axioms

*Article*

# Properties of Differential Subordination and Superordination for Multivalent Functions Associated with the Convolution Operators

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**Abstract:** Using convolution (or Hadamard product), we define the El-Ashwah and Drbuk linear operator, which is a multivalent function in the unit disk $U = [w : |w| < 1 \text{ and } w \in ¢]$, and satisfy its specific relationship to derive the subordination, superordination, and sandwich results for this operator by using properties of subordination and superordination concepts.

**Keywords:** multivalent functions; subordination; Hadamard product; superordination

**MSC:** 30C45



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## 1. Introduction and Definitions

The set $\Omega(U)$ denotes the class of all analytic functions in the open unit disk $U = [w : |w| < 1 \; anq \; w \in ¢]$ and $\Omega[a, k]$ as the subclass of $\Omega(U)$, which consists of the form functions

$$f(w) = a + a_k w^k + a_{k+1} w^{k+1} + \ldots, \; (a \in ¢, \; w \in U, \; k \in N). \tag{1}$$

With $\mathcal{A}_p$ as the class of all multivalent functions in open unit disk $U$ of the form

$$f(w) = w^p + \sum_{k=1+p}^{\infty} a_k w^k, \; w \in U, p \in N. \tag{2}$$

Additionally, we use $\mathcal{A} = \mathcal{A}_1$ to denote the class of analytic functions in the open unit disk $U$ and normalize them with $f(0) = 0$, $f'(0) = 1$.

Additionally, consider $\mathcal{S}$ as the class of the univalent function in $U$,

Let $\mathcal{S}^*(\varrho)$, $\mathcal{C}(\varrho)$ and $\mathcal{K}$ be the subclasses of $\mathcal{A}$ such that:

$\left\{ f \in \mathcal{S}^* : Re\left\{ \frac{wf'(w)}{f(w)} \right\} > \varrho \right\}, w \in U, \; (0 < \varrho < 1)$, then $f$ is a starlike function;

$\left\{ f \in \mathcal{C} : Re\left\{ 1 + \frac{wf''(w)}{f'(w)} \right\} > \varrho \right\}, w \in U, \; (0 < \varrho < 1)$, then $f$ is a convex function;

$\left\{ f \in \mathcal{K} : Re\left\{ \frac{f_1{}'(w)}{g'(w)} \right\} > 0 : g \in \mathcal{C} \right\}, w \in U$, then $f$ is a close-to-convex function.

If the functions $f$ and $g$ are analytic in $U$, then we say $f$ is subordinate to $g$ or $f$ is said to be superordinate to $f$ in $U$, written as $f \prec g$ or $f(w) \prec g(w)$ if there is a Schwarz function $v(w)$ analytic in $U$, with $|v(w)| < 1$, so that $f(w) = g(v(w))$ and $w \in U$. In particular, if the function $g$ is univalent in $U$, then the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ and $f(U) \subset g(U)$, (see [1–8]).

If $f, g \in \mathcal{A}_p$, where $f(w)$ is provided by (1) and $g(w)$ is defined by

$$g(w) = w^p + \sum_{k=1+p}^{\infty} a_k w^k, w \in U,$$

the Hadamard product (or convolution) of the function $f$ and $g$ is defined by

$$f(w) \times g(w) = w^p + \sum_{k=1+p}^{\infty} a_k b_k w^k, (w \in U) = (f \times g)(w). \tag{3}$$

Let $\delta > 0$, $a, c \in \mathbb{C}$ such that $Re(c-a) \geq 0$ and $Rea \geq -\delta p$, $p \in N$, $n \in \mathbb{Z}$, $\theta \geq 0$ and $\lambda > -p$.

El-Ashwah and Drbuk [5] introduced the linear operator $\mathcal{B}_{p,n}^{\theta,\lambda}(a,c,\delta) : \mathcal{A}_p \to \mathcal{A}_p$ defined by

$$\begin{aligned} &\mathcal{B}_{\theta,\lambda}^{n,p}(a,c,\delta)f(w) \\ &\quad = w^p + \frac{\Gamma(c+\delta p)}{\Gamma(c+\delta p)} \sum_{k=1+p}^{\infty} \left(\frac{p+\lambda+\theta(k-p)}{p+\lambda}\right)^n \frac{\Gamma(c+\delta p)}{\Gamma(c+\delta p)} a_k w^k. \end{aligned} \tag{4}$$

It is readily verified from (4) that

$$\begin{aligned} &\mathcal{B}_{\theta,\lambda}^{n+1,p}(a,c,\delta)f(w) = \left(1 - \frac{p\theta}{p+\lambda}\right)\mathcal{B}_{\theta,\lambda}^{n,p}(a,c,\delta)f(w) \\ &+\frac{\theta}{p+\lambda} w\left(\mathcal{B}_{\theta,\lambda}^{n,p}(a,c,\delta)f(w)\right)'. \end{aligned} \tag{5}$$

Putting $a = c$ in (4), we obtain the Prajapat operator $J_p^n(\theta, \lambda)$, see [9].

Additionally, when $n = 0$, we obtain the Erdelyi-Kober integral operator $I_{p,\delta}^{a,c}$, see [10].

**Definition 1.** *Let $Y$: $\mathbb{C}^3 \times U \to \mathbb{C}$ and $h(w)$ be univalent in $U$. If $p(w)$ is analytic in $U$, that fulfils the second-order differential subordination [11]:*

$$Y\left(p(w), wp'(w),\ w^2 p''(w); w\right) \prec h(w), \tag{6}$$

*then $p(w)$ is the differential subordination solution of (6).*

**Definition 2.** *Let $Y_1$: $\mathbb{C}^3 \times U \to \mathbb{C}$ and $h(w)$ be univalent in $U$. If $p(w)$ and $Y_1(p(w), wp'(w),\ w^2p''(w); w)$ are univalent in $U$ and $p(w)$ fulfill the second-order differential superordination [11]:*

$$h(w) \prec Y_1\left(p(w), wp'(w),\ w^2 p''(w); w\right), \tag{7}$$

*then $p(w)$ is the differential superordination solution of (7).*

**Definition 3.** *Let $Q$ be the collections of functions $f$ that are analytic and injective on $\overline{U} \backslash E(f)$,* when $E(f) = \left\{\varsigma \in \partial U\ : \lim\limits_{w \to \varsigma} f(w) = \infty\right\}$ *and $f'(w) \neq 0$ for $\varsigma \in\ \partial U \backslash E(f)$ [11].*

**Lemma 1.** *Let $p_1(w)$ be the univalent function in $U$ and let $\Sigma$ and $\vartheta$ be holomorphic in a domain $p_1(U) \subset D$, with $\vartheta(\omega) \neq 0$, when $\omega \in p_1(U)$. Set $\mathcal{O}(w) = w\, {p_1}'(w)\vartheta(p_1(w))$ and $\hbar(w) = \Sigma(p_1(w) + \mathcal{O}(w)$. Suppose that [12]*

*(i) $\mathcal{O}$ is starlike in $U$.*

*(ii) $Re\left(\frac{w\hbar'(w)}{\mathcal{O}(w)}\right) > 0, w \in\ U$.*

*If $p_2(w)$ is holomorphic in $U$ with $p_2(0) = p_1(0)$, $p_2(U) \subset D$, and $\Sigma(p_2(w)) + w{p_2}'(w)\vartheta(p_2(w)) \prec \Sigma(p_1(w)) + w{p_1}'(w)\vartheta(p_1(w))$, then $p_2(w) \prec\ p_1(w)$.*

**Lemma 2.** *Let $p_1(w)$ be convex in $U$ and $\beta_1 \in \mathbb{C}, \beta_2 \in \mathbb{C}^*$ with $Re\left(1 + \frac{{p_1}''(w)}{{p_1}'(w)}\right) > max\left\{0, -Re\frac{\beta_1}{\beta_2}\right\}$. If $p_2(w)$ is holomorphic in $U$ and $\beta_1 p_2(w) + \beta_2 w{p_2}'(w) \prec \beta_1 p_1(w) + \beta_2 w{p_1}'(w)$, then $p_2(w) \prec\ p_1(w)$ [11].*

**Lemma 3.** *Let $p_1(w)$ be convex univalent in $U$ and let $\Sigma$ and $\vartheta$ be holomorphic in a domain $D$, $p_1(U) \subset D$. Suppose that* [12]

*(i) $wp_1{}'(w)\vartheta(p_1(w))$ is starlike univalent in $U$.*

*(ii) $Re\left(\frac{\Sigma\,\prime(p_1(w))}{\vartheta(p_1(w))}\right) > 0, w \in U$.*

*If $p_2(w) \in \mathcal{A}[p_1(0), 1] \bigcap Q$, with $p_2(U) \subset D, \Sigma(p_2(w) + wp_2{}'(w)\vartheta(p_2(w))$ is univalent in $U$ and $\Sigma(p_1(w)) + wp_1{}'(w)\vartheta(p_1(w)) \prec \Sigma(p_2(w)) + wp_2{}'(w)\vartheta(p_2(w))$, then $p_1(w) \prec p_2(w)$.*

**Lemma 4.** *Let $p_1(w)$ be convex in $U$ and $Re(\beta) > 0$.*

*If $p_2(w) \in \mathcal{A}[p_1(0), 1] \bigcap Q$, $p_2(w) + \beta wp_2{}'(w)$ is univalent in $U$ and $p_1(w) + \beta wp_1{}'(w) \prec p_2(w) + \beta wp_2{}'(w)$, then $p_1(w) \prec p_2(w)$* [12].

## 2. Subordination Results

**Theorem 1.** *Let $b(w)$ be convex univalent in $U$, with $b(0) = 1$, $a_1 > 0, 0 \neq a_2 \in ¢$ and suppose*

$$Re\left(1 + \frac{b''(w)}{b'(w)}\right) > \max\left\{0, -Re\frac{a_1}{a_2}\right\}.$$

*If $f \in \mathcal{A}$, it satisfies the subordination:*

$$\left(1 - \frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} + \frac{a_2(p+\lambda)}{\theta}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)}{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}\right)^{a_1}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} \prec b(w) + \frac{a_2}{a_1}wb'(w),$$

*then*

$$\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} \prec b(w).$$

**Proof.** Consider

$$q(w) = \left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}.$$

Then

$$q'(w) = a_1\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}\left(\frac{w^p}{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}\right)$$
$$\left(\frac{w^p\left[\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right]' - pw^{p-1}\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}{(w^p)^2}\right)$$
$$= a_1\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}\left(\frac{\left[\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right]'}{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)} - \frac{p}{w}\right)$$

We have

$$\frac{wq'(w)}{q(w)} = a_1\left(\frac{w\left[\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right]'}{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)} - p\right).$$

By using (5) we obtain

$$\frac{wq'(w)}{q(w)}$$
$$= a_1\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right)+p\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)-\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}{\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}-p\right)$$
$$= a_1\frac{(p+\lambda)}{\theta}\left(\frac{\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right)}{\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}-1\right),$$

and
$$\frac{a_2wq'(w)}{a_1}=\frac{a_2(p+\lambda)}{\theta}\left(\frac{\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right)}{\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}\right)\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}$$
$$-\frac{a_2(p+\lambda)}{\theta}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}.$$

By using the hypothesis, we obtain $q(w)+\frac{a_2}{a_1}wq'(w)\prec b(w)+\frac{a_2}{a_1}wb'(w)$. Additionally, apply Lemma 2, when $\beta_1=1$ and $\beta_2=\frac{a_2}{a_1}$, then

$$\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}\prec b(w).$$

□

**Corollary 1.** *Let $b(w)$ be convex univalent in $U$, with $b(0)=1$, $a_1>0, 0\neq a_2\in ¢$ and suppose*

$$Re\left(1+\frac{b''(w)}{b'(w)}\right)>max\left\{0,-Re\frac{a_1}{a_2}\right\}.$$

*If $f\in\mathcal{A}$, it satisfies the subordination:*

$$\left(1-\frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{J_p^n(\theta,\lambda)f(w)}{w^p}\right)^{a_1}+\frac{a_2(p+\lambda)}{\theta}\left(\frac{J_p^{n+1}(\theta,\lambda)f(w)}{J_p^n(\theta,\lambda)f(w)}\right)^{a_1}\left(\frac{J_p^n(\theta,\lambda)f(w)}{w^p}\right)^{a_1}\prec b(w)+\frac{a_2}{a_1}wb'(w),$$

*then*
$$\left(\frac{J_p^n(\theta,\lambda)f(w)}{w^p}\right)^{a_1}\prec b(w).$$

**Theorem 2.** *Let $b$ be convex univalent in, $b(0)=1$, and $b(w)\neq 0$ for all $w\in U$, and suppose that $b$ satisfies:*

$$Re\left\{p+\frac{wt\sigma}{w^pa_2}+\frac{w\varepsilon(\sigma+1)}{w^pa_2}(w)+(\sigma-1)\frac{wb'(w)}{b(w)}+\frac{wb''(w)}{b'(w)}\right\}>0, \tag{8}$$

*where $\sigma,\varepsilon,t\in ¢, a_1>0, 0\neq a_2\in ¢$ and $w\in U$. Suppose that $w^p(b(w))^{\sigma-1}b'(w)$ is a starlike univalent in $U$.*

*If $f\in\mathcal{A}$ satisfies the subordination:*

$$\mathcal{M}(p,n,\lambda,\theta,\varepsilon,a_1,a_2;w)\prec(t+\varepsilon\, b(w))(b(w))^{\sigma}+a_2(b(w))^{\sigma-1}b'(w)$$

*where*

$$\mathcal{M}(p,n,\lambda,\theta,\varepsilon,a_1,a_2;w) = t\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1\sigma}$$
$$+\varepsilon\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1(\sigma+1)}$$
$$+a_2a_1\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1\sigma}$$
$$\left(\frac{w\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)'+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)'}{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}-p\right), \tag{9}$$

*then*

$$\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1} \prec b(w).$$

**Proof.** Let $H(\beta) = (t+\varepsilon\beta)\beta^{\sigma}$ and $L(\beta) = a_2(\beta)^{\sigma-1}$, $0 \neq \beta \in ¢$, when $H(\beta)$ and $L(\beta)$ are analytic in ¢. □

Then, we obtain $G(w) = wb'(w)L(b(w)) = a_2w^p(b(w))^{\sigma-1}b'(w)$ and $y(w) = H(b(w)) + G(w) = (t+\varepsilon(b(w)))(b(w))^{\sigma} + a_2w^p(b(w))^{\sigma-1}b'(w)$.
Since $w^p(b(w))^{\sigma-1}b'(w)$ is starlike, then $G(w)$ is starlike in $U$, and

$$Re\left(\frac{y'(w)}{G(w)}\right) = Re\left\{p + \frac{wt\sigma}{w^pa_2} + \frac{w\varepsilon(\sigma+1)}{w^pa_2}(w) + (\sigma-1)\frac{wb'(w)}{b(w)} + \frac{wb''(w)}{b'(w)}\right\} > 0$$

Additionally, consider

$$q(w) = \left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1}.$$

Then,

$$q'(w) = a_1\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1}$$
$$\left[\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)'+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)'}{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}-\frac{1}{w}\right].$$

We obtain

$$t\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1\sigma}$$

$$+\varepsilon\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1(\sigma+1)}$$

$$= t(q(w))^{\sigma} + \varepsilon\left[(q(w))^{\sigma}q(w)\right] = (t+\varepsilon q(w))(q(w))^{\sigma}.$$

Since

$$a_1\left(\frac{w\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)' + \left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)'}{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right) + \left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)} - p\right) = \frac{wq'(w)}{q(w)},$$

That

$$a_2 a_1\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1\sigma}$$

$$\left(\frac{w\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)'+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)'}{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\ p}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)\right)} - p\right)$$

$$= a_2\, w(q(w))^{\sigma-1}q'(w).$$

From (8) we obtain $(t+\varepsilon q(w))(q(w))^{\sigma} + a_2\ w(q(w))^{\sigma-1}q'(w) \prec (t+\varepsilon b(w))(b(w))^{\sigma} + a_2(b(w))^{\sigma-1}b'(w)$ and using Lemma 1 we obtain $q(w) \prec b(w)$.

**Corollary 2.** *Let $b$ be convex univalent in, $b(0)=1$, and $b(w)\neq 0$ for all $w\in U$, and suppose that $b$ satisfies:*

$$Re\left\{p + \frac{wt\sigma}{w^p a_2} + \frac{w\varepsilon(\sigma+1)}{w^p a_2}(w) + (\sigma-1)\frac{wb'(w)}{b(w)} + \frac{wb''(w)}{b'(w)}\right\} > 0,$$

*where $\sigma, \varepsilon, t \in ¢, a_1 > 0, 0 \neq a_2 \in ¢$ and $w \in U$.*

*Suppose that $w^p(b(w))^{\sigma-1}b'(w)$ is a starlike univalent in $U$.*

*If $f \in \mathcal{A}$, it satisfies the subordination:*

$$\mathcal{M}(\sigma, t, \varepsilon, h_{\mu}, \mu, a_1, a_2; w) \prec (t+\varepsilon\, b(w))(b(w))^{\sigma} + a_2(b(w))^{\sigma-1}b'(w),$$

*then*

$$\left(\frac{\frac{(p+\lambda)}{\theta}\left(J_p^{n+1}(\theta,\lambda)f(w)\right) + \left(1-\frac{(p+\lambda)}{\theta}\right)\left(J_p^{n}(\theta,\lambda)f(w)\right)}{w^p}\right)^{a_1} \prec b(w).$$

## 3. Superordination Results

**Theorem 3.** *Let $b(w)$ be convex in $U$, with $b(0)=1$, $a_1>0$, $Rea_2>0$, if $f\in\mathcal{A}$,*

$$\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\ p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} \in \Omega[q(0),1]\cap Q$$

*and*

$$\left(1-\frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1}+\frac{a_2(p+\lambda)}{\theta}\left(\frac{\mathcal{B}^{n+1,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}\right)^{a_1}\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1}$$

*is univalent in $U$ and satisfies the superordination.*

$$b(w)+\frac{a_2}{a_1}wb'(w)\prec\left(1-\frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1}+\frac{a_2(p+\lambda)}{\theta}\left(\frac{\mathcal{B}^{n+1,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}\right)^{a_1}\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1}$$

*then*

$$b(w)\prec\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1}.$$

**Proof.** Consider

$$q(w)=\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1},$$

then

$$q'(w)=a_1\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1-1}\left(\frac{w^p\left[\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)\right]'}{(w^p)^2}-\frac{pw^{p-1}\left(\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)\right)(w)}{(w^p)^2}\right).$$

We have

$$\frac{q'(w)}{q(w)}=a_1\left(\frac{\left[\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)\right]'}{\left(\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)\right)}-\frac{1}{w}\right),$$

with the same steps of Theorem 1 and using the hypothesis, we obtain

$$b(w)+\frac{a_2}{a_1}wb'(w)\prec\ b(w)+\frac{a_2}{a_1}wb'(w).$$

Apply Lemma 4 we obtain

$$b(w)\prec\left(\frac{\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)}{w^p}\right)^{a_1}.$$

□

**Corollary 3.** *Let $b(w)$ be convex in $U$, with $b(0)=1$, $a_1>0$, $Rea_2>0$, if $f\in\mathcal{A}$,*

$$\left(\frac{J^n_p(\theta,\lambda)f(w)}{w^p}\right)^{a_1}\in\Omega[q(0),1]\cap Q$$

*and*

$$\left(1-\frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{J^n_p(\theta,\lambda)f(w)}{w^p}\right)^{a_1}+\frac{a_2(p+\lambda)}{\theta}\left(\frac{J^{n+1}_p(\theta,\lambda)f(w)}{J^n_p(\theta,\lambda)f(w)}\right)^{a_1}\left(\frac{J^n_p(\theta,\lambda)f(w)}{w^p}\right)^{a_1}$$

*is univalent in U and satisfies the superordination*

$$b(w) + \tfrac{a_2}{a_1} w b'(w) \prec \left(1 - \tfrac{a_2(p+\lambda)}{\theta}\right)\left(\tfrac{J_p^n(\theta,\lambda)f(w)}{w^p}\right)^{a_1} + \tfrac{a_2(p+\lambda)}{\theta}\left(\tfrac{J_p^{n+1}(\theta,\lambda)f(w)}{J_p^n(\theta,\lambda)f(w)}\right)^{a_1}\left(\tfrac{J_p^n(\theta,\lambda)f(w)}{w^p}\right)^{a_1},$$

*then*

$$b(w) \prec \left(\frac{J_p^n(\theta,\lambda)f(w)}{w^p}\right)^{a_1}.$$

**Theorem 4.** *Let b be convex univalent in $b(0) = 1$ and $b(w) \neq 0$ for all $w \in U$ and suppose that b satisfies:*

$$Re\left\{\frac{t\sigma}{a_2} b'(w) + \frac{\varepsilon(\sigma+1)}{a_2} b(w) b'(w)\right\} > 0, \tag{10}$$

*where, $\varepsilon, t \in ¢, 0 \neq a_2 \in ¢^*$, $w \in U$, and $w(b(w))^{\sigma-1} b'(w)$ are all starlike univalent in U.*

*If $f \in \mathcal{A}$, satisfies the condition:*

$$\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right) + \left(1 - \frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1} \in \Omega[b(0),1] \bigcap Q,$$

*and $\mathcal{M}(p, n, \lambda, \theta, \varepsilon, a_1, a_2; w)$ is univalent in U.*

*If $(t + \varepsilon\, b(w))(b(w))^\sigma + a_2(b(w))^{\sigma-1} b'(w) \prec \mathcal{M}(\sigma, t, \varepsilon, h_\mu, \mu, a_1, a_2; w)$, then*

$$b(w) \prec \left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right) + \left(1 - \frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1}.$$

**Proof.** Let $H(\beta) = (t + \varepsilon\beta)\beta^\sigma$ and $L(\beta) = a_2(\beta)^{\sigma-1}$, $0 \neq \beta \in ¢$, when $H(\beta)$ is analytic in ¢ and $L(\beta) \neq 0$ is analytic in $¢/0$. Then, we obtain $G(w) = w^p b'(w) L(b(w)) = a_2 w^p (b(w))^{\sigma-1} b'(w)$. □

Since $w^p(b(w))^{\sigma-1} b'(w)$ is starlike, then $G(w)$ is starlike in $U$, and

$$Re\left(\frac{H'(b(w))}{L(b(w))}\right) = Re\left(\frac{\left[(t + \varepsilon(b(w)))(b(w))^\sigma\right]'}{a_2(b(w))^{\sigma-1}}\right) = Re\left\{\frac{t\sigma}{a_2} b'(w) + \frac{\varepsilon(\sigma+1)}{a_2} b(w) b'(w)\right\} > 0;$$

Now, let

$$q(w) = \left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right) + \left(1 - \frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1}.$$

From (8) we obtain

$$(t + \varepsilon\, b(w))(b(w))^\sigma + a_2(b(w))^{\sigma-1} b'(w) \prec (t + \varepsilon q(w))(q(w))^\sigma + a_2\, w(q(w))^{\sigma-1} q'(w).$$

Using Lemma 3 we obtain $b(w) \prec q(w)$.

**Corollary 4.** *Let b be convex univalent in U, $b(0) = 1$, and $b(w) \neq 0$ for all $w \in U$, and suppose that b satisfies:*

$$Re\left\{\frac{t\sigma}{a_2}b'(w) + \frac{\varepsilon(\sigma+1)}{a_2}b(w)b'(w)\right\}, > 0, \tag{11}$$

*where, $\varepsilon, t \in ¢, 0 \neq a_2 \in ¢^*$, $w \in U$, and $w(b(w))^{\sigma-1}b'(w)$ are starlike univalent in U.*

*Let $f \in \mathcal{A}$, satisfies the condition:*

$$\left(\frac{\frac{(p+\lambda)}{\theta}\left(J_p^{n+1}(\theta,\lambda)f(w)\right) + \left(1 - \frac{(p+\lambda)}{\theta}\right)\left(J_p^n(\theta,\lambda)f(w)\right)}{w^p}\right)^{a_1} \in \Omega[b(0),1]\bigcap Q,$$

*and $\mathcal{M}(p,n,\lambda,\theta,\varepsilon,a_1,a_2;w)$ is univalent in U.*

*If $(t + \varepsilon\, b(w))(b(w))^{\sigma} + a_2(b(w))^{\sigma-1}b'(w) \prec \mathcal{M}(\sigma,t,\varepsilon,h_\mu,\mu,a_1,a_2;w)$, then*

$$b(w) \prec \left(\frac{\frac{(p+\lambda)}{\theta}\left(J_p^{n+1}(\theta,\lambda)f(w)\right) + \left(1 - \frac{(p+\lambda)}{\theta}\right)\left(J_p^n(\theta,\lambda)f(w)\right)}{w^p}\right)^{a_1}.$$

## 4. Sandwich Results

By combining the above theories, we obtain the following two sandwich theories.

**Theorem 5.** *Let $b_1, b_2$ be convex univalent in U, with $b_1(0) = b_2(0) = 1$ $Rea_2 > 0$ and*

$$Re\left(1 + \frac{q''(w)}{q'(w)}\right) > \max\left\{0, -Re\frac{a_1}{a_2}\right\},$$

*where $a_1 > 0, 0 \neq a_2 \in ¢$.*

*If $f \in \mathcal{A}$ and*

$$\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} \in \Omega[1,1]\bigcap Q,$$

*and*

$$\left(1 - \frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} + \frac{a_2(p+\lambda)}{\theta}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)}{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}\right)^{a_1}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1}$$

*is univalent in U, it satisfies:*

$$b_1(w) + \frac{a_2}{a_1}wb'_1(w) \prec \left(1 - \frac{a_2(p+\lambda)}{\theta}\right)\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} + \frac{a_2(p+\lambda)}{\theta}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)}{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}\right)^{a_1}\left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} \prec b_2(w) + \frac{a_2}{a_1}wb'_2(w),$$

*then $b_1(w) \prec \left(\frac{\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)}{w^p}\right)^{a_1} \prec b_2(w)$.*

**Theorem 6.** *Let $b_1, b_2$ be convex univalent in U, with $b_1(0) = b_2(0) = 1$, and let $f \in \mathcal{A}$ satisfy the condition:*

$$\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}_{\theta,\lambda}^{n+1,\,p}(a,c,\delta)f(w)\right) + \left(1 - \frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}_{\theta,\lambda}^{n,\,p}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1} \in \Omega[1,1]\bigcap Q,$$

*and* $\mathcal{M}(p,n,\lambda,\theta,\varepsilon,a_1,a_2;w)$ *is univalent in* $U$.

*If*

$$(t+\varepsilon\, b_1(w))(b_1(w))^{\sigma}+a_2(b_1(w))^{\sigma-1}{b_1}'(w)\prec \mathcal{M}(p,n,\lambda,\theta,\varepsilon,a_1,a_2;w)$$
$$\prec (t+\varepsilon\, b_2(w))(b_2(w))^{\sigma}+a_2(b_2(w))^{\sigma-1}{b_2}'(w),$$

*then*

$$b_1(w)\prec\left(\frac{\frac{(p+\lambda)}{\theta}\left(\mathcal{B}^{n+1,\,p}_{\theta,\lambda}(a,c,\delta)f(w)\right)+\left(1-\frac{(p+\lambda)}{\theta}\right)\left(\mathcal{B}^{n,\,p}_{\theta,\lambda}(a,c,\delta)f(w)\right)}{w^p}\right)^{a_1}\prec b_2(w).$$

## 5. Conclusions

In this paper, using the convolution (or Hadamard product) we defined the El-Ashwah and Drbuk linear operator, which is a multivalent function in the unit disk $U$ and satisfied its specific relationship to derive the subordination, superordination, and some sandwich results for this operator using the properties of subordination and superordination concepts. The interesting results can be obtained for other operators using the same techniques of subordinations and superordinations.

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## References

1. Duren, P.L. Grundlehren der Mathematischen Wissenschaften. In *Univalent Functions*; Springer: New York, NY, USA; Berlin/Heidelberg, Germany; Tokyo, Japan, 1983; Volume 259.
2. El-Ashwah, R.M.; Kota, W.Y. Inclusion and Subordination Properties for Classes of Multivalent Functions Involving Differential-Integral Operator. *Inf. Sci. Lett.* **2022**, *11*, 11.
3. Miller, S.S.; Mocanu, P.T. Briot–Bouquet differential superordinations and sandwich theorems. *J. Math. Anal. Appl.* **2007**, *329*, 327–335. [CrossRef]
4. Cotîrlă, L.I.; Cătaş, A. A Differential Sandwich Theorem for Analytic Functions Defined by an Integral Operator. *Acta Univ. Apulensis* **2020**, *62*, 19–28.
5. Zayed, H.M.; Aouf, M.K.; Mostafa, A.O. Subordination properties for multivalent functions associated with a generalized fractional differential integral operator. *Axioms* **2018**, *7*, 27. [CrossRef]
6. El-Ashwah, R.M.; Drbuk, M.E. Subordination properties of p-valent functions defined by linear operators. *Br. J. Math. Comput. Sci.* **2014**, *21*, 3000–3013. [CrossRef]
7. Atshan, W.G.; Abbas, I.A.; Yalcin, S. New Concept on Fourth-Order Differential Subordination and Superordination with Some Results for Multivalent Analytic Functions. *J. Al-Qadisiyah Comput. Sci. Math.* **2020**, *12*, 96–107.
8. Al-Ameedee, S.A.; Atshan, W.G.; AL-Maamori, F.A.; Yalcin, S. Third-Order Sandwich Results of Multivalent Analytic Functions Involving Catas Operator. *An. Univ. Oradea Fasc. Mat.* **2021**, *28*, 111–129.
9. Prajapat, J. Subordination and superordination preserving properties for generalized multiplier transformation operator. *Math. Comput. Model.* **2012**, *55*, 1456–1465. [CrossRef]
10. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Pitman Research Notes in Mathematics Series; John Wiley & Sons, Inc.: New York, NY, USA, 1994; Volume 301.

11. Shanmugam, T.; Ravichandran, V.; Sivasubramanian, S. Differential sandwich theorems for subclasses of analytic functions. *Aust. J. Math. Anal. Appl.* **2006**, *3*, 1–11.
12. Miller, S.S.; Mocanu, P.T. Subordinates of differential superordinations. *Complex Var. Theory Appl.* **2003**, *48*, 815826.