes y$. D_T must be in the C^*-subalgebra of $\mathcal{B}(\mathscr{H})$ generated by $I_{\mathscr{H}}$ and the orthogonal projection $\frac{1}{\|y\|^2} y \otimes y$. We write it as a linear combination of $I_{\mathscr{H}}$ and $y \otimes y$ and identify the corresponding coefficients in order to obtain:

$$D_T = I_{\mathscr{H}} - \frac{1 - \sqrt{1 - \|x\|^2\|y\|^2}}{\|y\|^2} y \otimes y.$$

If T is a strict contraction (i.e., $\|T\| < 1$), we observe that 0 does not belong to the spectrum $\sigma(D_T)$ of D_T. Otherwise, by the spectral mapping theorem,

$$\frac{1}{\|x\|^2\|y\|^2} \in \sigma\Big(\frac{1}{\|y\|^2} y \otimes y\Big) = \{0, 1\},$$

which is a contradiction. It follows that D_T is invertible, so $\mathscr{D}_T = \mathscr{H}$ and $\delta_T = \dim \mathscr{H}$. On the other hand, if $\|x\|\|y\| = 1$, then D_T is an orthogonal projection $\mathscr{D}_T = \{y\}^\perp$ and $\delta_T = \dim \mathscr{H}$.

We also observe that, for every $\lambda \in \mathbb{D}$,

$$\begin{aligned} D_T(I_{\mathscr{H}} - \lambda T)^{-1} &= \sum_{n\geq 0} \lambda^n D_T T^n \\ &= D_T + \sum_{n\geq 1} \lambda^n \langle x, y\rangle^{n-1} \Big(I_{\mathscr{H}} - \frac{1-\sqrt{1-\|x\|^2\|y\|^2}}{\|y\|^2} y \otimes y \Big) x \otimes y \\ &= D_T + \frac{\lambda}{1-\lambda\langle x,y\rangle} x\otimes y - \frac{1-\sqrt{1-\|x\|^2\|y\|^2}}{\|y\|^2} \frac{\lambda\langle x,y\rangle}{1-\lambda\langle x,y\rangle} y \otimes y \\ &= I_{\mathscr{H}} + \frac{\lambda}{1-\lambda\langle x,y\rangle} x\otimes y - \frac{1-\sqrt{1-\|x\|^2\|y\|^2}}{\|y\|^2(1-\lambda\langle x,y\rangle)} y\otimes y. \end{aligned}$$

We express our findings in terms of the previous theorem.

"Let $\mathscr{H}, \mathscr{D}$ be infinite dimensional complex Hilbert spaces and $x, y \in \mathscr{H}$ be two linearly independent vectors. Then there exists a linear isometry $\pi : \mathscr{H} \to H^2_{\mathscr{D}}(\mathbb{D})$ such that $M_z^*\pi = \pi x \otimes y$ (i.e., $x \otimes y$ has a backward shift extension) if and only if $\|x\|\|y\| \leq 1$ and $\dim \mathscr{H} \leq \dim \mathscr{D}$. Any solution π of the equation $M_z^*\pi = \pi x \otimes y$ has the form $\pi = \pi_V$, where

$$(\pi_V h)(\lambda) := Vh + \frac{\lambda\langle h,y\rangle}{1-\lambda\langle x,y\rangle} Vx - \frac{(1-\sqrt{1-\|x\|^2\|y\|^2})\langle h,y\rangle}{\|y\|^2(1-\lambda\langle x,y\rangle)} Vy, \quad h \in \mathscr{H}, \lambda \in \mathbb{D}$$

and $V : \mathscr{D}_T \to \mathscr{D}$ is an arbitrary isometric operator. Here, $\mathscr{D}_T = \mathscr{H}$ or $\mathscr{D}_T = \{y\}^\perp$ according to whether T, is a strict contraction or not."

We describe the conditions under which the dilation (π_V, M_z) of T^* is minimal:

$$H^2_{\mathscr{D}}(\mathbb{D}) = \bigvee_{n\geq 0} M_z^n \pi_V \mathscr{H}.$$

Obviously, $\mathscr{F} := \bigvee_{n\geq 0} M_z^n \pi_V \mathscr{H} \subseteq H^2_{V\mathscr{D}_T}(\mathbb{D})$. We prove that the reverse inclusion is also true. Let $h \in \mathscr{H}$ and $\lambda \in \mathbb{D}$. Easy computations, based on the definition of π_V, show that:

$$VD_T h = (M_z \pi_V(Th) - \pi_V h)(\lambda).$$

One can successively apply the multiplication operator to prove that:

$$\lambda^m VD_T h = (M_z^{m+1}\pi_V(Th) - M_z^m \pi_V h)(\lambda) \text{ for every } m \geq 0.$$

We deduce that $H^2_{V\mathscr{D}_T}(\mathbb{D}) \subseteq \mathscr{F}$. Consequently, the dilation (π_V, M_z) of T^* is minimal if and only if $V\mathscr{D}_T = \mathscr{D}$.

We are now able to reformulate Theorem 5 in minimality conditions.

Theorem 6. *Let $\mathscr{H}, \mathscr{D}$ be complex Hilbert spaces and $T \in \mathcal{B}(\mathscr{H})$. Then there exists a linear isometry $\pi : \mathscr{H} \to H^2_{\mathscr{D}}(\mathbb{D})$ such that (π, M_z) is a minimal isometric dilation of T if and only if $\|T\| \leq 1, T \in \mathcal{C}_{\cdot 0}$ and $\delta_{T^*} = \dim \mathscr{D}$.*

Any solution π to this problem has the form $\pi = \pi_V$, where

$$(\pi_V h)(\lambda) := VD_{T^*}(I_{\mathscr{H}} - \lambda T^*)^{-1} h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D}$$

and $V : \mathscr{D}_{T^*} \to \mathscr{D}$ is an arbitrary unitary operator.

5. The General Case

At the beginning, we present two examples. The first one shows that the converse of the result presented in Remark 2 is not always valid.

Example 2. *Let $\mathscr{H}_0, \mathscr{H}_1$ be non-null complex Hilbert spaces, S a shift on $\mathscr{H}_0$ and $T = S^* \oplus 0$ acting on $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$.*

The operator V defined on $H^2_{\mathscr{H}}(\mathbb{D})$ by:

$$(Vf)(\lambda) := T^* f(0) + \lambda(f(\lambda) - P_{\mathscr{H}_0} f(0)), \quad \lambda \in \mathbb{D}, f \in H^2_{\mathscr{H}}(\mathbb{D})$$

is isometric:

$$\|Vf\|^2 = \|(S \oplus 0)P_{\mathscr{H}_0} f(0)\|^2 + \|P_{\mathscr{H}_1} f(0)\|^2 + \|M_z^* f\|^2 = \|f\|^2, \quad f \in H^2_{\mathscr{H}}(\mathbb{D})$$

and its adjoint can be computed according to the formula

$$(V^* f)(\lambda) = Tf(0) + P_{\mathscr{H}_1}(M_z^* f)(0) + \lambda(M_z^{*2} f)(\lambda), \quad \lambda \in \mathbb{D}, f \in H^2_{\mathscr{H}}(\mathbb{D}).$$

If π is the embedding of $\mathscr{H}$ into $H^2_{\mathscr{H}}(\mathbb{D})$:

$$(\pi h)(\lambda) := h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

then (π, V^*) extends T:

$$V^*(\pi h)(\lambda) = Th = \pi(Th)(\lambda), \quad h \in \mathscr{H}, \lambda \in \mathbb{D}.$$

In order to prove that the isometric dilation (π, V) of T^* is not minimal, we compute the smallest subspace, which is invariant under V and contains $\pi\mathscr{H}$:

$$\mathscr{K} := \bigvee_{n \geq 0} V^n \pi \mathscr{H}.$$

With this aim, we observe that, for every $h = (h_0, h_1) \in \mathscr{H}$ and $\lambda \in \mathbb{D}$, $V(\pi h)(\lambda) = (Sh_0, \lambda h_1)$ and, inductively,

$$V^n(\pi h)(\lambda) = (S^n h_0, \lambda^n h_1), \quad n \geq 0.$$

Obviously, for a given $h = (h_0, 0) \in \mathscr{H}$ with $h_0 \neq 0$, $M_z \pi h \perp V^n(\pi h')$ for every $h' \in \mathscr{H}$. This observation proves that $\mathscr{K}$ is strictly contained in $H^2_{\mathscr{H}}(\mathbb{D})$, so the dilation (π, V) is not minimal.

The second example shows that the minimality condition is necessary in order to obtain the conclusion of Remark 2.

Example 3. *We use the objects (spaces and operators) from the previous example. Let us firstly observe that $\mathscr{K}$ reduces V since, on the one hand, $V\mathscr{K} \subseteq \mathscr{K}$ and, on the other hand, for $h \in \mathscr{H}$ and $n \geq 0$,*

$$V^*(V^n(\pi h)) = \begin{cases} V^{n-1}(\pi h) & \text{if } n \geq 1 \\ V^*(\pi \mathscr{H}) = \pi(Th) & \text{if } n = 0 \end{cases}.$$

Let h_1 be a fixed element in $\mathscr{H}_1$ of unit norm and $h = (0, h_1)$. Then $\pi h \in \mathscr{K} \ominus V\mathscr{K} = \ker(V^*|_{\mathscr{K}})$. Consider also a fixed function $g \in \mathscr{K}^\perp := H^2_{\mathscr{H}}(\mathbb{D}) \ominus \mathscr{K}$ of norm 1. Let $P = P^{H^2_{\mathscr{H}}(\mathbb{D})}_{\mathscr{K}}$ and $P^\perp = I_{\mathscr{K}} - P$. The operator W, defined on $H^2_{\mathscr{H}}(\mathbb{D})$ by:

$$(Wf)(\lambda) := V(Pf)(\lambda) + (P^\perp f)(\lambda) + \langle P^\perp f, g\rangle(h - g(\lambda)), \quad f \in H^2_{\mathscr{H}}(\mathbb{D}), \lambda \in \mathbb{D}, \qquad (17)$$

is isometric. Indeed, for any $f \in H^2_{\mathscr{H}}(\mathbb{D})$, $V(Pf) \in V\mathscr{K} \subseteq \mathscr{K}$*, while* g *and* $P^{\perp} f$ *are elements of* $\mathscr{K}^{\perp}$*. Then*

$$\begin{aligned} \|Wf\|^2 &= \|V(Pf) + \langle P^{\perp} f, g\rangle \pi h\|^2 + \|P^{\perp} f - \langle P^{\perp} f, g\rangle g\|^2 \\ &= \|V(Pf)\|^2 + |\langle P^{\perp} f, g\rangle|^2 + 2\mathbb{R}e(\overline{\langle P^{\perp} f, g\rangle}\langle V(Pf), \pi h\rangle) \\ &\quad + \|P^{\perp} f\|^2 + |\langle P^{\perp} f, g\rangle|^2 - 2|\langle P^{\perp} f, g\rangle|^2 \\ &= \|V(Pf)\|^2 + \|P^{\perp} f\|^2 \\ &= \|Pf\|^2 + \|P^{\perp} f\|^2 = \|f\|^2. \end{aligned}$$

In this series of equalities, we used the fact that $\langle V(Pf), \pi h\rangle = 0$ *since* $V(Pf) \in V\mathscr{K}$ *and* $\pi h \in \mathscr{K} \ominus V\mathscr{K}$*. The adjoint of* W *can be computed according to the formula:*

$$W^* f = PV^* f + P^{\perp} f + \langle f, \pi h - g\rangle g, \quad f \in H^2_{\mathscr{H}}(\mathbb{D}).$$

Then, for any $h' \in \mathscr{H}$*, it holds:*

$$W^* \pi h' = PV^* \pi h' + \langle \pi h', \pi h\rangle g = \pi T h' + \langle h', h\rangle g.$$

We deduce that (π, W^*) *cannot be an extension of* T *since, otherwise,* $h_1 = 0$*, which contradicts its choice. We also observe that, by* (17), $W(\pi h') = V(\pi h')$, $W^2(\pi h') = WV(\pi h') = V^2(\pi h')$ *and, inductively,* $W^n(\pi h') = V^n(\pi h')$ *for every* $n \geq 0$ *and* $h' \in \mathscr{H}$*. Consequently,*

$$\pi^* W^n \pi h' = \pi^* V^n \pi h' = T^{*n} h', \quad h' \in \mathscr{H}, n \geq 0.$$

Hence, (π, W) *is a dilation of* T^**.*

We pass now to the study of the structure of coisometric extensions (π, V^*) of contraction operators T on a complex Hilbert space $\mathscr{H}$.

In the description of (π, V), the asymptotic limit associated with T plays an important role. Some of its main properties are collected in the following proposition.

Proposition 3 ([40], Chapter 3)**.** *Let* A_T *be the asymptotic limit associated with a contraction* $T \in \mathcal{B}(\mathscr{H})$*. Then:*

(a) $0 \leq A_T \leq I_{\mathscr{H}}$;
(b) $T^* A_T T = A_T$;
(c) $\ker A_T = \{h \in \mathscr{H} \mid T^n h \to 0 \text{ as } n \to \infty\}$;
(d) $\ker(I - A_T) = \{h \in \mathscr{H} \mid \|T^n h\| = \|h\| \text{ for all } n \geq 0\}$.

In view of condition (b), we can show that, for all $h \in \mathscr{H}$, the following equalities hold true:

$$\|A_T^{1/2} Th\|^2 = \langle T^* A_T Th, h\rangle = \langle A_T h, h\rangle = \|A_T^{1/2} h\|^2.$$

This formula allows us to define the isometric operator

$$\mathscr{A}_T \ni A_T^{1/2} h \mapsto V_T(A_T^{1/2} h) := A_T^{1/2} Th \in \mathscr{A}_T,$$

which will be called the *asymptotic isometry* associated with T.

According to the Wold–Halmos decomposition theorem (Theorem 1), the isometric operator V can be represented as the direct sum between a unitary operator and a shift. This is the reason why we study coisometric extensions (π, V^*), where V is the direct sum between a unitary operator U acting on a complex Hilbert space $\mathscr{E}$ and the operator M_z of multiplication by the independent variable on the Hardy space $H^2_{\mathscr{D}}(\mathbb{D})$.

Theorem 7. *Let* $\mathscr{H}, \mathscr{D}, \mathscr{E}$ *be complex Hilbert spaces,* $T \in \mathcal{B}(\mathscr{H})$ *and* $U \in \mathcal{B}(\mathscr{E})$ *a unitary operator. Then there exists a linear isometry* $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ *such that* $(\pi, (U \oplus M_z)^*)$

extends T if and only if $\|T\| \leq 1$, $\delta_T \leq \dim \mathscr{D}$ and there exists an isometric operator $Y : \mathscr{A}_T \to \mathscr{E}$ such that (Y, U^*) extends V_T.

Any solution π of the equation $\pi T = (U \oplus M_z)^* \pi$ has the form $\pi = \pi_{X,Y}$, where

$$\pi_{X,Y} h = (Y A_T^{1/2} h, W_X h), \quad h \in \mathscr{H}, \tag{18}$$

$Y \in \mathcal{B}(\mathscr{A}_T, \mathscr{E})$ being any isometry with $Y V_T = U^* Y$, while

$$(W_X h)(\lambda) = X D_T (I_{\mathscr{H}} - \lambda T)^{-1} h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D}, \tag{19}$$

$X \in \mathcal{B}(\mathscr{D}_T, \mathscr{D})$ being an arbitrary isometric operator.

Proof. Let $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ be an isometric operator such that:

$$\pi T = (U^* \oplus M_z^*) \pi. \tag{20}$$

If π has the matrix representation $\pi = \begin{pmatrix} Z \\ W \end{pmatrix}$, where $Z \in \mathcal{B}(\mathscr{H}, \mathscr{E})$ and $W \in \mathcal{B}(\mathscr{H}, H^2_{\mathscr{D}}(\mathbb{D}))$, then equation (20) can be translated by:

$$ZT = U^* Z \text{ and } WT = M_z^* W. \tag{21}$$

Furthermore,

$$\|Zh\|^2 + \|Wh\|^2 = \|h\|^2, \quad h \in \mathscr{H}, \tag{22}$$

due to the fact that π is isometric.

We proceed as in the proof of Theorem 5. Since W has the form:

$$(Wh)(\lambda) = \sum_{n \geq 0} \lambda^n T_n h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

where $\{T_n\}_{n \geq 0} \subset \mathcal{B}(\mathscr{H}, \mathscr{D})$, we deduce immediately that $T_n = T_0 T^n, n \geq 0$. Equation (22) becomes:

$$\|Zh\|^2 + \sum_{n \geq 0} \|T_0 T^n h\|^2 = \|h\|^2, \quad h \in \mathscr{H}. \tag{23}$$

We replace h with Th in (23) and obtain that:

$$\|ZTh\|^2 + \|h\|^2 = \|Th\|^2 + \|Zh\|^2 + \|T_0 h\|^2, \quad h \in \mathscr{H}.$$

We finally arrive at the equality

$$\|T_0 h\| = \|D_T h\|, \quad h \in \mathscr{H},$$

by (12) and the first condition of (21). Equivalently, there exists an isometric operator $X : \mathscr{D}_T \to \mathscr{D}$ such that $T_0 h = X D_T h, h \in \mathscr{H}$. This also implies that $\delta_T \leq \dim \mathscr{D}$. We replace our findings in (23) and use (12) again to obtain that, for every $h \in \mathscr{H}$,

$$\begin{aligned} \|Zh\|^2 &= \|h\|^2 - \sum_{n \geq 0} \|D_T T^n h\|^2 \\ &= \|h\|^2 - \lim_{n \to \infty} \sum_{k=0}^{n} (\|T^k h\|^2 - \|T^{k+1} h\|^2) \\ &= \lim_{n \to \infty} \|T^n h\|. \end{aligned}$$

Since $(T^{*n} T^n)_{n \geq 0}$ tends strongly to A_T as $n \to \infty$, it also tends weakly to the same limit. Thus,

$$\|Zh\|^2 = \lim_{n \to \infty} \|T^n h\|^2 = \lim_{n \to \infty} \langle T^{*n} T^n h, h \rangle = \langle A_T h, h \rangle = \|A_T^{1/2} h\|^2, \quad h \in \mathscr{H}.$$

Let $Y : \mathscr{A}_T \to \mathscr{E}$ be an isometric operator such that $YA_T^{1/2}h = Zh, h \in \mathscr{H}$. The equation $ZT = U^*Z$ can be rewritten as:

$$YA_T^{1/2}Th = U^*YA_T^{1/2}h, \quad h \in \mathscr{H},$$

or, equivalently, as:

$$YV_T = U^*Y.$$

This means that (Y, U^*) is a unitary extension of the isometric operator V_T. As a conclusion, π has the form $\pi_{X,Y}$ given by (18).

Conversely, if T is a contraction with its defect index $\delta_T \leq \dim \mathscr{D}$, there exists an isometric operator $X \in \mathcal{B}(\mathscr{D}_T, \mathscr{D})$. Let $Y \in \mathcal{B}(\mathscr{A}_T, \mathscr{E})$ be the isometric operator that intertwines V_T and U^*. The operator $\pi_{X,Y}$, introduced by (18) and (19), is well-defined and isometric since

$$\|YA_T^{1/2}h\|^2 + \sum_{n\geq 0} \|XD_T T^n h\|^2 = \|A_T^{1/2}h\|^2 + \|h\|^2 - \lim_{n\to\infty} \|T^n h\|^2 = \|h\|^2, \quad h \in \mathscr{H}.$$

In addition,

$$\begin{aligned}(U \oplus M_z)^* \pi_{X,Y} h &= (U^*YA_T^{1/2}h, M_z^* W_X h) \\ &= (YA_T^{1/2}Th, W_X(Th)) = \pi_{X,Y}(Th), \quad h \in \mathscr{H},\end{aligned}$$

which shows that $(\pi_{X,Y}, (U \oplus M_z)^*)$ extends T. $\square$

Sometimes, we can numerically express this extendability problem, as we can see in the following example.

Example 4. *Let $\mathscr{H}, \mathscr{D}, \mathscr{E}$ be complex Hilbert spaces, $\mathscr{H}_0$ a non-null subspace of $\mathscr{H}$, $U \in \mathcal{B}(\mathscr{E})$, a unitary operator and $\mu \in \mathbb{T}$ ($\mathbb{T}$ denotes the unit circle). If $P = P_{\mathscr{H}_0}$ denotes the orthogonal projection of $\mathscr{H}$ onto $\mathscr{H}_0$, then $T = \mu P$ has unit norm. Obviously, $D_T = I_{\mathscr{H}} - P$ and $T^{*n}T^n = P$, $n \geq 1$. Consequently, $\mathscr{D}_T = \mathscr{H}_0^{\perp}$, $\delta_T = \dim \mathscr{H}_0^{\perp}$, $A_T = A_T^{1/2} = P$, $\mathscr{A}_T = \mathscr{H}_0$ and $a_T = \dim \mathscr{H}_0$. We deduce immediately that $V_T = \mu I_{\mathscr{H}_0}$. Therefore, (Y, U^*) extends V_T for a certain given isometric operator $Y : \mathscr{H}_0 \to \mathscr{E}$ if and only if $\operatorname{ran} Y$ reduces U and $U|_{\operatorname{ran} Y} = \bar{\lambda} I_{\operatorname{ran} Y}$ (equivalently, $\operatorname{ran} Y \subseteq \ker(\bar{\mu} I_{\mathscr{E}} - U)$). In other words, there exists such an isometric operator Y if and only if $\dim \mathscr{H}_0 \leq \operatorname{nullity}(\bar{\mu} I_{\mathscr{E}} - U)$ (here, the nullity of an operator is the Hilbert dimension of its kernel). Moreover, for every $\lambda \in \mathbb{D}$,*

$$D_T(I_{\mathscr{H}} - \lambda T)^{-1} = \sum_{n\geq 0} \lambda^n (I_{\mathscr{H}} - P)P^n = I_{\mathscr{H}} - P.$$

With these computations, Theorem 7 takes the form:

"There exists a linear isometry $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ such that $(\pi, (U \oplus M_z)^*)$ extends $\mu P_{\mathscr{H}_0}$ if and only if $\dim \mathscr{H}_0^{\perp} \leq \dim \mathscr{D}$ and $\dim \mathscr{H}_0 \leq \operatorname{nullity}(\bar{\mu} I_{\mathscr{E}} - U)$.

Any solution π of the equation $\mu\pi P_{\mathscr{H}_0} = (U \oplus M_z)^*\pi$ has the form $\pi = \pi_{X,Y}$, where

$$\pi_{X,Y} h = (YP_{\mathscr{H}_0}h, W_X h), \quad h \in \mathscr{H},$$

while

$$(W_X h)(\lambda) = Xh - XP_{\mathscr{H}_0}h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

$X \in \mathcal{B}(\mathscr{H}_0^{\perp}, \mathscr{D})$ and $Y \in \mathcal{B}(\mathscr{H}_0, \ker(\bar{\mu} I_{\mathscr{E}} - U))$ being arbitrary isometric operators."

Remark 3. *The condition on the existence of an isometric operator $Y : \mathscr{A}_T \to \mathscr{E}$ with the property that (Y, U^*) is a unitary extension of V_T can be characterized by Theorem 3 as follows:*

(a) *There exists a subspace $\mathscr{E}_0$ of $\mathscr{E}$, which reduces U such that the spectral measures of the unitary operators $V_T|_{\bigcap_{n\geq 0} \overline{A_T^{1/2}T^n\mathscr{H}}}$ and $U^*|_{\mathscr{E}_0}$ possess identical multiplicity functions;*

(b) *There exists a subspace $\mathscr{E}_1$ of $\mathscr{E}$ that contains $\mathscr{E}_0$, and it is invariant under U^* such that the shifts $(V_T)_s$ and $(U^*|_{\bigvee_{n\geq 0}(\mathscr{E}_1 \ominus U^{*n}\mathscr{E}_1)})_s$ possess identical multiplicities.*

We describe the necessary and sufficient conditions under which the dilation $(\pi_{X,Y}, U \oplus M_z)$ of T^* is minimal:

$$\mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D}) = \bigvee_{n\geq 0} (U \oplus M_z)^n \pi_{X,Y}\mathscr{H}.$$

Our aim is to compute the subspace $\mathscr{F} := \bigvee_{n\geq 0}(U \oplus M_z)^n \pi_{X,Y}\mathscr{H}$. Let $h \in \mathscr{H}$. Then

$$(U \oplus M_z)(\pi_{X,Y}(Th)) = (UYV_T A_T^{1/2}h, W_X(Th)) = (YA_T^{1/2}h, W_X(Th)),$$

so

$$(U \oplus M_z)(\pi_{X,Y}(Th)) - \pi_{X,Y}h = (0, \pi_0(XD_T h)),$$

where $\pi_0 : \mathscr{D} \to H^2_{\mathscr{D}}(\mathbb{D})$ is the map:

$$(\pi_0 d)(\lambda) = d, \quad d \in \mathscr{D}, \lambda \in \mathbb{D}.$$

One can apply, successively, the operator $U \oplus M_z$ to this equation in order to obtain that:

$$(U \oplus M_z)^{n+1}(\pi_{X,Y}(Th)) - (U \oplus M_z)^n(\pi_{X,Y}h) = (0, M_z^n \pi_0(XD_T h)), \quad h \in \mathscr{H}, n \geq 0.$$

We deduce that $\{0\} \oplus H^2_{X\mathscr{D}_T}(\mathbb{D}) \subseteq \mathscr{F}$.

We proceed by computing the powers $(U \oplus M_z)^n \pi_{X,Y}h$ for every $h \in \mathscr{H}$ and $n \geq 0$. An inductive procedure shows that:

$$(U \oplus M_z)^n \pi_{X,Y}h = (U^n Y A_T^{1/2}h, f_n),$$

where $f_n \in H^2_{X\mathscr{D}_T}(\mathbb{D})$. Since $(0, f_n) \in \mathscr{F}$ by our remarks above, it follows that

$$\left(\bigvee_{n\geq 0} U^n Y \mathscr{A}_T\right) \oplus \{0\} \subseteq \mathscr{F}.$$

Finally, by a summation of subspaces,

$$\left(\bigvee_{n\geq 0} U^n Y \mathscr{A}_T\right) \oplus H^2_{X\mathscr{D}_T}(\mathbb{D}) \subseteq \mathscr{F}.$$

The converse inclusion is obvious.

We conclude that $\mathscr{F} = \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ if and only if $X\mathscr{D}_T = \mathscr{D}$ and $\bigvee_{n\geq 0} U^n Y \mathscr{A}_T = \mathscr{E}$. Equivalently, X is a unitary operator and the dilation (Y, U) of V_T^* is minimal.

With these findings, we can now reformulate Theorem 7 in order to obtain minimal isometric dilations.

Theorem 8. *Let $\mathscr{H}, \mathscr{D}, \mathscr{E}$ be complex Hilbert spaces, $T \in \mathcal{B}(\mathscr{H})$ and $U \in \mathcal{B}(\mathscr{E})$ a unitary operator. Then there exists a linear isometry $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ such that $(\pi, U \oplus M_z)$ is a minimal isometric dilation of T if and only if $\|T\| \leq 1$, $\delta_{T^*} = \dim \mathscr{D}$ and there exists an isometric operator $Y : \mathscr{A}_{T^*} \to \mathscr{E}$ such that (Y, U) is a minimal unitary dilation of $V_{T^*}^*$.*

Any solution π of this problem has the form $\pi = \pi_{X,Y}$, where

$$\pi_{X,Y}h = (YA_{T^*}^{1/2}h, W_X h), \quad h \in \mathscr{H},$$

$Y \in \mathcal{B}(\mathscr{A}_{T^*}, \mathscr{E})$ being any isometry with $YV_{T^*} = U^*Y$, while

$$(W_X h)(\lambda) = XD_{T^*}(I_{\mathscr{H}} - \lambda T^*)^{-1}h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

$X \in \mathcal{B}(\mathscr{D}_{T^*}, \mathscr{D})$ being an arbitrary unitary operator.

Remark 4. *If $Z : \mathscr{A}_{T^*} \oplus H^2_{\mathscr{A}_{T^*} \ominus \overline{A^{1/2}_{T^*} T^* \mathscr{H}}}(\mathbb{D}) \to \mathscr{E}$ is the unitary operator defined by:*

$$Z(h, f) := Yh + \sum_{n \geq 0} U^{n+1} Y(M_z^{*n} f)(0), \quad h \in \mathscr{A}_{T^*}, f \in H^2_{\mathscr{A}_{T^*} \ominus \overline{A^{1/2}_{T^*} T^* \mathscr{H}}}(\mathbb{D}),$$

then $(i_{\mathscr{A}_{T^*}}, Z^* U^* Z)$ is a unitary extension of V_{T^*}, the dilation $(i_{\mathscr{A}_{T^*}}, Z^* U Z)$ of $V^*_{T^*}$ is minimal and $Z^* U^* Z$ has the matrix representation:

$$Z^* U^* Z = \begin{pmatrix} V_{T^*} & E_0 \\ 0 & M_z^* \end{pmatrix}.$$

These observations are consequences of Theorem 4.

We also discuss the particular situation when U is the identity operator on $\mathscr{E}$. Then the condition that (Y, U^*) extends V_T can be expressed by $V_T = I_{\mathscr{A}_T}$. Equivalently, $A_T^{1/2} T = A_T^{1/2}$ or, as $A_T^{1/2}$ and A_T have identical closures $\mathscr{A}_T$ of their ranges, $A_T T = A_T$.

Corollary 1. *Let $\mathscr{H}, \mathscr{D}, \mathscr{E}$ be complex Hilbert spaces and $T \in \mathcal{B}(\mathscr{H})$. Then there exists a linear isometry $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ such that $(\pi, (I_{\mathscr{E}} \oplus M_z)^*)$ extends T if and only if $\|T\| \leq 1$, $\delta_T \leq \dim \mathscr{D}$, $a_T \leq \dim \mathscr{E}$ and $A_T T = A_T$.*

Any solution π of the equation $\pi T = (I_{\mathscr{E}} \oplus M_z)^* \pi$ has the form $\pi = \pi_{X,Y}$, where

$$\pi_{X,Y} h = (Y A_T^{1/2} h, W_X h), \quad h \in \mathscr{H}$$

and

$$(W_X h)(\lambda) = X D_T (I_{\mathscr{H}} - \lambda T)^{-1} h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

$X \in \mathcal{B}(\mathscr{D}_T, \mathscr{D})$ and $Y \in \mathcal{B}(\mathscr{A}_T, \mathscr{E})$ being arbitrary isometric operators.

Corollary 2. *Let $\mathscr{H}, \mathscr{D}, \mathscr{E}$ be complex Hilbert spaces and $T \in \mathcal{B}(\mathscr{H})$. Then there exists a linear isometry $\pi : \mathscr{H} \to \mathscr{E} \oplus H^2_{\mathscr{D}}(\mathbb{D})$ such that $(\pi, I_{\mathscr{E}} \oplus M_z)$ is a minimal isometric dilation of T if and only if $\|T\| \leq 1$, $\delta_{T^*} = \dim \mathscr{D}$, $a_{T^*} = \dim \mathscr{E}$ and $T A_{T^*} = A_{T^*}$.*

Any solution π of this problem has the form $\pi = \pi_{X,Y}$, where

$$\pi_{X,Y} h = (Y A_{T^*}^{1/2} h, W_X h), \quad h \in \mathscr{H}$$

and

$$(W_X h)(\lambda) = X D_{T^*} (I_{\mathscr{H}} - \lambda T^*)^{-1} h, \quad h \in \mathscr{H}, \lambda \in \mathbb{D},$$

$X \in \mathcal{B}(\mathscr{D}_{T^*}, \mathscr{D})$ and $Y \in \mathcal{B}(\mathscr{A}_{T^*}, \mathscr{E})$ being arbitrary unitary operators.

6. Conclusions

Given a bounded linear operator T acting on a complex Hilbert space $\mathscr{H}$ and an isometric operator V, which is defined on the complex Hilbert space $\mathscr{K}$, we completely describe the conditions that should be imposed on T in order to ensure the existence of an isometry $\pi \in \mathcal{B}(\mathscr{H}, \mathscr{K})$ such that $\pi T = V^* \pi$. We provide, in addition, parametrizations for the set of all solutions π of this problem. We also discuss the connection with the problem of finding linear isometries $\pi : \mathscr{H} \to \mathscr{K}$ such that $T^{*n} = \pi^* V^n \pi$ for every positive integer n. The two problems are actually equivalent in minimality conditions, i.e., when $\mathscr{K}$ is the smallest subspace, which is invariant under V and contains $\pi \mathscr{H}$. This means that a result on extensions can be translated into a result on dilations and the other way around. However, when the minimality assumption is not satisfied, dilations are more general than extensions. We formulated an example in which (π, V) is a dilation of T^*, but (π, V^*) is not an extension of T. Another example shows that even if (π, V^*) is an extension of T, the dilation (π, V) of T^* is not always minimal.

When $\mathscr{H}$ is a subspace of $\mathscr{K}$ and π is the corresponding inclusion $i_{\mathscr{H}}$ (so, π^* is the orthogonal projection $P_{\mathscr{H}}$ of $\mathscr{K}$ onto $\mathscr{H}$), the first equation becomes an extension problem, i.e., $V^*|_{\mathscr{H}} = T$, while the second one is the classical dilation problem proposed by Sz.-Nagy, i.e., $T^{*n} = P_{\mathscr{H}} V^n|_{\mathscr{H}}$ for every $n \geq 0$. In other words, we propose generalized versions of the classical results and provide some clarification on the structure of these dilations or extensions. Our approach is, in some sense, different from the one usually followed in dilation theory. We have the operator V, with all its structure, and we study the properties of T such that it can be π-extended to V^*. Finally, we precisely describe the operator π.

The first particular situation that was taken into consideration in this study is when V is a unitary operator. In this case, T must be a linear isometry. Our characterization is exclusively numeric, and it involves the multiplicities of certain unilateral shifts and the multiplicity functions associated with the spectral measures of some unitary operators. It is also connected with the invariant subspace theory since it requires the existence of some subspaces that are reducing for V or that are invariant under V. Passing to the case when V is a given shift, we obtain similar results as the one obtained by Foiaș in the classical case, i.e., T must be a contraction of class $\mathcal{C}_{0\cdot}$. We present, in addition, the exact form of π. In full generality, by the Wold–Halmos theorem, a linear isometry V can be written as the direct sum between a unitary operator U on a Hilbert space $\mathscr{E}$ and a unilateral shift M_z (i.e., the multiplication by the independent variable on a certain $\mathscr{D}$-valued Hardy space on the unit disc). We obtain that T must be a contraction, its defect index is less than the Hilbert dimension of $\mathscr{D}$ and U^* is a unitary extension of the asymptotic isometry V_T associated with T. For the last assertion, in order to obtain a more precise description, we are able to use the results of Section 3. One can also note that the isometry V_T associated with the asymptotic limit A_T has been extensively studied in the literature, so we have a good chance of clarifying the connection between T, A_T and V_T. We also gave an example in which U has a particular form; namely, it is the identity operator $I_{\mathscr{E}}$. In this case, $V_T = I_{\mathscr{A}_T}$ and $A_T T = A_T$.

While the classical results were widely studied and are useful in many domains, e.g., in invariant subspace theory, interpolation theory, prediction, control, and so on, it is natural to expect that these generalizations would provide larger classes of applications. Some steps forward in this direction were already made recently (see, e.g., [15,16,41,42]). A natural way to extend these results is to pass to the multivariable case, i.e., to systems $(T_1, \ldots, T_n)$ of commuting contractions acting on the same Hilbert space $\mathscr{H}$. The starting point in this direction is a result of Curto and Vasilescu [43] (a refined version has been proposed in [44]), who provided necessary and sufficient conditions on such a tuple in order to possess a backward multishift extension. This subject will be treated elsewhere.

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