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Article

A Note on the Volume Conserving Solution to Simultaneous Aggregation and Collisional Breakage Equation

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Abstract: A new population balance model is introduced, in which a pair of particles can coagulate into a larger one if their encounter is a completely inelastic collision; otherwise, one of them breaks into multiple fragments (two or more) due to the elastic collision. Mathematically, coagulation and breakage models both manifest nonlinearity behavior. We prove the global existence and uniqueness of the solution to this model for the compactly supported kinetic kernels and an unbounded breakage distribution function. A further investigation dealt with the volume conservation property (necessary condition) of the solution.

Keywords: coagulation; collisional breakage; existence and uniqueness; volume conservation; compact support

MSC: 35Q70; 45K05; 45G05

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1. Introduction

Aggregation (coagulation) and fragmentation are fundamental mechanisms that occur in particulate processes such as granulation and crystallization in the pharmaceutical industry [1]. When two particles merge to form a larger one, this process is defined as aggregation. In reverse, fragmentation leads to the formation of smaller particles after the breakup of the mother particle. The aggregation process is inherently nonlinear, while fragmentation is of two types (a) linear, and (b) nonlinear. If fragmentation is spontaneous and driven by an external agent then the process is linear. However, if the process occurs due to the interactions (collisions) between the particles in the system, then it is recognized as a nonlinear fragmentation. The byproducts of the original fragmentation undergo repeated collisions and breakages to drive this process forward. The collisional-induced fragmentation can also be observed in various fields of science and engineering, including the formation of raindrops [2], communication systems [3] and milling processes [4]. Both aggregation and fragmentation mechanisms have been intensively used in the literature for developing mathematical models corresponding to granulation processes [1].

Mathematically, both aggregation and collisional-induced fragmentation mechanisms are represented by a nonlinear integro-partial differential equation. The mathematical expression for tracking the changes in the distribution $\varphi(x,t)$ via these mechanisms can be written as:

$$\begin{aligned}\frac{\partial\varphi(x,t)}{\partial t} =& \frac{1}{2}\int_0^x \mathscr{C}(x-y,y)\varphi(x-y,t)\varphi(y,t)\mathrm{d}y - \varphi(x,t)\int_0^\infty \mathscr{C}(x,y)\varphi(y,t)\mathrm{d}y \\ &+ \int_0^\infty\int_x^\infty \mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,t)\varphi(z,t)\mathrm{d}y\mathrm{d}z - \varphi(x,t)\int_0^\infty \mathscr{K}(x,y)\varphi(y,t)\mathrm{d}y\end{aligned} \quad (1)$$

with the initial data

$$\varphi(x,0) = \varphi_0(x)(\geq 0), \quad \text{for all} \quad x \in \mathbb{R}_+ = (0,\infty). \tag{2}$$

Here, ∂_t stands for the partial derivative with respect to the time t. φ is the number density function for particles of volume x at time t. The kernel $\mathscr{C}(x,y)$ is the aggregation rate at which two particles with particle properties x and y combine to form a larger cluster. $\mathscr{K}(x,y)$ is the collision kernel which describes the rate at which particles of properties x and y are colliding. It is worth noting that both the kernels $\mathscr{C}(x,y)$ and $\mathscr{K}(x,y)$ are symmetric,that is, $\mathscr{C}(x,y) = \mathscr{C}(y,x)$ and $\mathscr{K}(x,y) = \mathscr{K}(y,x)$. $\mathcal{B}(x|y;z)$ is the rate at which particles of property y breaks into fragments of property x due to its impact with a particle of property z. The breakage kernel $\mathcal{B}$ satisfies the following properties.

(i) $\mathcal{B}(x,y;z)$ is non negative and symmetric with respect to y and z, that is

$$\mathcal{B}(x,y;z) = \mathcal{B}(x,z;y).$$

(ii) Volume conservation law

$$\int_0^y x\mathcal{B}(x,y;z)\mathrm{d}x = y \quad \textit{and} \quad \mathcal{B}(x,y;z) = 0 \quad \text{for all} \quad y \leq x; \tag{3}$$

(iii) Number of particles after fragmentation

$$\int_0^y \mathcal{B}(x,y;z)\mathrm{d}x = \nu(y,z) \leq \bar{N} < \infty \quad \text{for all} \quad y > 0, z > 0. \tag{4}$$

The first integral on the right-hand side of Equation (1) represents the formation of the particle property x due to the merging of particles of properties $(x-y)$ and y. The second term denotes the disappearance of the particle property x from the system. The third integral describes the formation of the particle property x from y due to its collision with another particle z at a specific breakup rate $\mathcal{B}(x,y;z)$. In this term, there is no restriction on the particle property z, which acts as a catalyst, as it collides with the fragmenting particle property y, which leads to the formation of x. The final term explains the disappearance of particle property x due to their collision with the other particles present in the system at a specific collision rate $\mathscr{K}(x,y)$.

To represent the full dynamical systems (specifically granulation and crystallization), it is also required to identify the integral properties such as the total number of particles, total volume in the system and total area of the particles. For this reason, the moments of number density $\varphi(x,t)$ must also be defined. Let $\mathcal{M}_k(t)$ denote the k^{th} order moment of the number density function $\varphi(x,t)$, and it is defined as follows:

$$\mathcal{M}_k(t) = \mathcal{M}_k(\varphi(x,t)) := \int_0^\infty x^k \varphi(x,t)dx. \tag{5}$$

The zeroth order moment gives the total number of particles, whereas the total volume in the system is given by the first order moment. The property of volume conservation is expected to hold during both aggregation and fragmentation events.

Smoluchowski [5] was the first to develop an aggregation kinetics discrete model, now known as the discrete Smoluchowski coagulation equation (SCE). Müller [6] proposed a continuous model for the volume distribution of particles, which included other phenomena such as particle fragmentation. Dubovskiĭ and Stewart [7] established the existence and uniqueness of the solution for this continuous model. In 1988, Cheng and Redner [8,9] were the first to formulate a model on the nonlinear breakage equation. The analytical solutions of the general nonlinear breakage equation were studied by Kostoglou and Karabelas [10]. Ernst and Pagonabarraga [11] studied the collision-induced nonlinear fragmentations caused by binary interactions. Vigil et al. [12] and Ke et al. [13] provided the extensive analysis on coagulation with collision-induced fragmentation. Some other existence and uniqueness studies can also be found in [14,15]. Various numerical approaches in 1D and 2D for solving these models have been discussed in detail by [16–22].

In the SCE, the only possibility for the clusters is to continue growing due to the aggregation mechanism, that is, smaller particles cannot be formed in the system. This restricts the application of only the coagulation process in the granulation process, however, it is still useful for polymerization process. This completely eliminates the possibility of the system to reach a steady state or equilibrium solution. Thus, this presents an opportunity for studying the Smoluchowski equation along with the fragmentation process, allowing the system to reach equilibrium. We have highlighted some of the works conducted in this regard in the above literature review. Our work in this article is another extension of the previously mentioned articles, albeit with the establishment of a new model.

In the present work, we introduce an entirely new model for continuous coagulation with collisional breakage. Earlier works have analysed equations with collsional breakage but this is the first time that such a model has been studied. The model mentioned includes the coagulation terms from the continuous SCE and the fragmentation process is represented by the third and fourth terms in (1). This allows us to study the existence of an equilibrium solution for these mechanisms and discuss the well-posedness of Equation (1). The current research work is majorly focused on establishing this well-posedness for compactly supported kernels. Furthermore, it is hypothesized that the breakage distribution function has the structure of a power law. The volume conservation law and uniqueness of the solution will also be proven to hold true.

Let us now mention the spaces considered in this article. For a fixed $T(>0)$, consider a strip

$$\mathcal{W} := \{(x,t) : 0 < x < \infty, 0 \leq t \leq T\}$$

and define $\Psi_{r,\sigma}(T)$ to be the space of all continuous functions φ with the norm

$$\|\varphi\|_{\Psi} := \sup_{0\leq t\leq T} \int_0^\infty \left(x^r + \frac{1}{x^{2\sigma}} \right) |\varphi(x,t)| \mathrm{d}x, \quad r \geq 1,\ \sigma \geq 0. \tag{6}$$

Furthermore, consider $\Psi^+_{r,\sigma}(T)$ the set of all non-negative functions from $\Psi_{r,\sigma}(T)$. In this article, we prove the existence of strong solutions for the coagulation fragmentation of Equation (1) and (2) under the following assumptions over the kinetic kernels;

(A_1) $\mathscr{K}(x,y)$ is a non-negative and continuous function on $\mathbb{R}_+ \times \mathbb{R}_+$.
(A_2) $\mathcal{B}(x,y;z)$ is a non-negative, continuous function satisfying the condition

$$\int_0^y x^{-\theta\sigma} \mathcal{B}(x,y;z) \mathrm{d}x \leq \Phi(y), \quad \text{where} \quad \Phi(y) = \eta y^{-\theta\sigma},$$

where η and θ are considered to be positive constants.

A breakdown of the various sections of this paper is as follows: In Section 2, we state and provide a detailed proof of the existence of solutions for the IVP (1) and (2). In Section 3, the theoretical results for the volume conservation property of the solution is provided. Meanwhile in Section 4, the uniqueness of the solution is proved. The last section is devoted to some important remarks and conclusions.

2. Existence of Solutions

Theorem 1. *Let the functions $\mathscr{C}(x,y)$, $\mathscr{K}(x,y)$ and $\mathcal{B}(x,y;z)$ be nonnegative and continuous on $\mathbb{R}_+ \times \mathbb{R}_+$, $\mathbb{R}_+ \times \mathbb{R}_+$ and $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$ respectively, and satisfy the conditions (A_1), (A_2). Moreover, the kernel $\mathscr{C}$ and $\mathscr{K}$ have compact support for each time $0 \leq t \leq T$. Then, the IVP (1) and (2) has at least one solution $\varphi \in \Psi^+_{r,\sigma}(T)$.*

Proof. We prove the theorem in the following steps;

- Local existence of the solution, that is, there exists a $\tau > 0$ such that the IVP (1) and (2) has at least one solution $\varphi \in \Psi^+_{r,\sigma}(\tau)$;

- Nonnegativity of the local solution;
- Global existence of the unique solution to the space $\Psi^{+}_{r,\sigma}(T)$.

Existence of local solution: Let us consider that there is a fixed $R(>0)$, the coagulation and fragmentation kernels $\mathscr{C}(x,y)$ and $\mathscr{K}(x,y)$ have compact supports in the intervals $[0,R]\times[0,R]$ for each $t\in[0,T]$. Followed from Equation (1), we have

$$\varphi(x,t)=\varphi_0(x)+\int_0^t\Big[\tfrac{1}{2}\int_0^x\mathscr{C}(x-y,y)\varphi(x-y,\xi)\varphi(y,\xi)\mathrm{d}y-\int_0^\infty\mathscr{C}(x,y)\varphi(x,\xi)\varphi(y,\xi)\mathrm{d}y \\ +\int_0^\infty\int_x^\infty\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d}z-\varphi(x,\xi)\int_0^\infty\mathscr{K}(x,y)\varphi(y,\xi)\mathrm{d}y\Big]\mathrm{d}\xi. \quad (7)$$

Hence, the solution to (1) and (2) for $x>2R$ takes the value

$$\varphi(x,t)=\varphi_0(x). \quad (8)$$

The relation (8) provides an approximate solution function beyond the right hand side of the compact domain, where the *tails* of the solution $\varphi(x,t)$, that is, larger size particles, does not alter at all and matches with the *tails* of the initial distribution $\varphi_0(x)$. Let us now focus to show that the local existence of a unique solution for $0<x\leq 2R$.

In this regard, let us define the integral operator $\mathcal{H}$ as follows;

$$\mathcal{H}(\varphi)(x,t):=\quad \text{right hand side of Equation (7).}$$

Since $\mathscr{C}$ and $\mathscr{K}$ have compact supports and φ_0 is a nonnegative continuous function,the integral operator $\mathcal{H}$ is well-defined on $\Psi_{r,\sigma}(\tau)$. This result will be proven via the contraction mapping principle. We began this exercise by showing that for small $\tau>0$ there exists a closed ball in $\Psi_{r,\sigma}(\tau)$, which is invariant relatively to the mapping $\mathcal{H}$. Let $L_0(>0)$ be a constant such that

$$\|\varphi\|_{\Psi}^{(\tau)}:=\sup_{0\leq t\leq\tau}\int_0^\infty\left(x^r+\frac{1}{x^{2\sigma}}\right)|\varphi(x,t)|\mathrm{d}x\leq L_0. \quad (9)$$

Multiplying Equation (7), with $\left(x^r+\frac{1}{x^{2\sigma}}\right)$ on both hand sides and after performing the integration over x, we reached

$$\|\mathcal{H}(\varphi)\|_{\Psi}^{(\tau)}\leq\|\varphi_0\|_{\Psi}^{(\tau)}+\int_0^t\left[\frac{1}{2}\int_0^\infty\left(x^r+\frac{1}{x^{2\sigma}}\right)\int_0^x\mathscr{C}(x-y,y)\varphi(x-y,\xi)\varphi(y,\xi)\mathrm{d}y\mathrm{d}x\right. \\ +\int_0^\infty\left(x^r+\frac{1}{x^{2\sigma}}\right)\int_0^\infty\int_x^\infty\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d}z\mathrm{d}x \\ \left.-\int_0^\infty\left(x^r+\frac{1}{x^{2\sigma}}\right)\varphi(x,\xi)\int_0^\infty[\mathscr{C}(x,y)+\mathscr{K}(x,y)]\varphi(y,\xi)\mathrm{d}y\mathrm{d}x\right]\mathrm{d}\xi. \quad (10)$$

Further, we use the application of the Fubini theorem followed by changing the order of integration and considering $\mu:=\max\{\bar{N},\eta\}$, then, one can obtain the following

$$\begin{aligned}&\int_0^\infty\int_0^\infty\int_x^\infty\left(x^r+\frac{1}{x^{2\sigma}}\right)\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d}z\mathrm{d}x\\ &\quad=\int_0^\infty\int_0^\infty\int_0^y\left(x^r+\frac{1}{x^{2\sigma}}\right)\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad\leq\int_0^\infty\int_0^\infty\int_0^y y^r\mathcal{B}(x,y;z)\mathscr{K}(y,z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad+\int_0^\infty\int_0^\infty\int_0^y x^{-2\sigma}\mathcal{B}(x,y;z)\mathscr{K}(y,z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\ &\quad\leq\int_0^\infty\int_0^\infty\left[\bar{N}y^r+\eta y^{-2\sigma}\right]y^r\mathscr{K}(y,z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d}z\end{aligned}$$

$$\leq \mu \int_0^\infty \int_0^\infty \left(y^r + \frac{1}{y^{2\sigma}}\right) \mathscr{K}(y,z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d}z$$

Since $\mathscr{C}$ and $\mathscr{K}$ both have compact support, their supremiums exist. Let $\kappa_1 = \sup_{\frac{\sigma}{R}\leq x,y\leq R} \mathscr{C}(x,y)$ and $\kappa_2 = \sup_{\frac{\sigma}{R}\leq x,y\leq R} \mathscr{K}(x,y)$. Applying this inequality in (10), we obtain

$$\begin{aligned}\|\mathcal{H}(\varphi)\|_\Psi^{(\tau)} &\leq \|\varphi_0\|_\Psi^{(\tau)} + (2^r\kappa_1 + \mu\kappa_2)\int_0^t \int_0^\infty \int_0^\infty \left(y^r + \frac{1}{y^{2\sigma}}\right)\left(z^r + \frac{1}{z^{2\sigma}}\right)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d}z\mathrm{d}\xi \\ &\leq \|\varphi_0\|_\Psi^{(\tau)} + (2^r\kappa_1 + \mu\kappa_2)\tau L_0^2 \end{aligned} \tag{11}$$

Further, let $\zeta_1 := \max\{\|\varphi_0\|_\Psi^{(\tau)}, (2^r\kappa_1 + \mu\kappa_2)\}$; then, the expression (11) reduces to

$$\|\mathcal{H}(\varphi)\|_\Psi^{(\tau)} \leq \zeta_1\left(1 + \tau L_0^2\right).$$

Hence, $\|\mathcal{H}(\varphi)\|_\Psi^{(\tau)} \leq L_0$, if $\zeta_1(1+\tau L_0^2) \leq L_0$. This inequality holds if $\tau < \frac{1}{4\zeta_1^2}$ and

$$\frac{1-\sqrt{1-4\zeta_1^2\tau}}{2\zeta_1\tau} \leq L_0 \leq \frac{1+\sqrt{1-4\zeta_1^2\tau}}{2\zeta_1\tau}. \tag{12}$$

Presently, our focus will be to demonstrate that the mapping of $\mathcal{H}$ is contracting. Using the relation in (7), we have

$$\begin{aligned}\|\mathcal{H}(\varphi)-\mathcal{H}(\psi)\|_\Psi^{(\tau)} \leq \int_0^t \Bigg[&\frac{1}{2}\int_0^\infty \left(x^r + \frac{1}{x^{2\sigma}}\right)\int_0^x \mathscr{C}(x-y,y)|\mathcal{A}(x-y,y,\xi)|\mathrm{d}y\mathrm{d}x \\ &+ \int_0^\infty \int_0^\infty \int_x^\infty \left(x^r + \frac{1}{x^{2\sigma}}\right)\mathscr{K}(y,z)\mathcal{B}(x,y;z)|\mathcal{A}(y,z,s)|\mathrm{d}y\mathrm{d}z\mathrm{d}x \\ &+ \int_0^\infty \int_0^\infty \left(x^r + \frac{1}{x^{2\sigma}}\right)(\mathscr{C}(x,y)+\mathscr{K}(x,y))|\mathcal{A}(x,y,s)|\mathrm{d}y\mathrm{d}x\Bigg]\mathrm{d}s \end{aligned} \tag{13}$$

where $\mathcal{A}(x,y,s) = \varphi(x,s)\varphi(y,s) - \psi(x,s)\psi(y,s)$.

The first expression in the above inequality (13) can be estimated, as follows

$$\frac{1}{2}\int_0^\infty \left(x^r + \frac{1}{x^{2\sigma}}\right)\int_0^x \mathscr{C}(x-y,y)|\mathcal{A}(x-y,y,\xi)|\mathrm{d}y\mathrm{d}x \leq 2^r\kappa_1\|\varphi-\psi\|_\Psi^{(\tau)}\left[\|\varphi\|_\Psi^{(\tau)} + \|\psi\|_\Psi^{(\tau)}\right]$$

Furthermore, the second expression in the above inequality (13) is simplified using the Fubini's theorem with respect to z and x followed by interchanging the order of integration with respect to y and x, which gives the following expression

$$\begin{aligned}&\int_0^\infty \int_0^\infty \int_x^\infty \left(x^r + \frac{1}{x^{2\sigma}}\right)\mathscr{K}(y,z)\mathcal{B}(x,y;z)|\mathcal{A}(y,z,s)|\mathrm{d}y\mathrm{d}z\mathrm{d}x \\ &\qquad \leq \int_0^\infty \int_0^\infty \int_0^y x^r \mathscr{K}(y,z)\mathcal{B}(x,y;z)|\mathcal{A}(y,z,s)|\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\qquad + \int_0^\infty \int_0^\infty \int_0^y \frac{1}{x^{2\sigma}}\mathscr{K}(y,z)\mathcal{B}(x,y;z)|\mathcal{A}(y,z,s)|\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &\qquad \leq \mu\int_0^\infty \int_0^\infty \left(y^r + \frac{1}{y^{2\sigma}}\right)\mathscr{K}(y,z)|\mathcal{A}(y,z,s)|\mathrm{d}y\mathrm{d}z \\ &\qquad \leq \mu\kappa_2\int_0^\infty \int_0^\infty \left(y^r + \frac{1}{y^{2\sigma}}\right)|\varphi(z,s)(\varphi(y,s)-\psi(y,s)) \\ &\qquad\qquad + g(y,s)(\varphi(z,s)-\psi(z,s))|\mathrm{d}y\mathrm{d}z \\ &\qquad \leq \kappa_2\mu\|\varphi-\psi\|_\Psi^{(\tau)}\left[\|\varphi\|_\Psi^{(\tau)} + \|\psi\|_\Psi^{(\tau)}\right]\end{aligned}$$

Using this estimation on the relation (13), the following is obtained

$$\|\mathcal{H}(\varphi)-\mathcal{H}(\psi)\|_{\Psi}^{(\tau)} \leq \tau(\kappa_1(2^r+1)+\kappa_2(\mu+1))\|\varphi-\psi\|_{\Psi}^{(\tau)}\left[\|\varphi\|_{\Psi}^{(\tau)}+\|\psi\|_{\Psi}^{(\tau)}\right] \tag{14}$$

Further, let $\zeta_2 := 2[\kappa_1(2^r+1)+\kappa_2(\mu+1)]$, then the inequality (14) reduces to

$$\|\mathcal{H}(\varphi)-\mathcal{H}(\psi)\|_{\Psi}^{(\tau)} \leq \tau\zeta_2 L_0\|\varphi-\psi\|_{\Psi}^{(\tau)}. \tag{15}$$

Thus, the mapping $\mathcal{H}$ is contractive on $\Psi^+_{r,\sigma}(\tau)$ for $\tau < [\zeta_2 L_0]^{-1}$. Using this result together with the inequality (12), there exists an invariant ball of radius L_0 for sufficiently small $\tau > 0$ and in this ball, $\mathcal{H}$ is contractive. Consequently, the ball contains a fixed point of $\mathcal{H}$.

Nonnegativity: Case I: Consider $\varphi_0(x) > 0$ for all $x \in (0, R)$. Since φ is continuous, there exists a small strip $\{(x,t) : 0 < x < R, t \in [0,t_0)\}$, where φ is strictly positive. For a particular t_0, we can find an $x_0 \in (0, R)$ such that (x_0, t_0) is the point with the property that

$$\varphi(x_0,t_0) = 0 \text{ and } \varphi(x,t) \neq 0 \text{ for all } 0 < x < \max\{x_0, R\},\ t \in [0,t_0) \tag{16}$$

Since the solution is continuous and satisfies (7) it must be continuously differentiable with respect to t. Therefore,

$$\begin{aligned}\partial_t\varphi(x,t)|_{(x_0,t_0)} =& \frac{1}{2}\int_0^{x_0}\mathscr{C}(x_0-y,y)\varphi(x_0-y,t_0)\varphi(y,t_0)\mathrm{d}y \\ &+\int_0^R\int_{x_0}^R\mathscr{K}(y,z)\mathcal{B}(x_0,y;z)\varphi(y,t)\varphi(z,t)\mathrm{d}y\mathrm{d}z,\end{aligned} \tag{17}$$

- If $x_0 \leq R$, then $\varphi(x,t) > 0$ for all $0 < x \leq R$ and $0 \leq t < t_0$. The positivity of the right hand side of (17) implies $\partial_t\varphi(x,t)|_{(x_0,t_0)} > 0$.
- If $x_0 > R$, we use the property (3) of the breakage function to obtain

$$\begin{aligned}\int_0^R\int_{x_0}^R\mathscr{K}(y,z)\mathcal{B}(x_0,y;z)\varphi(y,t)\varphi(z,t)\mathrm{d}y\mathrm{d}z &= -\int_0^R\int_{x_0}^R\mathscr{K}(y,z)\mathcal{B}(x_0,y;z)\varphi(y,t)\varphi(z,t)\mathrm{d}y\mathrm{d}z \\ &= 0\end{aligned}$$

Thus, from the Equation (17), we have $\partial_t\varphi(x,t)|_{(x_0,t_0)} > 0$.

The positive value of the time derivative establishes that there exists a point (x_0, t), with $t < t_0$ such that $\varphi(x_0, t) < 0$. However, this counters the hypothesis that (x_0, t_0) is a point bearing a property provided by relation (16). Hence, the point (x_0, t_0) where the solution vanishes does not exist.

Further, when $x \geq R$ by (7) and the compactly supported kernels $\mathscr{C}$ and $\mathscr{K}$, the solution coincides with the initial data. Hence, again it becomes positive. Consequently, $\varphi(x,t)$ is strictly positive provided that the initial distribution is strictly positive.

Case II: Suppose φ_0 is not strictly positive. Then, we construct the sequence $\{\varphi_0^n\}$ of the positive function to satisfy the conditions listed in Theorem 1, which then converges to φ_0 uniformly in $\Psi_{r,\sigma}(\tau)$ with respect to $t \in [0,\tau]$. We have established earlier that the family of operators $\mathcal{H}_n : \Psi_{r,\sigma}(\tau) \to \Psi_{r,\sigma}(\tau)$, defined as

$$\begin{aligned}\mathcal{H}_n(\varphi)(x,t) =& \varphi_0^n(x) + \int_0^t\Bigg[\frac{1}{2}\int_0^x\mathscr{C}(x-y,y)\varphi(x-y,\xi)\varphi(y,\xi)\mathrm{d}y - \int_0^\infty\mathscr{C}(x,y)\varphi(x,\xi)\varphi(y,\xi)\mathrm{d}y \\ &\int_0^\infty\int_x^\infty\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,\xi)\varphi(z,\xi)\mathrm{d}y\mathrm{d} - \varphi(x,\xi)\int_0^\infty\mathscr{K}(x,y)\varphi(y,\xi)\mathrm{d}y\Bigg]\mathrm{d}\xi\end{aligned}$$

is a contraction mapping. Therefore, as $n \to \infty$, we have

$$\sup_{\|\varphi\|_{\Psi}^{(\tau)} \leq L} \|\mathcal{H}_n(\varphi) - \mathcal{H}(\varphi)\|_{\Psi}^{(\tau)} \leq \int_0^\infty \left(x^r + \frac{1}{x^{2\sigma}}\right)|\varphi_0^n(x) - \varphi_0(x)|\mathrm{d}x \to 0.$$

Since the mapping is contractive in $\Psi_{r,\sigma}(\tau)$, therefore

$$\begin{aligned}\|\varphi^n - \varphi\|_{\Psi}^{(\tau)} = \|\mathcal{H}_n(\varphi^n) - \mathcal{H}(\varphi)\|_{\Psi}^{(\tau)} &\leq \|\mathcal{H}_n(\varphi^n) - \mathcal{H}(\varphi^n)\|_{\Psi}^{(\tau)} + \|\mathcal{H}(\varphi^n) - \mathcal{H}(\varphi)\|_{\Psi}^{(\tau)} \\ &\leq \|\mathcal{H}_n(\varphi^n) - \mathcal{H}(\varphi^n)\|_{\Psi}^{(\tau)} + \bar{\zeta}\|\varphi^n - \varphi\|_{\Psi}^{(\tau)},\end{aligned}$$

which implies

$$(1-\bar{\zeta})\|\varphi^n - \varphi\|_{\Psi}^{(\tau)} = \|\mathcal{H}_n(\varphi^n) - \mathcal{H}(\varphi^n)\|_{\Psi}^{(\tau)} \to 0 \quad \text{whenever} \quad n \to \infty.$$

This shows that for a positive initial data, the solution φ is also positive.

Global existence of unique solution: Let us first discuss the boundedness of the moments

$$\mathcal{M}_k(t) = \int_0^\infty x^k \varphi(x,t)\mathrm{d}x; \quad \text{where} \quad 0 \leq k \leq r \quad \text{and} \quad k = -2\sigma,$$

for compactly supported kernels. Simple calculations will lead us to the following results:

$$\mathcal{M}_1(t) \leq \bar{m}_1, \quad \mathcal{M}_{-2\sigma}(t) \leq \bar{m}_{-2\sigma}, \quad \mathcal{M}_0(t) \leq \bar{m}_0, \quad \mathcal{M}_2(t) \leq \bar{m}_2, \tag{18}$$

and so on. Here, terms $\bar{m}_k$, $k = -2\sigma, 0, 1, \ldots, r$ are all constants. Furthermore, it is important to note that the boundedness of the k^{th} moment ensures the boundedness of the $(k+1)^{th}$ moment for $k = 2, 3, \ldots, r$. Thus, using the aforementioned results, we can conclude that the

$$\|\varphi\|_{\Psi} \leq \bar{m}_r + \bar{m}_{-2\sigma}.$$

implies that the solution of IVP (1) and (2) is bounded in the norm $\|.\|_{\Psi}$. Taking into account the positivity/nonnegativity of the local solution, it is easy to extend it for $0 \leq t \leq T$. Recalling Theorem 2.2 of [23], the global existence of the unique solution belonging to $\Psi^+_{r,\sigma}(T)$ can easily be proved. □

3. Conservation of Volume

In order to show the volume conservation law, let us multiply equation (1) by the x by performing integration over x; the following is obtained

$$\begin{aligned}\frac{\mathrm{d}\mathcal{M}(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty x\varphi(x,t)\mathrm{d}x = &\underbrace{\frac{1}{2}\int_0^\infty\int_0^x x\mathscr{C}(x-y,y)\varphi(x-y,t)\varphi(y,t)\mathrm{d}y}_{M_1} \\ &\underbrace{\int_0^\infty\int_0^\infty\int_x^\infty x\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,t)\varphi(z,t)\mathrm{d}y\mathrm{d}z\mathrm{d}x}_{M_2} \\ &-\underbrace{\int_0^\infty\int_0^\infty x(\mathscr{C}(x,y)+\mathscr{K}(x,y))\varphi(x,t)\varphi(y,t)\mathrm{d}y\mathrm{d}x}_{M_3}\end{aligned} \tag{19}$$

Under a suitable transformation, we can estimate the integral M_1, as follows

$$\begin{aligned}M_1 &= \frac{1}{2}\int_0^\infty\int_0^\infty (x+y)\mathscr{C}(x,y)\varphi(x,t)\varphi(y,t)\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty\int_0^\infty x\mathscr{C}(x,y)\varphi(x,t)\varphi(y,t)\mathrm{d}y\mathrm{d}x\end{aligned} \tag{20}$$

For the integral $\mathcal{N}_1$, using the application of the Fubini's theorem followed by a change in the order of integration with respect to y and x, and using (3), obtains

$$\begin{aligned} M_2 &= \int_0^\infty \int_0^\infty \int_0^y x\mathscr{K}(y,z)\mathcal{B}(x,y;z)\varphi(y,t)\varphi(z,t)\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \int_0^\infty \int_0^\infty y\mathscr{K}(y,z)\varphi(y,t)\varphi(z,t)\mathrm{d}y\mathrm{d}z \\ &= \int_0^\infty \int_0^\infty x\mathscr{K}(x,y)\varphi(x,t)\varphi(y,t)\mathrm{d}x\mathrm{d}y \end{aligned} \tag{21}$$

Adding the estimations (20) and (21), $M_1 + M_2 = M_3$ are obtained. Hence, by using this relation on (19), we can conclude the volume conservation property of the existing solution.

4. Uniqueness Theory

Theorem 2. *Let the assumptions of Theorem 1 hold true, then the IVP (1) and (2) has a unique solution in $\Psi^+_{r,\sigma}(T)$.*

Proof. Let $t \neq 0$, $\varphi_1(x,t)$ and $\varphi_2(x,t)$ be two distinct solutions of (1), (2) along with $\varphi_1(x,0) = \varphi_2(x,0)$. Further suppose $\mathcal{Q}(x,t) := \varphi_1(x,t) - \varphi_2(x,t)$, and we construct an auxiliary function

$$\mathcal{P}(t) := \int_0^\infty |\mathcal{Q}(x,t)|\mathrm{d}x.$$

Since both the solutions $\varphi_1(x,t)$ and $\varphi_2(x,t)$ satisfy the Equation (7), we have

$$\begin{aligned} \mathcal{P}(t) \leq \int_0^t \Bigg[& \underbrace{\frac{1}{2}\int_0^\infty \int_0^x \mathscr{C}(x-y,y)|\mathcal{A}(x-y,y,\xi)|\mathrm{d}y\mathrm{d}x}_{J_0} \\ & + \underbrace{\int_0^\infty \int_0^\infty \int_x^\infty \mathscr{K}(y,z)\mathcal{B}(x,y;z)|\mathcal{A}(y,z,\xi)|\mathrm{d}y\mathrm{d}z\mathrm{d}x}_{J_1} \\ & + \underbrace{\int_0^\infty \int_0^\infty (\mathscr{C}(x,y) + \mathscr{K}(x,y))|\mathcal{A}(x,y,\xi)|\mathrm{d}y\mathrm{d}x}_{J_2} \Bigg] \mathrm{d}\xi \end{aligned} \tag{22}$$

Further performing the change in the order of integration followed by the application of Fubini's theorem, the integrals J_0 and J_1 can be estimated as

$$\begin{aligned} J_0 &\leq \frac{1}{2}k_1(\|\varphi_1\|_\Psi + \|\varphi_2\|_\Psi)\mathcal{P}(s). \\ J_1 &\leq k_2\bar{N}(\|\varphi_1\|_\Psi + \|\varphi_2\|_\Psi)\mathcal{P}(s). \end{aligned}$$

Similar operations apply for the integral J_2, and when using the relation (22), we obtain

$$\mathcal{N}(t) \leq \Lambda(\|\varphi_1\|_\Psi + \|\varphi_2\|_\Psi) \int_0^t \mathcal{P}(s)\mathrm{d}s., \tag{23}$$

where Λ is a positive constant depending only on k_1, k_2 and $\bar{N}$. Since φ_1 and φ_2 both belong to the space $\Psi^+_{r,\sigma}(T)$, the norms $\|\varphi_1\|_\Psi$ and $\|\varphi_2\|_\Psi$ are uniformly bounded with respect to $0 \leq t \leq T$. Then, by applying Grownwall's inequality on (23), we obtain

$$\mathcal{P}(t) = 0. \quad \text{for all} \quad 0 \leq t \leq T,$$

which concludes the proof. □

5. Concluding Remarks

A new population balance model, including the nonlinear coagulation and fragmentation, was introduced in this paper. The model accounts for a completely inelastic collision between a pair of particles, which leads to the formation of a larger particle. If their encounter is not completely inelastic, then there is a possibility of the formation of smaller particles when they collide. A proof has been given to obtain the existence and the uniqueness of a solution to the purely nonlinear model for a set of kernels with compact support. The results of the existence and uniqueness are further supported by providing the theoretical outcome of the volume conservation law.

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