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Some Properties for Subordinations of Analytic Functions

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Abstract: Let the class of functions of $f(z)$ of the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, which are denoted by $\mathcal{A}$ and called analytic functions in the open-unit disk. There are many interesting properties of the functions $f(z)$ in the class $\mathcal{A}$ concerning the subordinations. Applying the three lemmas for $f(z) \in \mathcal{A}$ provided by Miller and Mocanu and by Nunokawa, we consider many interesting properties of $f(z) \in \mathcal{A}$ with subordinations. Furthermore, we provide simple examples for our results. We think it is very important to consider examples of the results.

Keywords: analytic function; starlike function of order α; convex function of order α; subordination; differential subordinaton

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open-unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Given the two analytic functions $f(z)$ and $g(z)$, the function $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$ and written as $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$ analytic such that $f(z) = g(w(z))$ with $w(0) = 0$ and $|w(z)| \leq 1, z \in \mathbb{U}$. In particular, if $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subseteq g(\mathbb{U})$ (cf. [1,2]). We note that if $f(z) \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in \mathbb{U}) \tag{2}$$

for some real α $(0 \leq \alpha < 1)$, then $f(z)$ is said to be the starlike function of order α in $\mathbb{U}$ and, if $f(z) \in \mathcal{A}$ satisfies

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{1+(1-2\alpha)z}{1-z} \quad (z \in \mathbb{U}) \tag{3}$$

for some real α $(0 \leq \alpha < 1)$, then $f(z)$ is said to be the convex of order α in $\mathbb{U}$.

Furthermore, let $p(z)$ be analytic in $\mathbb{U}$ and $p(0) = 1$. Then, if $p(z)$ satisfies

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}) \tag{4}$$

for some real α $(\alpha > 0)$, then $p(z)$ satisfies

$$|argp(z)| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}). \tag{5}$$

If $f(z) \in \mathcal{A}$ satisfies

$$\left|argf'(z)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}) \tag{6}$$

for some real α $(0 < \alpha \leq 1)$, then we say that $f(z)$ is the strongly univalent function of order α in $\mathbb{U}$. If $f(z) \in \mathcal{A}$ satisfies

$$\left|arg\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}) \tag{7}$$

for some real α $(0 < \alpha \leq 1)$, then we say that $f(z)$ is the strongly starlike function of order α in $\mathbb{U}$. Further, if $f(z) \in \mathcal{A}$ satisfies

$$\left|arg\left(1 + \frac{zf''(z)}{f'(z)}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}) \tag{8}$$

for some real α $(0 < \alpha \leq 1)$, then we say that $f(z)$ is the strongly convex function of order α in $\mathbb{U}$ (cf. [2]).

2. Some Applications of Differential Subordinations

To consider some applications for subordinations, we introduce the following lemma from Miller and Mocanu [3].

Lemma 1. *Let $\beta_0 = 1.21872\ldots$ be the solution of $\beta\pi = \frac{3}{2}\pi - Tan^{-1}\beta$ and let $\alpha = \alpha(\beta) = \beta + 2Tan^{-1}\left(\frac{\beta}{\pi}\right)$ for $0 < \beta < \beta_0$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \; ; \; (z \in \mathbb{U}) \tag{9}$$

implies that

$$p(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \; ; \; (z \in \mathbb{U}). \tag{10}$$

Remark 1. *If $\beta = 1$ in Lemma 1, then $\alpha(1) = \frac{3}{2}$. Thus, Lemma 1 says that if the function $p(z)$ satisfies the following subordination:*

$$p(z) + zp'(z) \prec \left(\frac{1+z}{1-z}\right)^{\frac{3}{2}} \; ; \; (z \in \mathbb{U}) \tag{11}$$

then

$$p(z) \prec \frac{1+z}{1-z} \; ; \; (z \in \mathbb{U}). \tag{12}$$

Now, we prove the following theorem.

Theorem 1. *Let $\beta_0 = 1.21872\ldots$ be the solution of $\beta\pi = \frac{3}{2}\pi - Tan^{-1}\beta$ and let $\alpha = \alpha(\beta) = \beta + 2Tan^{-1}\left(\frac{\beta}{\pi}\right)$ for $0 < \beta < \beta_0$. If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$, then*

$$p(z) + zp'(z) \prec \gamma + (1-\gamma)\left(\frac{1+z}{1-z}\right)^{\alpha} \; ; \; (z \in \mathbb{U}) \tag{13}$$

implies that

$$p(z) \prec \gamma + (1-\gamma)\left(\frac{1+z}{1-z}\right)^{\beta} \; ; \; (z \in \mathbb{U}) \tag{14}$$

where $0 \leq \gamma < 1$.

Proof. Let us define a function $F(z)$ using

$$F(z) = \frac{p(z) - \gamma}{1 - \gamma} \ , \ (z \in \mathbb{U}). \tag{15}$$

Then, $F(z)$ is analytic in $\mathbb{U}$ with $F(0) = 1$ and

$$zF'(z) = \frac{zp'(z)}{1 - \gamma}. \tag{16}$$

Therefore, Lemma 1 implies that if

$$F(z) + zF'(z) = \frac{p(z) + zp'(z) - \gamma}{1 - \gamma} \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \ , \ (z \in \mathbb{U}) \tag{17}$$

then

$$F(z) = \frac{p(z) - \gamma}{1 - \gamma} \prec \left(\frac{1+z}{1-z}\right)^{\beta} \ , \ (z \in \mathbb{U}). \tag{18}$$

The subordination (17) implies (13) and the subordination (18) is the same as (14). □

Letting $\beta = 1$ in Theorem 1, we obtain the following corollary.

Corollary 1. *If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = 1$ satisfies*

$$p(z) + zp'(z) \prec \gamma + (1-\gamma)\left(\frac{1+z}{1-z}\right)^{\frac{3}{2}} \ ; \ (z \in \mathbb{U}) \tag{19}$$

for some real γ $(0 \leq \gamma < 1)$ then

$$p(z) \prec \frac{1 + (1-2\gamma)z}{1-z} \ ; \ (z \in \mathbb{U}) \tag{20}$$

and $Rep(z) > \gamma$ $(z \in \mathbb{U})$.

In Corollary 1, considering $p(z) = \frac{f(z)}{z}$ for the function $f(z)$ in the class $\mathcal{A}$, we have the following.

Corollary 2. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$f'(z) \prec \gamma + (1-\gamma)\left(\frac{1+z}{1-z}\right)^{\frac{3}{2}} \ ; \ (z \in \mathbb{U}) \tag{21}$$

for some real γ $(0 \leq \gamma < 1)$ then

$$\frac{f(z)}{z} \prec \frac{1 + (1-2\gamma)z}{1-z} \ ; \ (z \in \mathbb{U}) \tag{22}$$

and $Re\left(\frac{f(z)}{z}\right) > \gamma$ $(z \in \mathbb{U})$.

In Corollary 1, ensuring $p(z) = f'(z)$ for the function $f(z)$ in the class $\mathcal{A}$, we have the following.

Corollary 3. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$f'(z) + zf''(z) \prec \gamma + (1-\gamma)\left(\frac{1+z}{1-z}\right)^{\frac{3}{2}} \ ; \ (z \in \mathbb{U}) \tag{23}$$

for some real γ $(0 \leq \gamma < 1)$, *then*

$$f'(z) \prec \frac{1+(1-2\gamma)z}{1-z} \; ; \; (z \in \mathbb{U}) \tag{24}$$

and $Re(f'(z)) > \gamma$ $(z \in \mathbb{U})$.

Further, in Corollary 1, letting $p(z) = \frac{zf'(z)}{f(z)}$ for the function $f(z)$ in the class $\mathcal{A}$, we have the following corollary.

Corollary 4. *If the function* $f(z)$ *in the class* $\mathcal{A}$ *satisfies*

$$2\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2 \prec \gamma + (1-\gamma)\left(\frac{1+z}{1-z}\right)^{\frac{3}{2}} \; ; \; (z \in \mathbb{U}) \tag{25}$$

for some real γ $(0 \leq \gamma < 1)$, *then*

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\gamma)z}{1-z} \; ; \; (z \in \mathbb{U}) \tag{26}$$

and $f(z)$ *is the starlike function of order* γ *in* $\mathbb{U}$.

To consider the next problem, let $\mathcal{P}_n$ be the class of functions $p(z)$ that are analytic in $\mathbb{U}$ with

$$p(z) = 1 + \sum_{k=n}^{\infty} c_n z^n \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}). \tag{27}$$

For $p(z) \in \mathcal{P}_n$, Nunokawa [4,5] derives the following lemma.

Lemma 2. *Let a function* $p(z)$ *be in the class* $\mathcal{P}_n$. *If there exists a point* z_0 $(|z_0| < 1)$ *such that*

$$|arg(p(z))| < \frac{\pi}{2}\beta \quad (|z| < |z_0|) \tag{28}$$

and

$$|arg(p(z_0))| = \frac{\pi}{2}\beta \tag{29}$$

for some real $\beta > 0$, *then*

$$\frac{z_0 p'(z_0)}{p(z_0)} = \frac{2im}{\pi} arg(p(z_0)) \tag{30}$$

for some $m \geq \frac{n}{2}\left(a + \frac{1}{a}\right) > n$, *where*

$$(p(z_0))^{\frac{1}{\beta}} = \pm ia \quad (a > 0). \tag{31}$$

Applying the Lemma 2, we derive the following theorem.

Theorem 2. *If the function* $p(z)$ *in the class* $\mathcal{P}_n$ *satisfies*

$$\left\{\frac{p(z)-\alpha}{1-\alpha} + \frac{zp'(z)}{p(z)-\alpha}\right\}^2 - 1 \prec \frac{4(n+1)^2 z}{(1-z)^2} \; , \; (z \in \mathbb{U}) \tag{32}$$

for some real α $(0 \leq \alpha < 1)$, *then*

$$Re(p(z)) > \alpha \; , \; (z \in \mathbb{U}). \tag{33}$$

Proof. We suppose that there exists a point z_0 $(|z_0| < 1)$ such that

$$\mathrm{Re}\left\{\frac{p(z)-\alpha}{1-\alpha}\right\} > 0 \quad (|z| < |z_0| < 1) \tag{34}$$

and

$$\mathrm{Re}\left\{\frac{p(z_0)-\alpha}{1-\alpha}\right\} = 0. \tag{35}$$

If

$$\frac{p(z_0)-\alpha}{1-\alpha} \neq 0, \tag{36}$$

then Lemma 2 provides

$$\begin{aligned} \frac{z_0 p'(z_0)}{p(z_0)-\alpha} &= \frac{2im}{\pi} arg\left(\frac{p(z_0)-\alpha}{1-\alpha}\right) \\ &= \frac{2im}{\pi} arg(p(z_0)-\alpha) \end{aligned} \tag{37}$$

for some real $m \geq \frac{n}{2}\left(a + \frac{1}{a}\right) > n$ with

$$\left(\frac{p(z_0)-\alpha}{1-\alpha}\right)^{\frac{1}{\beta}} = \pm ia \;\; (a > 0). \tag{38}$$

It follows from the above that

$$\begin{aligned} \left\{\frac{p(z_0)-\alpha}{1-\alpha} + \frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right\}^2 - 1 &= (\pm ia \pm im)^2 - 1 \\ &\leq -\left(a + \frac{n(a^2+1)}{2a}\right)^2 - 1. \end{aligned} \tag{39}$$

We consider a function $h(a)$ provided by

$$h(a) = a + \frac{n(a^2+1)}{2a} \;\; (a > 0). \tag{40}$$

Then, $h(a)$ satisfies

$$h(a) \geq h\left(\sqrt{\frac{n}{n+2}}\right) = \sqrt{n(n+2)}. \tag{41}$$

This implies that

$$\begin{aligned} \left\{\frac{p(z_0)-\alpha}{1-\alpha} + \frac{z_0 p'(z_0)}{p(z_0)-\alpha}\right\}^2 - 1 &\leq -\left(a + \frac{n(a^2+1)}{2a}\right)^2 - 1 \\ &= -(n+1)^2. \end{aligned} \tag{42}$$

On the other hand, we consider a function $g(z)$ provided by

$$g(z) = \frac{4(n+1)^2 z}{(1-z)^2} \;\; (z \in \mathbb{U}). \tag{43}$$

The function $g(z)$ maps $\mathbb{U}$ onto the domain with the slit $(-\infty, -(n+1)^2)$. This contradicts our condition (31). Therefore, we have that

$$\mathrm{Re}\left(\frac{p(z)-\alpha}{1-\alpha}\right) > 0 \tag{44}$$

for all $z \in \mathbb{U}$. This shows us that

$$\mathrm{Re}(p(z)) > \alpha \ , \ (z \in \mathbb{U}). \tag{45}$$

□

Considering $p(z) = \frac{f(z)}{z}$ for the function $f(z)$ in the class $\mathcal{A}$, we have the following corollary.

Corollary 5. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$\left\{ \frac{f(z) - \alpha}{z(1-\alpha)} + \frac{zf'(z) - f(z)}{f(z) - z} \right\}^2 - 1 \prec \frac{16z}{(1-z)^2} \ ; \ (z \in \mathbb{U}) \tag{46}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Re\frac{f(z)}{z} > \alpha \ ; \ (z \in \mathbb{U}). \tag{47}$$

Causing $p(z) = f'(z)$ for the function $f(z)$ in the class $\mathcal{A}$, thus we obtain the following corollary.

Corollary 6. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$\left\{ \frac{f'(z) - \alpha}{1-\alpha} + \frac{zf''(z)}{f'(z) - \alpha} \right\}^2 \prec \frac{16z}{(1-z)^2} \ ; \ (z \in \mathbb{U}) \tag{48}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Ref'(z) > \alpha \ ; \ (z \in \mathbb{U}). \tag{49}$$

Using $p(z) = \frac{zf'(z)}{f(z)}$ for the function $f(z)$ in the class $\mathcal{A}$, we have the following corollary.

Corollary 7. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$\left\{ \frac{1}{f(z)} \left(\frac{zf'(z) - \alpha f(z)}{1-\alpha} + \frac{z(f(z)f'(z) + zf(z)f''(z) - z(f'(z))^2)}{zf'(z) - \alpha f(z)} \right) \right\}^2 - 1 \prec \frac{16z}{(1-z)^2} \tag{50}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Re\left(\frac{zf'(z)}{f(z)} \right) > \alpha \ ; \ (z \in \mathbb{U}). \tag{51}$$

Next, we derive the following theorem.

Theorem 3. *If the function $p(z)$ in the class $\mathcal{P}_n$ satisfies*

$$\frac{p(z) - \alpha}{1-\alpha} + \frac{zp'(z)}{p(z) - \alpha} \prec \frac{1+z}{1-z} \ , \ (z \in \mathbb{U}) \tag{52}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Re(p(z)) > \alpha \ , \ (z \in \mathbb{U}). \tag{53}$$

Proof. We consider that there exists a point z_0 $(|z_0| < 1)$ such that

$$\mathrm{Re}\left\{ \frac{p(z) - \alpha}{1-\alpha} \right\} > 0 \ \ (|z| < |z_0| < 1) \tag{54}$$

and

$$\mathrm{Re}\left\{\frac{p(z_0)-\alpha}{1-\alpha}\right\}=0. \tag{55}$$

If

$$\frac{p(z_0)-\alpha}{1-\alpha}\neq 0, \tag{56}$$

using Lemma 2 we have

$$\frac{z_0p'(z_0)}{p(z_0)-\alpha}=\frac{2im}{\pi}arg(p(z_0)-\alpha) \tag{57}$$

for some real $m\geq\frac{n}{2}\left(a+\frac{1}{a}\right)>n$ with

$$\frac{p(z_0)-\alpha}{1-\alpha}=\pm ia\ (a>0). \tag{58}$$

This provides

$$\begin{aligned}\frac{p(z_0)-\alpha}{1-\alpha}+\frac{z_0p'(z_0)}{p(z_0)-\alpha}&=\pm ia\pm im\\&=\pm i(a+m).\end{aligned} \tag{59}$$

Noting that

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right)>0\ \ (z\in\mathbb{U}), \tag{60}$$

we say that

$$\frac{p(z)-\alpha}{1-\alpha}+\frac{zp'(z)}{p(z)-\alpha}\ \textit{is not subordinate to}\ \frac{1+z}{1-z}\ \ (z\in\mathbb{U}). \tag{61}$$

Therefore, there is no z_0 $(|z_0|<1)$ as in (34) and (35). This implies that

$$\left(\frac{p(z)-\alpha}{1-\alpha}\right)>0\ \ (z\in\mathbb{U}) \tag{62}$$

that is

$$\mathrm{Re}(p(z))>\alpha\ \ ,\ (z\in\mathbb{U}). \tag{63}$$

□

Example 1. *Let us consider a function $p(z)$ provided by*

$$p(z)=\frac{1}{1-z}\in\mathcal{P}_1 \tag{64}$$

and $\alpha=0$. Then, $p(z)$ satisfies

$$p(z)+\frac{zp'(z)}{p(z)}=\frac{1+z}{1-z}\ \ ,\ (z\in\mathbb{U}). \tag{65}$$

Thus, $p(z)$ satisfies the subordination (52) for $\alpha=0$. For such $p(z)$, we have that

$$Re(p(z))>\frac{1}{2}>0\ \ ,\ (z\in\mathbb{U}). \tag{66}$$

Corollary 8. *If the function $p(z)$ in the class $\mathcal{P}_n$ satisfies*

$$\left| arg\left(\frac{p(z)-\alpha}{1-\alpha} + \frac{zp'(z)}{p(z)-\alpha} \right) \right| < \frac{\pi}{2} \ , \ (z \in \mathbb{U}) \tag{67}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Re(p(z)) > \alpha \ , \ (z \in \mathbb{U}). \tag{68}$$

Corollary 9. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$\frac{f'(z)-\alpha}{1-\alpha} + \frac{zf''(z)}{f'(z)-\alpha} \prec \frac{1+z}{1-z} \ ; \ (z \in \mathbb{U}) \tag{69}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Ref'(z) > \alpha \ ; \ (z \in \mathbb{U}). \tag{70}$$

Corollary 10. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$\frac{f(z)}{z} + \frac{zf'(z)}{f(z)} - 1 \prec \frac{1+z}{1-z} \ ; \ (z \in \mathbb{U}) \tag{71}$$

for some real α $(0 \leq \alpha < 1)$, then

$$Re\left(\frac{f(z)}{z}\right) > 0 \ ; \ (z \in \mathbb{U}). \tag{72}$$

3. Applications of Miller–Mocanu Lemma

In this section, we would like to apply the Miller–Mocanu lemma [1,6] (also from Jack [7]).

Lemma 3. *Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0) = 0$. Then, if $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in \mathbb{U}$, then we have*

$$z_0 w'(z_0) = m w(z_0) \tag{73}$$

and

$$Re\left(1 + \frac{z_0 w''(z_0)}{w'(z_0)}\right) \geq m \tag{74}$$

where $m \geq 1$.

Theorem 4. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$\frac{f(z)}{z} \prec \frac{\alpha(1+z)}{\alpha + (2-\alpha)z} \ ; \ (z \in \mathbb{U}) \tag{75}$$

for some real α $(\alpha > 1)$, then

$$\left| \frac{f(z)}{z} - \frac{\alpha}{2} \right| < \frac{\alpha}{2} \ ; \ (z \in \mathbb{U}). \tag{76}$$

Proof. Let us define a function $w(z)$ using

$$\frac{f(z)}{z} = \frac{\alpha(1+w(z))}{\alpha + (2-\alpha)w(z)} \ ; \ (z \in \mathbb{U}). \tag{77}$$

Then, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$. Letting

$$g(z) = \frac{f(z)}{z}, \tag{78}$$

we see that

$$|w(z)| = \left| \frac{\alpha(g(z) - 1)}{\alpha - (2 - \alpha)g(z)} \right| < 1\ ;\ (z \in \mathbb{U}). \tag{79}$$

It follows from (79) that

$$2|g(z)|^2 - \alpha(g(z) + \overline{g(z)}) < 0\ ;\ (z \in \mathbb{U}) \tag{80}$$

and that

$$\left| g(z) - \frac{\alpha}{2} \right| < \frac{\alpha}{2}\ ;\ (z \in \mathbb{U}). \tag{81}$$

□

Next, we have the following theorem.

Theorem 5. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$f'(z) \prec \frac{\alpha(1 + z)}{\alpha + (2 - \alpha)z}\ ;\ (z \in \mathbb{U}) \tag{82}$$

for some real α $(\alpha > 1)$, then

$$\left| f'(z) - \frac{\alpha}{2} \right| < \frac{\alpha}{2}\ ;\ (z \in \mathbb{U}). \tag{83}$$

Proof. Considering a function $w(z)$ such that

$$f'(z) = \frac{\alpha(1 + w(z))}{\alpha + (2 - \alpha)w(z)}\ ;\ (z \in \mathbb{U}), \tag{84}$$

we prove the theorem. □

Remark 2. *The inequality (76) implies that*

$$0 < Re\left(\frac{f(z)}{z}\right) < \alpha\ ;\ (z \in \mathbb{U}) \tag{85}$$

and the inequality (83) implies that

$$0 < Ref'(z) < \alpha\ ;\ (z \in \mathbb{U}). \tag{86}$$

The following theorem is our next result.

Theorem 6. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{\alpha - 1}{2\delta}\ ;\ (z \in \mathbb{U}) \tag{87}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{zf'(z)}{f(z)}\right) < 1 + \frac{1}{2\delta(\alpha - 1)}\ ;\ (z \in \mathbb{U}) \tag{88}$$

for some real α $(\alpha > 2)$, then

$$\left| \left(\frac{f(z)}{z} \right)^{\delta} - \frac{\alpha}{2} \right| < \frac{\alpha}{2} \; ; \; (z \in \mathbb{U}) \tag{89}$$

where $0 < \delta \leq 1$.

Proof. We define a function $w(z)$ provided by

$$\left(\frac{f(z)}{z} \right)^{\delta} = \frac{\alpha(1 + w(z))}{\alpha + (2 - \alpha)w(z)} \; ; \; (z \in \mathbb{U}) \tag{90}$$

for $0 < \delta \leq 1$. Then, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$. This function $w(z)$ satisfies

$$\frac{zf'(z)}{f(z)} - 1 = \frac{zw'(z)}{\delta w(z)} \left(\frac{w(z)}{1 + w(z)} - \frac{(2 - \alpha)w(z)}{\alpha + (2 - \alpha)w(z)} \right). \tag{91}$$

Suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0| = 1. \tag{92}$$

Then, Lemma 3 shows us that

$$z_0 w'(z_0) = m w(z_0) \;\; (m \geq 1) \tag{93}$$

and $w(z_0) = e^{i\theta}$ $(0 \leq \theta < 2\pi)$. It follows from the above that

$$\begin{aligned} \mathrm{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)} \right) &= \mathrm{Re}\left(1 + \frac{m}{\delta} \left(\frac{e^{i\theta}}{1 + e^{i\theta}} - \frac{(2 - \alpha)e^{i\theta}}{\alpha + (2 - \alpha)e^{i\theta}} \right) \right) \\ &= 1 + \frac{m}{\delta} \left(\frac{1}{2} + \frac{(\alpha - 2)(2 - \alpha + \alpha cos\theta)}{\alpha^2 + (2 - \alpha)^2 + 2\alpha(2 - \alpha)cos\theta} \right). \end{aligned} \tag{94}$$

We consider a function $g(t)$ provided by

$$g(t) = \frac{2 - \alpha + \alpha t}{\alpha^2 + (2 - \alpha)^2 + 2\alpha(2 - \alpha)t} \;\; (t = cos\theta). \tag{95}$$

It follows from (95) that

$$g'(t) = \frac{4\alpha(\alpha - 1)}{(\alpha^2 + (2 - \alpha)^2 + 2\alpha(2 - \alpha)t)^2} > 0 \tag{96}$$

for $\alpha > 1$. Since $g(t)$ is increasing for $t = cos\theta$, we have

$$\mathrm{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)} \right) \geq 1 + \frac{m}{\delta} \left(\frac{1}{2} + \frac{\alpha - 2}{2} \right) \geq 1 + \frac{\alpha - 1}{2\delta} \tag{97}$$

for $1 < \alpha \leq 2$ and

$$\mathrm{Re}\left(\frac{z_0 f'(z_0)}{f(z_0)} \right) \geq 1 + \frac{m}{\delta} \left(\frac{1}{2} - \frac{\alpha - 2}{2(\alpha - 1)} \right) \geq 1 + \frac{1}{2\delta(\alpha - 1)} \tag{98}$$

for $\alpha > 2$. Thus, inequalities (97) and (98) contradict the conditions (87) and (88). Therefore, we say that there is no $w(z)$ such that $w(0) = 0$ and $|w(z_0)| = 1$ for $z_0 \in \mathbb{U}$. This implies that $|w(z)| < 1$ for all $z \in \mathbb{U}$, that is

$$|w(z)| = \left| \frac{\alpha\left(\left(\frac{f(z)}{z}\right)^{\delta} - 1\right)}{\alpha - (2-\alpha)\left(\frac{f(z)}{z}\right)^{\delta}} \right| < 1\ ;\ (z \in \mathbb{U}). \tag{99}$$

This completes the proof of the theorem. □

Using $\delta = 1$ in Theorem 6, we have the following corollary.

Corollary 11. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{\alpha+1}{2}\ ;\ (z \in \mathbb{U}) \tag{100}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{2\alpha-1}{2(\alpha-1)}\ ;\ (z \in \mathbb{U}) \tag{101}$$

for some real α $(\alpha > 2)$, then

$$\left|\frac{f(z)}{z} - \frac{\alpha}{2}\right| < \frac{\alpha}{2}\ ;\ (z \in \mathbb{U}). \tag{102}$$

Letting $\delta = \frac{1}{2}$, we have the following corollary.

Corollary 12. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \alpha\ ;\ (z \in \mathbb{U}) \tag{103}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{\alpha}{\alpha-1}\ ;\ (z \in \mathbb{U}) \tag{104}$$

for some real α $(\alpha > 2)$, then

$$\left|\sqrt{\frac{f(z)}{z}} - \frac{\alpha}{2}\right| < \frac{\alpha}{2}\ ;\ (z \in \mathbb{U}). \tag{105}$$

Theorem 7. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha-1}{2\delta}\ ;\ (z \in \mathbb{U}) \tag{106}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1}{2\delta(\alpha-1)}\ ;\ (z \in \mathbb{U}) \tag{107}$$

for some real α $(\alpha > 2)$, then

$$\left|(f'(z))^{\delta} - \frac{\alpha}{2}\right| < \frac{\alpha}{2}\ ;\ (z \in \mathbb{U}) \tag{108}$$

where $0 < \delta \leq 1$.

Proof. Let us consider a function $w(z)$ provided by

$$\left(f'(z)\right)^{\delta} = \frac{\alpha(1+w(z))}{\alpha+(2-\alpha)w(z)} \;;\; (z \in \mathbb{U}) \tag{109}$$

for $0 < \delta \leq 1$. Then, $w(z)$ is analytic in $\mathbb{U}$ with $w(0) = 0$ and satisfies

$$\frac{zf''(z)}{f'(z)} = \frac{zw'(z)}{\delta w(z)}\left(\frac{w(z)}{1+w(z)} - \frac{(2-\alpha)w(z)}{\alpha+(2-\alpha)w(z)}\right). \tag{110}$$

Therefore, applying Lemma 3 as the proof of Theorem 6, we prove the theorem. □

Using $\delta = 1$, we have the following corollary.

Corollary 13. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{\alpha-1}{2} \;;\; (z \in \mathbb{U}) \tag{111}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1}{2(\alpha-1)} \;;\; (z \in \mathbb{U}) \tag{112}$$

for some real α $(\alpha > 2)$, then

$$\left|f'(z) - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \;;\; (z \in \mathbb{U}). \tag{113}$$

Example 2. *We consider a function $f(z) \in \mathcal{A}$ provided by*

$$f(z) = \frac{\alpha}{2-\alpha}\left(z + \frac{2(1-\alpha)}{2-\alpha} log\left(1 + \frac{2-\alpha}{\alpha} z\right)\right). \tag{114}$$

Then, we see

$$f'(z) = \frac{\alpha(1+z)}{\alpha+(2-\alpha)z} \tag{115}$$

and

$$f'(z) - \frac{\alpha}{2} = \frac{\alpha((2-\alpha)+\alpha z)}{2(\alpha+(2-\alpha)z)}. \tag{116}$$

It follows from (116) that

$$\left|f'(z) - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \;;\; (z \in \mathbb{U}). \tag{117}$$

On the other hand, $f(z)$ implies that

$$\frac{zf''(z)}{f'(z)} = \frac{z}{1+z} - \frac{(2-\alpha)z}{\alpha+(2-\alpha)z}. \tag{118}$$

Thus, $f(z)$ satisfies the conditions (111) and (112) of Corollary 13.

Causing $\delta = \frac{1}{2}$ in Theorem 7, we have the following corollary.

Corollary 14. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(\frac{zf''(z)}{f'(z)}\right) < \alpha - 1 \;;\; (z \in \mathbb{U}) \tag{119}$$

for some real α $(1 < \alpha \leq 2)$ *or*

$$Re\left(\frac{zf''(z)}{f'(z)}\right) < \frac{1}{\alpha - 1} \ ; \ (z \in \mathbb{U}) \tag{120}$$

for some real α $(\alpha > 2)$, *then*

$$\left|\sqrt{f'(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \ ; \ (z \in \mathbb{U}). \tag{121}$$

Further, we obtain the following theorem.

Theorem 8. *If the function* $f(z)$ *in the class* $\mathcal{A}$ *satisfies*

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{\alpha - 1}{2\delta} \ ; \ (z \in \mathbb{U}) \tag{122}$$

for some real α $(1 < \alpha \leq 2)$ *or*

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{1}{2\delta(\alpha - 1)} \ ; \ (z \in \mathbb{U}) \tag{123}$$

for some real α $(\alpha > 2)$, *then*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^{\delta} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \ ; \ (z \in \mathbb{U}) \tag{124}$$

where $0 < \delta \leq 1$.

Letting $\delta = 1$, we obtain the following corollary.

Corollary 15. *If the function* $f(z)$ *in the class* $\mathcal{A}$ *satisfies*

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{\alpha - 1}{2} \ ; \ (z \in \mathbb{U}) \tag{125}$$

for some real α $(1 < \alpha \leq 2)$ *or*

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{1}{2(\alpha - 1)} \ ; \ (z \in \mathbb{U}) \tag{126}$$

for some real α $(\alpha > 2)$, *then*

$$\left|\frac{zf'(z)}{f(z)} - \frac{\alpha}{2}\right| < \frac{\alpha}{2} \ ; \ (z \in \mathbb{U}). \tag{127}$$

Example 3. *We consider a function* $f(z) \in \mathcal{A}$ *provided by*

$$f(z) = z(\alpha + (2 - \alpha)z)^{\frac{2(\alpha-1)}{2-\alpha}} \ , \ \ (\alpha \neq 2). \tag{128}$$

It follows from (128) that

$$\frac{zf'(z)}{f(z)} = \frac{\alpha(1 + z)}{\alpha + (2 - \alpha)z} \tag{129}$$

and

$$1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} = \frac{z}{1 + z} - \frac{(2 - \alpha)z}{\alpha + (2 - \alpha)z}. \tag{130}$$

With (130), we know that $f(z)$ satisfies the inequalities (125) and (126). Furthermore, by (129) we see that $f(z)$ satisfies the inequality (127).

Letting $\delta = \frac{1}{2}$ in Theorem 8, we have the following corollary.

Corollary 16. *If the function $f(z)$ in the class $\mathcal{A}$ satisfies*

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \alpha - 1 \ ; \ (z \in \mathbb{U}) \tag{131}$$

for some real α $(1 < \alpha \leq 2)$ or

$$Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) < \frac{1}{\alpha - 1} \ ; \ (z \in \mathbb{U}) \tag{132}$$

for some real α $(\alpha > 2)$, then

$$\left| \sqrt{\frac{zf'(z)}{f(z)}} - \frac{\alpha}{2} \right| < \frac{\alpha}{2} \ ; \ (z \in \mathbb{U}). \tag{133}$$

In addition to our results given above, we can add the following:

In Theorem 3, we prove that if $p(z) \in \mathcal{P}_n$ satisfies the subordination (52), then $p(z)$ satisfies the inequality (53). We know that

$$\mathrm{Re}\left(\frac{1+z}{1-z}\right) > 0 \ ; \ (z \in \mathbb{U}) \tag{134}$$

and

$$\frac{1+e^{i\theta}}{1-e^{i\theta}} = icot\frac{\theta}{2} \ ; \ (0 \leq \theta < 2\pi). \tag{135}$$

Furthermore, Equation (59) implies that

$$\mathrm{Re}\left\{ \frac{p(z_0) - \alpha}{1 - \alpha} + \frac{z_0 p'(z_0)}{p(z_0) - \alpha} \right\} = 0 \ ; \ (z \in \mathbb{U}). \tag{136}$$

Thus, we see that

$$\left| arg\left\{ \frac{p(z_0) - \alpha}{1 - \alpha} + \frac{z_0 p'(z_0)}{p(z_0) - \alpha} \right\}^{\beta} \right| > \left| arg\left(\frac{1+z}{1-z}\right)^{\gamma} \right| \tag{137}$$

for some real β and γ $(0 < \beta \leq \gamma \leq 2)$.

With the above comment, we derive the following theorem.

Theorem 9. *If $p(z) \in \mathcal{P}_n$ satisfies*

$$\left(\frac{p(z) - \alpha}{1 - \alpha} + \frac{zp'(z)}{p(z) - \alpha}\right)^{\beta} \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \ , \ (z \in \mathbb{U}) \tag{138}$$

for some real α $(0 \leq \alpha < 1)$ and for some real β and γ $(0 < \beta \leq \gamma \leq 2)$, then

$$Re(p(z)) > \alpha \ , \ (z \in \mathbb{U}). \tag{139}$$

Corollary 17. *If the function $p(z)$ in the class $\mathcal{P}_n$ satisfies*

$$\left| arg\left(\frac{p(z) - \alpha}{1 - \alpha} + \frac{zp'(z)}{p(z) - \alpha}\right)^{\beta} \right| < \frac{\pi}{2}\beta \ , \ (z \in \mathbb{U}) \tag{140}$$

for some real α $(0 \leq \alpha < 1)$ *and* β $(0 < \beta \leq 2)$, *then*

$$Re(p(z)) > \alpha \ , \ (z \in \mathbb{U}). \tag{141}$$

Corollary 18. *If the function* $f(z)$ *in the class* $\mathcal{A}$ *satisfies*

$$\left| arg\left(\frac{f'(z) - \alpha}{1 - \alpha} + \frac{zf''(z)}{f'(z) - \alpha} \right)^{\beta} \right| < \frac{\pi}{2}\beta \ ; \ (z \in \mathbb{U}) \tag{142}$$

for some real α $(0 \leq \alpha < 1)$ *and* β $(0 < \beta \leq 2)$, *then*

$$Ref'(z) > \alpha \ ; \ (z \in \mathbb{U}). \tag{143}$$

Example 4. *We consider a function* $f(z) \in \mathcal{A}$ *provided by*

$$f(z) = log\left(\frac{1}{1 - z} \right). \tag{144}$$

Then, we have that

$$\begin{aligned} Re\left(\frac{f'(z) - \alpha}{1 - \alpha} + \frac{zf''(z)}{f'(z) - \alpha} \right) &= Re\left(\frac{1}{1 - \alpha}\left(\frac{1}{1 - z} - \alpha \right) \right) \\ &> \frac{1 - 2\alpha}{1 - \alpha} \geq 0 \ ; \ (z \in \mathbb{U}) \end{aligned} \tag{145}$$

with α $(0 \leq \alpha \leq \frac{1}{2})$. *This provides*

$$\left| arg\left(\frac{f'(z) - \alpha}{1 - \alpha} + \frac{zf''(z)}{f'(z) - \alpha} \right)^{\beta} \right| < \frac{\pi}{2}\beta \ ; \ (z \in \mathbb{U}) \tag{146}$$

with α $(0 \leq \alpha \leq \frac{1}{2})$ *and* β $(0 < \beta \leq 2)$. *Furthermore, we have that*

$$Ref'(z) = Re\left(\frac{1}{1 - z} \right) > \frac{1}{2} \geq \alpha \ ; \ (z \in \mathbb{U}). \tag{147}$$

4. Conclusions

There are many interesting properties of functions $f(z)$ that are analytic in the open-unit disk concerning subordinations. In this paper, we consider many interesting properties of $f(z)$ that are analytic in the open-unit disk with subordinations by applying the three lemmas for $f(z)$ provided by Miller and Mocanu and by Nunokawa. Furthermore, we provide simple examples for our results since we think it is very important to consider examples of the obtained results.

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