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An Improved Intuitionistic Fuzzy Decision-Theoretic Rough Set Model and Its Application

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Abstract: The Decision-Theoretic Rough Set model stands as a compelling advancement in the realm of rough sets, offering a broader scope of applicability. This approach, deeply rooted in Bayesian theory, contributes significantly to delineating regions of minimal risk. Within the Decision-Theoretic Rough Set paradigm, the universal set undergoes a tripartite division, where distinct regions emerge and losses are intelligently distributed through the utilization of membership functions. This research endeavors to present an enhanced and more encompassing iteration of the Decision-Theoretic Rough Set framework. Our work culminates in the creation of the Generalized Intuitionistic Decision-Theoretic Rough Set (GI-DTRS), a fusion that melds the principles of Decision-Theoretic Rough Sets and intuitionistic fuzzy sets. Notably, this synthesis bridges the gaps that exist within the conventional approach. The innovation lies in the incorporation of an error function tailored to the hesitancy grade inherent in intuitionistic fuzzy sets. This integration harmonizes seamlessly with the contours of the membership function. Furthermore, our methodology deviates from established norms by constructing similarity classes based on similarity measures, as opposed to relying on equivalence classes. This shift holds particular relevance in the context of aggregating information systems, effectively circumventing the challenges associated with the process. To demonstrate the practical efficacy of our proposed approach, we delve into a concrete experiment within the information technology domain. Through this empirical exploration, the real-world utility of our approach becomes vividly apparent. Additionally, a comprehensive comparative analysis is undertaken, juxtaposing our approach against existing techniques for aggregation and decision modeling. The culmination of our efforts is a well-rounded article, punctuated by the insights, recommendations, and future directions delineated by the authors.

Keywords: intuitionistic fuzzy sets; decision-theoretic fuzzy rough set model; three-way decision model; decision making; efficiency; optimization

MSC: 03E72; 94D05

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1. Introduction

Vagueness is a critical problem that many scientists are currently working on [1]. The discovery of effective knowledge from ambiguous data has become a major area of research [2]. To address this challenge, several techniques have been developed for identifying uncertain information, such as fuzzy set (FS) theory [3], quotient space theory [4], and rough set theory (RST) [5]. These theories aim to handle issues arising from ambiguity and uncertainty.

The three-way decision (3WD) model, introduced by Yao, is an extension of RST that focuses on uncertain classification problems [6–8]. Using a set of thresholds, 3WD divides the universe into three distinct zones: acceptance, deferment, and rejection [9]. Therefore, three-way decision theory has become a valuable tool for solving challenging problems in various fields [10–12]. For example, in medical diagnosis, clinicians often have three options based on a patient's symptoms: treat the patient, do nothing, or observe the patient longer [13]. In email filtering [14], texts are assigned to corresponding topics within the framework of text classification [15]. Wang et al. [16] have worked on the application of 3WD based on probabilistic dominance relations under IF data. Chen et al. [17] have proposed a novel model of three-way decision where the relationship between attributes plays a significant role. Additionally, great progress has been made in related fields [18–21].

Decision-theoretic rough sets (DTRSs), an expanded version of RS in the light of the Bayesian decision technique, have significantly enhanced three-way decisions [22]. DTRSs were proposed by Yao et al. [20,23], and the approach includes rational decision semantics and reflects relevant risks. TWD with DTRSs is obtained from the minimum of the total risk. Zhang et al. [24] have developed a technique for ranking the alternatives based on DTRSs. Zhao et al. [25] have explored the area of 3WD with DTRSs in multiset-valued information tables. Qian et al. [26] have expanded the above-mentioned concept to the multiorganization of DTRSs. Liang et al. [27] have presented an approach for dual hesitant fuzzy sets to establish three-way decisions with decision-theoretic rough sets. Moreover, Liu et al. [28] have designed a new model of the environment of the q-rung orthopair fuzzy rough set, which establishes three-way decisions with decision-theoretic rough sets. Liu et al. [29] have contributed to the field by adding probabilistic model criteria with DTRSs. By considering the modern trend of research to connect and combine different theories, Liu et al. [18] have presented the notion of fuzzy data with three-way decision-theoretic rough sets, and Ali et al. [19] have worked on the DTRSs with single-valued neutrosophic data. Additionally, several models and approaches from experts for DTRSs have been described [23,24,28,30].

Based on the research reported in [31–35], intuitionistic fuzzy sets (IFSs) have emerged as a valuable evaluation format for describing uncertainty. IFSs, as described by Atanassov [36,37], are a more useful concept for representing uncertainty than fuzzy sets due to their duality characteristic, which is expressed by both a membership degree and a non-membership degree. IFSs are primarily utilized in conjunction with intuitionistic fuzzy numbers (IFNs) for decision-making purposes [38–40]. For example, Senapati et al. [41] proposed a novel aggregation operator to rank alternatives based on multiple attributes of intuitionistic fuzzy information. Gohain et al. [42] used similarity measures for IFSs to solve various problems, while Singh et al. [43] combined the theory of IFS and RS to offer intuitionistic fuzzy rough sets and their application. Liang et al. [44,45] developed the notion of intuitionistic decision-theoretic rough sets (DTRS) by fusing the idea of three-way decision-theoretic rough sets with intuitionistic fuzzy sets and demonstrated its application with examples. Researchers have explored the 3WD approach of DTRS with various applications [12,13,46–49].

Motivation for This Study

Qinghua et al. [50] proposed a novel model to connect the theories of DTRS and IFS. They proposed a 3WD and sequential 3WD model with intuitionistic fuzzy numbers, which reshapes the 3WD theory based on the Bayesian theory of risk and loss function. In this approach, the author considered cost parameters as well as attribute values of a universal set based on intuitionistic fuzzy numbers. They investigated the sequential 3WD model with IFNs by considering both the IF indices (membership and non-membership indices). While the theory offers a broad range of applications, there are some areas where it could be improved. Our motivation in this paper is to improve upon this approach and preserve the genuine nature of the DTRS theory. To achieve this, we propose a better and expanded version of the theory of Generalized Intuitionistic Decision-Theoretic Rough Sets (GI-DTRS). Our contributions are as follows:

i We will generalize the concept of a three-way decision based on a decision-theoretic rough set for intuitionistic fuzzy numbers.
ii We will employ similarity classes instead of equivalence classes, which expands the scope of the approach.
iii We will retain the concept of conditional probabilities, which was ignored in [50], from the parental theory of the decision-theoretic rough set [22].
iv We will use our proposed approach to show the validity and effectiveness of solving real-life issues. For this purpose, we will discuss the model of an electronic device for the special person and use the proposed approach for taking decisions.
v We will deeply discuss the comparative analysis of the developed model and some existing techniques and show our preference for the mentioned approach.

The rest of this article is distributed as follows: Section 2 contains a brief review of IFS, 3WD theory, and a decision-theoretic rough set of Yao [22]. Section 3 consists of an explained review of the existing model of a 3WD with a decision-theoretic model with IFNs presented by Qinghua et al. [50]; in Section 4, the novel model of generalized IFN-based DTRS (GI−DTRS) is established. In Section 5, the practical and real-life use of gadgets is prepared based on the proposed approach for decision-making, which shows the validity and confirmation of results. Finally, Section 6 concluded the comments and future plans of the author.

2. Preliminaries

In this section, the theories and structures of the IFSs, 3WD, and DTRS models are briefly reviewed. Table 1 is added to show the symbols and their descriptions.

Table 1. Symbols and Descriptions.

Symbols	Description	Symbols	Description
IFSs	Intuitionistic Fuzzy Sets	DTRS	Decision-Theoretic Rough Sets
IFNs	Intuitionistic Fuzzy Numbers	DRs	Decision Rules
3WD	Three-Way Decision	TWDM	Three-Way Decision Making

2.1. IFSs: A Brief Overview

Intuitionistic fuzzy sets [36] can be viewed as a powerful tool for indicating hesitancy involving both membership and non-membership of a component of a set [0, 1]. To be more precise, intuitionistic fuzzy sets need not adhere to the fundamental tenet of the FS model that states that if we choose a real integer from [0, 1] to represent the degree of membership of an element in a fuzzy collection, say a, then the degree of its non-membership is automatically determined as $1 - a$. It is assumed in the IFS model that non-membership should not exceed $1 - a$. The detailed concepts are discussed below.

Definition 1 ([36]). *An intuitionistic fuzzy set Nis defined over a universal set X as follows*:

$$N = \{(e,\ \alpha_N(e), \beta_N(e))|\alpha_N(e) + \beta_N(e) \leq 1\ \forall\ e\ \in\ X\}$$

where $\alpha_N(e) : X \to [0,1]$ and $\beta_N(e) : X \to [0,1]$ represent the grades of membership and non-membership of e to N, respectively.

The noteworthy property here is that the total of membership and non-membership grades is less than or equal to 1 and greater than or equal to 0. The hesitation grade for e to N is defined as $\psi_N(e) = 1 - \alpha_N(e) - \beta_N(e)$. For brevity, an IFN can be written as $N(e) = (\alpha_N(e), \beta_N(e))$.

Definition 2. *Let $N(e_1) = (\alpha_N(e_1), \beta_N(e_1))$ and $N(e_2) = (\alpha_N(e_2), \beta_N(e_2))$ be two IFNs, the following are some basic operations defined over them.*

(1) $N(e_1) \oplus N(e_2) = (\alpha_N(e_1) + \alpha_N(e_2) - \alpha_N(e_1)\alpha_N(e_2),\ \beta_N(e_1)\ \beta_N(e_2))$;
(2) $N(e_1) \otimes N(e_2) = (a_N(e_1)\alpha_N(e_2),\ \beta_N(e_1) + \beta_N(e_2) - \beta_N(e_1)\beta_N(e_2)\ )$;
(3) $\mathscr{k}N(e_1) = 1 - (1 - a_N(e_1))^{\mathscr{k}}\ , (\beta_N(e_1))^{\mathscr{k}})$; where $\mathscr{k}$ is a scalar.
(4) $N^c(e_1) = (\beta_N(e_1), a_N(e_1))$.

2.2. 3WD Based on Rough Set-Theory and DTRS Model

Yao [8] introduced a three-way decision model, and this model consists of two basic tasks. A trisection, or tri-partition, of the universal set is one task that involves dividing the set into three pair-wise disjoint areas. The secondary target is to use the right techniques to respond to situations in one or more regions. A collection of efficient techniques and strategies known as 3WD are frequently applied in human information processing and problem-solving. To partition the universal set, Yao designed the approximation classes and three regions based on a rough set.

We begin this section by reviewing the fundamental ideas related to rough sets and 3WD.

Definition 3 ([7])**.** *An information system $I = (X,\ \mathscr{A}t, \mathscr{V}l, \mathscr{M})$ consisting of a non-empty finite set X, a set of attributes $\mathscr{A}t$, a set of attribute values $\mathscr{V}l$, and $\mathscr{M} : X \to \mathscr{V}l$. An indiscernibility relation $\mathscr{I}_C$ where $C \subseteq \mathscr{A}t$, is defined as follows:*

$$\mathscr{I}_C = \left\{(e, m) : (e, m) \in X^2\ for\ every\ b \in C\ (b(e) = b(m))\right\}$$

$\mathscr{I}_C$ is an equivalence relation that generates the partition $X/\mathscr{I}_C = \left\{[e]_{\mathscr{I}_C} \middle| e \in X\right\}$. Here, $[e]_{\mathscr{I}_C}$ is an equivalence class of e. The equivalence class $[e]_{\mathscr{I}_C}$ is abbreviated to $[e]$.

Definition 4. *Given an information system $I = (X,\ \mathscr{A}t, \mathscr{V}l, \mathscr{M})$ and a subset of attributes $C \subseteq \mathscr{A}t$, the structural positive, negative, and boundary zones of a concept $\Omega \subseteq X$ are respectively described by:*

$$POS_C(\Omega) = \{[e]_C \in X/\mathscr{I}_C : [e]_C \subseteq \Omega\}$$
$$NEG_C(\Omega) = \{[e]_C \in X/\mathscr{I}_C : [e]_C \subseteq \Omega'\ \}$$
$$BND_C(\Omega) = \{[e]_C \in X/\mathscr{I}_C : \neg([e]_C \subseteq \Omega) \wedge \neg([e]_C \subseteq \Omega')\}$$

where Ω' denotes the complement of Ω.

Definition 5 ([22])**.** *For an information system $I = (X,\ \mathscr{A}t, \mathscr{V}l, \mathscr{M})$ and a subset $C \subseteq \mathscr{A}t$, the rough membership function for $\Omega \subseteq X$ is a mapping $\delta_\Omega^{\mathscr{I}_C} : X \to [0, 1]$ defined by*

$$\delta_\Omega^{\mathscr{I}_C}(e) = \mathscr{P}r(\Omega|[e]\,) = \frac{|\Omega \cap [e]|}{|[e]|} \text{ for all } e \in X,$$

where, $\mathscr{P}r(\Omega|[e]\,)$ is the conditional probability of classification. Here, $|\ .\ |$ indicates the order of the set.

DTRS is a famous 3WD model based on Bayesian decision theory that minimizes the risk of several decisions [20]. The result is similar to hypothesis testing in statistics. A hypothesis is accepted if there is convincing evidence supporting it, rejected if there is convincing evidence refuting it, and neither accepted nor rejected but needs to be further evaluated if there is no convincing evidence supporting or refuting it. The interpretation justifies three-way decision-making based on the risk or cost of different decisions. It needs an understanding of the cost of acquiring and applying evidence.

The 3WDM with DTRS theory [20] is succinctly explained here. It begins with a set of states $(\Omega, -\Omega)$ designating, respectively, that components are in Ω and not in Ω. In both of these states, a series of actions is taken as $\mathscr{A}c = \{\mathscr{a}_{\mathrm{P}}, \mathscr{a}_{\mathrm{B}}, \mathscr{a}_{\mathrm{N}}\}$, where $\mathscr{a}_{\mathrm{P}}$, $\mathscr{a}_{\mathrm{B}}$ and $\mathscr{a}_{\mathrm{N}}$, respectively, represent the classification of an object e's acceptance $(\mathrm{e} \in \mathscr{P}os(\Omega))$, deferment $(\mathrm{e} \in \mathscr{B}nd(\Omega))$, and rejection $(\mathrm{e} \in \mathscr{N}eg(\Omega))$ decision. The positive region $\mathscr{P}os(\Omega)$, boundary region $\mathscr{B}nd(\Omega)$, and negative region $\mathscr{N}eg(\Omega)$ are three disjoint regions. Moreover, as indicated in Table 2, a matrix $\mathscr{M} = \{\xi_{\sigma\tau}\}_{3\times 2}$ $(\sigma = \mathrm{P, B, N}, \textit{ and } \tau = \mathrm{P, N})$ provides the cost parameters. The costs associated with the actions $\mathscr{a}_{\mathscr{P}}$, $\mathscr{a}_{\mathscr{B}}$, and $\mathscr{a}_{\mathscr{N}}$ when an element goes to Ω are $\xi_{\mathscr{P}\mathscr{P}}$, $\xi_{\mathscr{B}\mathscr{P}}$, and $\xi_{\mathscr{N}\mathscr{P}}$. However, the expenses for the corresponding three actions are denoted by $\xi_{\mathscr{P}\mathscr{N}}$, $\xi_{\mathscr{B}\mathscr{N}}$, and $\xi_{\mathscr{N}\mathscr{N}}$ when an item does not belong to Ω. The classification losses $\mathbb{R}(\mathscr{a}_{\sigma}|[\mathrm{e}])$ associated with the three actions are expressed as follows:

$$\mathbb{R}(\mathscr{a}_{\mathscr{P}}|[\mathrm{e}]) = \xi_{\mathscr{P}\mathscr{P}}\mathscr{P}r(\Omega|[e]) + \xi_{\mathscr{P}\mathscr{N}}\mathscr{P}r(\neg\Omega|[e])$$

$$\mathbb{R}(\mathscr{a}_{\mathscr{B}}|[\mathrm{e}]) = \xi_{\mathscr{B}\mathscr{P}}\mathscr{P}r(\Omega|[e]) + \xi_{\mathscr{B}\mathscr{N}}\mathscr{P}r(\neg\Omega|[e])$$

$$\mathbb{R}(\mathscr{a}_{\mathscr{N}}|[\mathrm{e}]) = \xi_{\mathscr{N}\mathscr{P}}\mathscr{P}r(\Omega|[e]) + \xi_{\mathscr{N}\mathscr{N}}\mathscr{P}r(\neg\Omega|[e])$$

Table 2. Cost parameter matrix.

Actions\States	$\mathbf{\Omega}$	$-\mathbf{\Omega}$
$\mathscr{a}_{\mathscr{P}}$	$\xi_{\mathscr{P}\mathscr{P}}$	$\xi_{\mathscr{P}\mathscr{N}}$
$\mathscr{a}_{\mathscr{B}}$	$\xi_{\mathscr{B}\mathscr{P}}$	$\xi_{\mathscr{B}\mathscr{N}}$
$\mathscr{a}_{\mathscr{N}}$	$\xi_{\mathscr{N}\mathscr{P}}$	$\xi_{\mathscr{N}\mathscr{N}}$

For minimum-loss decisions, the DTRS theory presents the following decision rules:

(1) If $\mathbb{R}(\mathscr{a}_{\mathrm{P}}|[\mathrm{e}]) \leq \mathbb{R}(\mathscr{a}_{\mathrm{B}}|[\mathrm{e}])$ and $\mathbb{R}(\mathscr{a}_{\mathrm{P}}|[\mathrm{e}]) \leq \mathbb{R}(\mathscr{a}_{\mathrm{N}}|[\mathrm{e}])$, then $\mathrm{e} \in \mathscr{P}os(\Omega)$.
(2) If $\mathbb{R}(\mathscr{a}_{\mathrm{B}}|[\mathrm{e}]) \leq \mathbb{R}(\mathscr{a}_{\mathrm{P}}|[\mathrm{e}])$ and $\mathbb{R}(\mathscr{a}_{\mathrm{B}}|[\mathrm{e}]) \leq \mathbb{R}(\mathscr{a}_{\mathrm{N}}|[\mathrm{e}])$, then $\mathrm{e} \in \mathscr{B}nd(\Omega)$.
(3) If $\mathbb{R}(\mathscr{a}_{\mathrm{N}}|[\mathrm{e}]) \leq \mathbb{R}(\mathscr{a}_{\mathrm{P}}|[\mathrm{e}])$ and $\mathbb{R}(\mathscr{a}_{\mathrm{N}}|[\mathrm{e}]) \leq \mathbb{R}(\mathscr{a}_{\mathrm{B}}|[\mathrm{e}])$, then $\mathrm{e} \in \mathscr{N}eg(\Omega)$.

Given the prerequisites of $\xi_{\mathscr{P}\mathscr{P}} \leq \xi_{\mathscr{B}\mathscr{P}} \leq \xi_{\mathscr{N}\mathscr{P}}, \xi_{\mathscr{N}\mathscr{N}} \leq \xi_{\mathscr{B}\mathscr{N}} \leq \xi_{\mathscr{P}\mathscr{N}}$, and $\mathscr{P}r(\Omega|[e]) + \mathscr{P}r(-\Omega|[e]) = 1$, the rules $1-3$ can be refined as $(\mathscr{T}\mathscr{P}) - (\mathscr{T}\mathscr{N})$ using two thresholds μ and ν $(0 \leq \nu < \mu \leq 1)$ as below:

$$(\mathscr{T}\mathscr{P}) \text{ If } \mathscr{P}r(\Omega|[e]) \geq \mu, \text{ then } e \in \mathscr{P}os(\Omega).$$

$$(\mathscr{T}\mathscr{B}) \text{ If } \nu < \mathscr{P}r(\Omega|[e]) < \mu, \text{ then } e \in \mathscr{B}nd(\Omega).$$

$$(\mathscr{T}\mathscr{N}) \text{ If } \mathscr{P}r(\Omega|[e]) \leq \nu, \text{ then } e \in \mathscr{N}eg(\Omega),$$

where, $\mu = \frac{\xi_{\mathscr{P}\mathscr{N}} - \xi_{\mathscr{B}\mathscr{N}}}{(\xi_{\mathscr{P}\mathscr{N}} - \xi_{\mathscr{B}\mathscr{N}}) + (\xi_{\mathscr{B}\mathscr{P}} - \xi_{\mathscr{P}\mathscr{P}})}$, $\nu = \frac{\xi_{\mathscr{B}\mathscr{N}} - \xi_{\mathscr{N}\mathscr{N}}}{(\xi_{\mathscr{B}\mathscr{N}} - \xi_{\mathscr{N}\mathscr{N}}) + (\xi_{\mathscr{N}\mathscr{P}} - \xi_{\mathscr{B}\mathscr{P}})}$.

3. 3WD Based on DTRS Model with IFNs: Existing Model

The three-way decision is a very famous and realistic theory that has gained the attention of many scholars. However, Qinghua et al. [50] extended the model of 3WD under DTRS based on IFNs.

This section presents a comprehensive revision of the said approach [50]. Certain shortcomings in the available approach are also highlighted. This section will lead to the development of the next section, where a novel approach to designing DTRS with IFNs is proposed. Membership, hesitation, and non-membership degrees are the three characteristics used by IFSs to define ambiguous notions. The membership grade is always seen as a degree of supporting an object whose aspect is ambiguous, and the non-membership grade is a degree of opposing an object subject to a particular concept when dealing with intuitionistic fuzzy environments. The index that remains after deducting the membership degree and non-membership degree from one is called the hesitation margin. This parameter depicts the neutrality of an idea. Utilizing the three indices, Qinghua et al. [50] designed IFNs-based DTRS as below.

The intuitionistic fuzzy information system $IFI = (X, \mathscr{A}t, I\mathscr{V}l, \mathscr{M})$ consists of a non-empty finite set X, a collection of attributes $\mathscr{A}t$ and a collection of IF attribute values $I\mathscr{V}l$. Thus, $I\mathscr{V}l$ contains all IFNs $N(e) = (\alpha_N(e), \beta_N(e))$. Moreover, $\mathscr{M} : X \to I\mathscr{V}l$ is used to assign IFNs to the elements of X. Furthermore, an IF cost parameter matrix is taken as $IM = \{N(\xi_{\sigma\tau}) = (\alpha_N(\xi_{\sigma\tau}), \beta_N(\xi_{\sigma\tau}))\}_{3\times 2}(\sigma = \mathscr{P}, \mathscr{B}, \mathscr{N}, \textit{ and } \tau = \mathscr{P}, \mathscr{N})$ is shown in Table 3. Distinct from the costs $\xi_{\sigma\tau}$ in matrix M, IFNs $N(\xi_{\sigma\tau})$ are the results of the cost parameters in IM. To present a superior analysis, $N(\xi_{\mathscr{P}\mathscr{P}})$ which is expressed by the membership degree $\alpha_N(\xi_{\mathscr{P}\mathscr{P}})$ and the non-membership degree $\beta_N(\xi_{\mathscr{P}\mathscr{P}})$, is the cost if an object takes $a_{\mathscr{P}}$ lying in the positive zone.

Table 3. Intuitionistic Fuzzy Cost Parameter Matrix.

Actions\States	Ω	$-\Omega$
$a_{\mathscr{P}}$	$N(\xi_{\mathscr{P}\mathscr{P}}) = (\alpha_N(\xi_{\mathscr{P}\mathscr{P}}), \beta_N(\xi_{\mathscr{P}\mathscr{P}}))$	$N(\xi_{\mathscr{P}\mathscr{N}}) = (\alpha_N(\xi_{\mathscr{P}\mathscr{N}}), \beta_N(\xi_{\mathscr{P}\mathscr{N}}))$
$a_{\mathscr{B}}$	$N(\xi_{\mathscr{B}\mathscr{P}}) = (\alpha_N(\xi_{\mathscr{B}\mathscr{P}}), \beta_N(\xi_{\mathscr{B}\mathscr{P}}))$	$N(\xi_{\mathscr{B}\mathscr{N}}) = (\alpha_N(\xi_{\mathscr{B}\mathscr{N}}), \beta_N(\xi_{\mathscr{B}\mathscr{N}}))$
$a_{\mathscr{N}}$	$N(\xi_{\mathscr{N}\mathscr{P}}) = ((\xi_{\mathscr{N}\mathscr{P}}), \beta_N(\xi_{\mathscr{N}\mathscr{P}}))$	$N(\xi_{\mathscr{N}\mathscr{N}}) = (\alpha_N(\xi_{\mathscr{N}\mathscr{N}}), \beta_N(\xi_{\mathscr{N}\mathscr{N}}))$

The cost parameters meet the relationships stated below:

$$\begin{aligned}
&\alpha_N(\xi_{\mathscr{P}\mathscr{P}}) < \alpha_N(\xi_{\mathscr{B}\mathscr{P}}) < \alpha_N(\xi_{\mathscr{N}\mathscr{P}}) \\
&\beta_N(\xi_{\mathscr{N}\mathscr{P}}) < \beta_N(\xi_{\mathscr{B}\mathscr{P}}) < \beta_N(\xi_{\mathscr{P}\mathscr{P}}) \\
&\alpha_N(\xi_{\mathscr{N}\mathscr{N}}) < \alpha_N(\xi_{\mathscr{B}\mathscr{N}}) < \alpha_N(\xi_{\mathscr{P}\mathscr{N}}) \\
&\beta_N(\xi_{\mathscr{P}\mathscr{N}}) < \beta_N(\xi_{\mathscr{B}\mathscr{N}}) < \beta_N(\xi_{\mathscr{N}\mathscr{N}})
\end{aligned} \tag{1}$$

The classification losses of e based on IFNs are shown as follows:

$$\begin{aligned}
\mathbb{R}(a_{\mathscr{P}}|[\mathrm{e}]) &= N(\xi_{\mathscr{P}\mathscr{P}})\alpha(e) \oplus N(\xi_{\mathscr{P}\mathscr{N}})\beta(e) \\
\mathbb{R}(a_{\mathscr{B}}|[\mathrm{e}]) &= N(\xi_{\mathscr{B}\mathscr{P}})\alpha(e) \oplus N(\xi_{\mathscr{B}\mathscr{N}})\beta(e) \\
\mathbb{R}(a_{\mathscr{N}}|[\mathrm{e}]) &= N(\xi_{\mathscr{N}\mathscr{P}})\alpha(e) \oplus N(\xi_{\mathscr{N}\mathscr{N}})\beta(e)
\end{aligned} \tag{2}$$

Because $\alpha(e) + \beta(e) + \psi(e) = 1$, losses in (2) can be written as

$$\begin{aligned}
\mathbb{R}(a_{\mathscr{P}}|[\mathrm{e}]) &= N(\xi_{\mathscr{P}\mathscr{P}})\alpha(e) \oplus N(\xi_{\mathscr{P}\mathscr{N}})(1 - \beta(e) - \psi(e)) \\
\mathbb{R}(a_{\mathscr{B}}|[\mathrm{e}]) &= N(\xi_{\mathscr{B}\mathscr{P}})\alpha(e) \oplus N(\xi_{\mathscr{B}\mathscr{N}})(1 - \beta(e) - \psi(e)) \\
\mathbb{R}(a_{\mathscr{N}}|[\mathrm{e}]) &= N(\xi_{\mathscr{N}\mathscr{P}})\alpha(e) \oplus N(\xi_{\mathscr{N}\mathscr{N}})(1 - \beta(e) - \psi(e))
\end{aligned} \tag{3}$$

Next, based on Bayesian decision theory, the minimum-loss decision rules are as follows:

(1) If $\alpha(\mathbb{R})_{\mathscr{P}} \leq \alpha(\mathbb{R})_{\mathscr{B}}$ and $\alpha(\mathbb{R})_{\mathscr{P}} \leq \alpha(\mathbb{R})_{\mathscr{N}}$, then $e \in \mathscr{P}os(\Omega)$
(2) If $\alpha(\mathbb{R})_{\mathscr{B}} \leq \alpha(\mathbb{R})_{\mathscr{P}}$ and $\alpha(\mathbb{R})_{\mathscr{B}} \leq \alpha(\mathbb{R})_{\mathscr{N}}$ then $e \in \mathscr{B}nd(\Omega)$
(3) If $\alpha(\mathbb{R})_{\mathscr{N}} \leq \alpha(\mathbb{R})_{\mathscr{P}}$ and $\alpha(\mathbb{R})_{\mathscr{N}} \leq \alpha(\mathbb{R})_{\mathscr{B}}$ then $e \in \mathscr{N}eg(\Omega)$

Utilizing Equations (1) and (3), the above decision rules can be rewritten as $(P1) - (N1)$.

$$(P1) \text{ If } \alpha(e) \geq \theta_1 \text{ and } \alpha(e) \geq \sigma_1, \text{ then } e \in \mathscr{P}os(\Omega)$$

$$(B1) \text{ If } \alpha(e) \leq \theta_1 \text{ and } \alpha(e) \geq \tau_1, \text{ then } e \in \mathscr{B}nd(\Omega)$$

$$(N1)\text{ If } \alpha(e) \leq \omega_1 \text{ and } \alpha(e) \leq \sigma_1,\text{ then } e \in \mathscr{N}eg(\Omega),$$

where,

$$\theta_1 = (1-\psi(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}\times\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{4}$$

$$\sigma_1 = (1-\psi(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}\times\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{5}$$

$$\tau_1 = (1-\psi(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}\times\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right)} \tag{6}$$

The decision rules (DRs) $(P1)-(N1)$ have so far been characterized by utilizing three thresholds $\theta_1(e)$, $\sigma_1(e)$ and $\tau_1(e)$ from the membership degree perspective. Moreover, DRs (4)–(6) from the perspective of non-membership degree are defined in [50] as below:

(4) If $\beta(\mathbb{R})_{\mathscr{P}} \geq \beta(\mathbb{R})_{\mathscr{B}}$ and $\beta(\mathbb{R})_{\mathscr{P}} \geq \beta(\mathbb{R})_{\mathscr{N}}$, then $e \in \mathscr{P}os(\Omega)$
(5) If $\beta(\mathbb{R})_{\mathscr{B}} \geq \beta(\mathbb{R})_{\mathscr{P}}$ and $\beta(\mathbb{R})_{\mathscr{B}} \geq \beta(\mathbb{R})_{\mathscr{N}}$ then $e \in \mathscr{B}nd(\Omega)$
(6) If $\beta(\mathbb{R})_{\mathscr{N}} \geq \beta(\mathbb{R})_{\mathscr{P}}$ and $\beta(\mathbb{R})_{\mathscr{N}} \geq \beta(\mathbb{R})_{\mathscr{B}}$ then $e \in \mathscr{N}eg(\Omega)$

Because $\beta(e) = 1-\alpha(e)-\psi(e)$, the thresholds in decision rules $(P1)-(N1)$ can be derived based on non-membership. Utilizing Equation (3), the DRs can thus be re-stated as below: Obviously, the DRs are basically modified as $(P2)-(N2)$.

$$(P2)\text{ If } \alpha(e) \geq \theta_2 \text{ and } \alpha(e) \geq \sigma_2,\text{ then } e \in \mathscr{P}os(\Omega)$$

$$(B2)\text{ If } \alpha(e) \leq \theta_2 \text{ and } \alpha(e) \geq \tau_2,\text{ then } e \in \mathscr{B}nd(\Omega)$$

$$(N2)\text{ If } \alpha(e) \leq \tau_2 \text{ and } \alpha(e) \leq \sigma_2,\text{ then } e \in \mathscr{N}eg(\Omega),$$

where,

$$\theta_2 = (1-\psi(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}\times\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{7}$$

$$\sigma_2 = (1-\psi(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}\times\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{8}$$

$$\tau_2 = (1-\psi(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}\times\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}\right)} \tag{9}$$

Qinghua et al. [50] thus devised the following 3WD-making rules using the IF environment:

(7) If $\alpha(e) \geq \theta_j$, then take $a_{\mathscr{P}}$
(8) If $\tau_j < \alpha(e) < \theta_j$, then take $a_{\mathscr{B}}$
(9) If $\alpha(e) \leq \tau_j$, then take $a_{\mathscr{N}}$.

4. Generalized Intuitionistic Fuzzy-Based DTRS (GI-DTRS) Model

Extending the concept of DTRS via IFNs in [50], many important components have been ignored or replaced with unjustified notions. Below, we briefly discuss these issues and show how the novel approach proposed in this paper helps to address them.

4.1. Some Concerns in the Existing IF-Based DTRS Model

i The clustering of elements via equivalence classes is a restrictive condition. Stating differently, for two elements to be in the same cluster, their feature values in all the features should be exactly similar. Even when all other feature values are the same, a slight variation in one feature value may cause two components to be in distinct clusters. Relaxing this restriction, we introduce similarity classes in the DTRS model. The threshold is determined by how much similarity between the elements is required.

ii The essence of DTRS lies in defining the conditional probabilities of elements for the given concept. In [22], these conditional probabilities have been replaced with intuitionistic fuzzy degrees. Probabilities and fuzzy degrees are totally different concepts that cannot be interchanged. Probability describes how likely an event is to occur, while fuzzy and IF degrees are linguistic information-based concepts used to manage partial truths. To retain the true essence of DTRS theory, we use conditional probabilities defined by Yao in [22]. These probabilities are a generalization of the equivalence-class-based conditional probabilities.

iii The classical DTRS starts with the set of states that are to be approximated. These states are actually subsets of the universe. In [50], these states are considered external components that have no link with the universe. As a result, the theory outlined in [22] significantly deviates from the fundamental idea of DTRS. On the other hand, we nevertheless adhere to the classical approach's interpretation of the concept of states. This makes our model more reliable.

4.2. Generalized DTRS Based on IFNs

In this section, the novel approach GI-DTRS is described, which is very efficient as compared to the existing approach.

In IFS [36], there are membership grades, non-membership grades, and hesitancy grades, which show the position of an element of the universal set in a close 0,1 interval with the condition that the total sum is 0. At this point, we use the conditional probability as a membership grade [22], the complement of probability as a non-membership grade, and the error value as a hesitancy grade. Moreover, the total probability, complement of probability, and error value are 1.

$$\mathscr{P}r + \mathscr{P}r' + \Delta(e) = 1$$

Additionally, it is necessary to partition the information table for 3WD. To fulfill this requirement, equivalence classes play a vital role. But, in this article, we utilized the cosine similarity measure [51] and obtained similarity classes instead of equivalence classes to partition the information system. Based on the above thought, a novel approach of 3W-DTRS under the environment of IFNs is designed and explained in detail.

Similarity measures are important instruments for determining the similarity degree between two elements. In literature, there are many kinds of similarity measures. We straight-forwardly utilized the cosine similarity measure in this article.

Definition 6 ([51])**.** *The cosine similarity measure proposed by Ye et al. for IFNs* (L, M)

$$\mathbb{S}_e(L,M) = \frac{1}{m}\sum_{i=1}^{m} \frac{(\alpha_L(e_i)\alpha_M(e_i) + \beta_L(e_i)\beta_M(e_i))}{\sqrt{\alpha_L^2(e_i) + \beta_L^2(e_i)}\sqrt{\alpha_M^2(e_i) + \beta_M^2(e_i)}} \tag{10}$$

After utilizing the novel approach, the DTRS model will change, and Equation (2) can be written as the classification losses of e are presented as below:

$$\begin{aligned}\mathbb{R}(\mathscr{a}_{\mathscr{P}}|\mathbb{S}_e) &= N(\xi_{\mathscr{P}\mathscr{P}})\mathscr{P}r(\Omega|\mathbb{S}_e) \oplus N(\xi_{\mathscr{P}\mathscr{N}})\mathscr{P}r(\neg\Omega|\mathbb{S}_e)\\ \mathbb{R}(\mathscr{a}_{\mathscr{B}}|\mathbb{S}_e) &= N(\xi_{\mathscr{B}\mathscr{P}})\mathscr{P}r(\Omega|\mathbb{S}_e) \oplus N(\xi_{\mathscr{B}\mathscr{N}})\mathscr{P}r(\neg\Omega|\mathbb{S}_e)\\ \mathbb{R}(\mathscr{a}_{\mathscr{N}}|\mathbb{S}_e) &= N(\xi_{\mathscr{N}\mathscr{P}})\mathscr{P}r(\Omega|\mathbb{S}_e) \oplus N(\xi_{\mathscr{N}\mathscr{N}})\mathscr{P}r(\neg\Omega|\mathbb{S}_e)\end{aligned} \tag{11}$$

Because $\mathscr{P}r(\Omega|\mathbb{S}_e) + \mathscr{P}r(\neg\Omega|\mathbb{S}_e) + \Delta(e) = 1$, gives (11) can be written as

$$\begin{aligned}\mathbb{R}(\mathscr{a}_{\mathscr{P}}|\mathbb{S}_e) &= N(\xi_{\mathscr{P}\mathscr{P}})\mathscr{P}r(\Omega|\mathbb{S}_e) \oplus N(\xi_{\mathscr{P}\mathscr{N}})[1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)]\\ \mathbb{R}(\mathscr{a}_{\mathscr{B}}|\mathbb{S}_e) &= N(\xi_{\mathscr{B}\mathscr{P}})\mathscr{P}r(\Omega|\mathbb{S}_e) \oplus N(\xi_{\mathscr{B}\mathscr{N}})[1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)]\\ \mathbb{R}(\mathscr{a}_{\mathscr{N}}|\mathbb{S}_e) &= N(\xi_{\mathscr{N}\mathscr{P}})\mathscr{P}r(\Omega|\mathbb{S}_e) \oplus N(\xi_{\mathscr{N}\mathscr{N}})[1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)\end{aligned} \tag{12}$$

By Definition 1 and the concept of DTRS, Equation (12) helps with the classification losses, which are defined below.

$$\mathbb{R}(\mathscr{a}_{\mathscr{P}}|\mathbb{S}_e) = \Big[1-(1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}}))^{\mathscr{P}r(\Omega|\mathbb{S}_e)}(1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}}))^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)},\ \beta_N(\xi_{\mathscr{P}\mathscr{P}})^{\mathscr{P}r(\Omega|\mathbb{S}_e)}\beta_N(\xi_{\mathscr{P}\mathscr{N}})^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}\Big]$$

$$\mathbb{R}(\mathscr{a}_{\mathscr{B}}|\mathbb{S}_e) = \Big[1-(1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}}))^{\mathscr{P}r(\Omega|\mathbb{S}_e)}(1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}}))^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)},\ \beta_N(\xi_{\mathscr{B}\mathscr{P}})^{\mathscr{P}r(\Omega|\mathbb{S}_e)}\beta_N(\xi_{\mathscr{B}\mathscr{N}})^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}\Big]$$

$$\mathbb{R}(\mathscr{a}_{\mathscr{B}}|\mathbb{S}_e) = \Big[1-(1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}}))^{\mathscr{P}r(\Omega|\mathbb{S}_e)}(1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}}))^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)},\ \beta_N(\xi_{\mathscr{B}\mathscr{P}})^{\mathscr{P}r(\Omega|\mathbb{S}_e)}\beta_N(\xi_{\mathscr{B}\mathscr{N}})^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}\Big]$$

Let $\alpha(\mathbb{R})_\sigma = 1-(1-\alpha_N(\xi_{\sigma\mathscr{P}}))^{\mathscr{P}r(\Omega|\mathbb{S}_e)}(1-\alpha_N(\xi_{\sigma\mathscr{N}}))^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}$ and $\beta(\mathbb{R})_\sigma = \beta_N(\xi_{\sigma\mathscr{P}})^{\mathscr{P}r(\Omega|\mathbb{S}_e)}\beta_N(\xi_{\sigma\mathscr{N}})^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}$
here $\sigma = \mathscr{P},\ \mathscr{B},\ \mathscr{N}$.

This classification losses $\alpha(\mathbb{R})_\sigma$ and $\beta(\mathbb{R})_\sigma$ are respectively determined from the membership and non-membership degrees of the cost parameter, respectively. Further, based on Bayesian decision theory, the new DRs are clearly examined by $\alpha(\mathbb{R})_\sigma$ with respect to the minimum-loss classifications.

(10) If $\alpha(\mathbb{R})_{\mathscr{P}} \le \alpha(\mathbb{R})_{\mathscr{B}}$ and $\alpha(\mathbb{R})_{\mathscr{P}} \le \alpha(\mathbb{R})_{\mathscr{N}}$, then $e \in \mathscr{P}os(\Omega)$
(11) If $\alpha(\mathbb{R})_{\mathscr{B}} \le \alpha(\mathbb{R})_{\mathscr{P}}$ and $\alpha(\mathbb{R})_{\mathscr{B}} \le \alpha(\mathbb{R})_{\mathscr{N}}$ then $e \in \mathscr{B}nd(\Omega)$
(12) If $\alpha(\mathbb{R})_{\mathscr{N}} \le \alpha(\mathbb{R})_{\mathscr{P}}$ and $\alpha(\mathbb{R})_{\mathscr{N}} \le \alpha(\mathbb{R})_{\mathscr{B}}$ then $e \in \mathscr{N}eg(\Omega)$

According to classification losses, if $\alpha_N(\mathbb{R})_{\mathscr{P}} \le \alpha_N(\mathbb{R})_{\mathscr{B}}$, then

$$\begin{aligned}&\ln\Big[(1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}}))^{\mathscr{P}r(\Omega|\mathbb{S}_e)}(1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}}))^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}\Big]\\ &\ge \ln\Big[(1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}}))^{\mathscr{P}r(\Omega|\mathbb{S}_e)}(1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}}))^{1-\mathscr{P}r(\Omega|\mathbb{S}_e)-\Delta(e)}\Big]\end{aligned}$$

By (1)

$$\mathscr{P}r(\Omega|\mathbb{S}_e) \ge (1-\Delta(e))\frac{\ln\Big[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}\Big]}{\ln\Big(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})} * \frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\Big)}$$

Similarly

$$\begin{aligned}&\alpha_N(\mathbb{R})_{\mathscr{P}} \le \alpha_N(\mathbb{R})_{\mathscr{N}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e)\\ &\qquad \ge (1-\Delta(e))\frac{\ln\Big[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\Big]}{\ln\Big(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})} * \frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\Big)}\end{aligned}$$

$$\alpha_N(\mathbb{R})_{\mathscr{B}} \leq \alpha_N(\mathbb{R})_{\mathscr{P}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \leq (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right)}$$

$$\alpha_N(\mathbb{R})_{\mathscr{B}} \leq \alpha_N(\mathbb{R})_{\mathscr{N}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \geq (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right)}$$

$$\alpha_N(\mathbb{R})_{\mathscr{N}} \leq \alpha_N(\mathbb{R})_{\mathscr{P}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \leq (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right)}$$

$$\alpha_N(\mathbb{R})_{\mathscr{N}} \leq \alpha_N(\mathbb{R})_{\mathscr{B}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \leq (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right)}$$

New DRs (10)–(12) can be rephrased as $(P2)-(N2)$.
$(P2)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \chi_1$ and $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \psi_1$, then $e \in \mathscr{P}os(\Omega)$
$(B2)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \chi_1$ and $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \omega_1$, then $e \in \mathscr{B}nd(\Omega)$
$(N2)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \omega_1$ and $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \psi_1$, then $e \in \mathscr{N}eg(\Omega)$,
here

$$\chi_1 = (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{13}$$

$$\psi_1 = (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{14}$$

$$\omega_1 = (1-\Delta(e))\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right]}{\ln\left(\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right)} \tag{15}$$

The DRs $(P2)-(N2)$ have so far been characterized by utilizing three thresholds $\chi_1(e)$, $\psi_1(e)$ and $\omega_1(e)$ from the membership degree perspective. Further, DRs (13)–(15) from the perspective of a non-membership degree are discussed.

(13) If $\beta(\mathbb{R})_{\mathscr{P}} \geq \beta(\mathbb{R})_{\mathscr{B}}$ and $\beta(\mathbb{R})_{\mathscr{P}} \geq \beta(\mathbb{R})_{\mathscr{N}}$, then $e \in \mathscr{P}os(\Omega)$
(14) If $\beta(\mathbb{R})_{\mathscr{B}} \geq \beta(\mathbb{R})_{\mathscr{P}}$ and $\beta(\mathbb{R})_{\mathscr{B}} \geq \beta(\mathbb{R})_{\mathscr{N}}$ then $e \in \mathscr{B}nd(\Omega)$
(15) If $\beta(\mathbb{R})_{\mathscr{N}} \geq \beta(\mathbb{R})_{\mathscr{P}}$ and $\beta(\mathbb{R})_{\mathscr{N}} \geq \beta(\mathbb{R})_{\mathscr{B}}$ then $e \in \mathscr{N}eg(\Omega)$

Because $\mathscr{P}r(\Omega|\mathbb{S}_e) = 1 - \mathscr{P}r(-\Omega|\mathbb{S}_e) - \Delta(e)$, using Equation (3), DRs are expressed based on complement of conditional probability as below.

If $\beta(\mathbb{R})_{\mathscr{P}} \geq \beta(\mathbb{R})_{\mathscr{B}}$, then

$$\mathscr{P}r(\Omega|\mathbb{S}_e)\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right) \geq (1-\Delta(e))\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}$$

Thus

$$\mathscr{P}r(\Omega|\mathbb{S}_e) \geq (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)}$$

Similarly

$$\beta(\mathbb{R})_{\mathscr{P}} \geq \beta(\mathbb{R})_{\mathscr{N}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \geq (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)}$$

$$\beta(\mathbb{R})_{\mathscr{B}} \geq \beta(\mathbb{R})_{\mathscr{P}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \leq (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)}$$

$$\beta(\mathbb{R})_{\mathscr{B}} \geq \beta(\mathbb{R})_{\mathscr{N}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \geq (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}\right)}$$

$$\beta(\mathbb{R})_{\mathscr{N}} \geq \beta(\mathbb{R})_{\mathscr{P}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \leq (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)}$$

$$\beta(\mathbb{R})_{\mathscr{N}} \geq \beta(\mathbb{R})_{\mathscr{B}} \Rightarrow \mathscr{P}r(\Omega|\mathbb{S}_e) \leq (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}\right)}$$

Obviously, the decision rules are easily revised as $(P2)-(N2)$.
$(P2)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \chi_2$ and $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \psi_2$, then $e \in \mathscr{P}os(U)$
$(B2)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \chi_2$ and $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \omega_2$, then $e \in \mathscr{B}nd(U)$
$(N2)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \omega_2$ and $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \psi_2$, then $e \in \mathscr{N}eg(U)$,
where,

$$\chi_2 = (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{16}$$

$$\psi_2 = (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}\right)} \tag{17}$$

$$\omega_2 = (1-\Delta(e))\frac{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}}{\ln\left(\frac{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}*\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}\right)} \tag{18}$$

Noticeably, providing the IFN cost parameters are also fulfilled,

$$\frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}\right]}{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]} < \frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}\right]}{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right]}$$

And

$$\frac{\ln\frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}} < \frac{\ln\frac{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}}{\ln\frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}}$$

The $\omega_j(e) < \psi_j(e) < \chi_j(e)$ is obtained where $\chi_j(e) \in (0,1]$, $\omega_j(e) \in (0,1]$, and $\psi_j(e) \in (0,1]$, $(j=1,2)$. Therefore, the general DRs $(P3)-(N3)$ can be described as below
$(P3)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \chi_j$, then $e \in \mathscr{P}os(U)$
$(B3)$ If $\omega_j < \mathscr{P}r(\Omega|\mathbb{S}_e) < \chi_j$, then $e \in \mathscr{B}nd(U)$

$(N3)$ If $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \omega_j$, then $e \in \mathscr{N}eg(U)$,

From the above-obtained results, the 3WD made under IFNs is described according to the Bayesian DRs as below.

(16) If $\mathscr{P}r(\Omega|\mathbb{S}_e) \geq \chi_j$, then take $a_{\mathscr{P}}$
(17) If $\omega_j < \mathscr{P}r(\Omega|\mathbb{S}_e) < \chi_j$, then take $a_{\mathscr{B}}$
(18) If $\mathscr{P}r(\Omega|\mathbb{S}_e) \leq \omega_j$, then take $a_{\mathscr{N}}$,

In TWDM-IFNs, two pairs of thresholds $(\chi_1(e),\ \omega_1(e))$ and $(\chi_2(e),\ \omega_2(e))$ are obtained from various viewpoints in (13)–(18). Thus, actions are taken while $\mathscr{P}r(\Omega|\mathbb{S}_e)$ is in corresponding thresholds of the positive, boundary, and negative regions. Then, the relationship among the four thresholds $\chi_1(e),\ \omega_1(e),\ \chi_2(e)$, and $\omega_2(e)$ can be revealed and proven.

Theorem 1. *Given an intuitionistic fuzzy information table $IS = (X,\ \mathscr{A}t, I\mathscr{V}l, \mathscr{M})$; an intuitionistic fuzzy cost parameter matrix $IM = \{N(\xi_{\sigma\tau}) = (\alpha_N(\xi_{\sigma\tau}), \beta_N(\xi_{\sigma\tau}))\}_{3\times 2}(\sigma = \mathscr{P},\ \mathscr{B},\ \mathscr{N},$ and $\tau = \mathscr{P},\ \mathscr{N})$; and two pair thresholds $(\chi_1(e),\ \omega_1(e))$ and $(\chi_2(e),\ \omega_2(e))$, where $\omega_1(e) < \chi_1(e)$ and $\omega_2(e) < \chi_2(e)$ for any $e \in X$, then $\chi_2(e) > \chi_1(e)$ and $\omega_2(e) > \omega_1(e)$, when*

$$\frac{\ln \frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln \frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}} < \frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}\right]}{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}$$

and

$$\frac{\ln \frac{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}{\beta_N(\xi_{\mathscr{N}\mathscr{P}})}}{\ln \frac{\beta_N(\xi_{\mathscr{N}\mathscr{N}})}{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}} < \frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{N}\mathscr{P}})}\right]}{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{N}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}\right]}$$

Proof. Let $T = 1 - \alpha_N(\xi_{\mathscr{B}\mathscr{N}})$, $U = 1 - \alpha_N(\xi_{\mathscr{P}\mathscr{N}})$, $V = 1 - \alpha_N(\xi_{\mathscr{P}\mathscr{P}})$, and $W = 1 - \alpha_N(\xi_{\mathscr{B}\mathscr{P}})$. Furthermore, let $T' = \beta_N(\xi_{\mathscr{B}\mathscr{N}})$, $U' = \beta_N(\xi_{\mathscr{P}\mathscr{N}})$, $V' = \beta_N(\xi_{\mathscr{P}\mathscr{P}})$, and $W' = \beta_N(\xi_{\mathscr{B}\mathscr{P}})$.

From the perspective of membership degree

$$\chi_1 = (1 - \Delta(e))\frac{\ln \frac{T}{U}}{\ln\left(\frac{V}{W} * \frac{T}{U}\right)}$$

and $(1/\chi_1(e)) = (1 + (lnV - lnW/lnT - lnU))$. Moreover, from the perspective of membership degree, then

$$\chi_2 = (1 - \Delta(e))\frac{\ln \frac{T'}{U'}}{\ln\left(\frac{V'}{W'} * \frac{T'}{U'}\right)}$$

and $(1/\chi_2(e)) = \left(1 - \Delta(e)^{-1}\right)(1 + (lnV' - lnW'/lnT' - lnU'))$

with respect to

$$\frac{\ln \frac{\beta_N(\xi_{\mathscr{P}\mathscr{P}})}{\beta_N(\xi_{\mathscr{B}\mathscr{P}})}}{\ln \frac{\beta_N(\xi_{\mathscr{B}\mathscr{N}})}{\beta_N(\xi_{\mathscr{P}\mathscr{N}})}} < \frac{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{P}\mathscr{P}})}{1-\alpha_N(\xi_{\mathscr{B}\mathscr{P}})}\right]}{\ln\left[\frac{1-\alpha_N(\xi_{\mathscr{B}\mathscr{N}})}{1-\alpha_N(\xi_{\mathscr{P}\mathscr{N}})}\right]}$$

The $(1/\chi_2(e)) < (1/\chi_1(e))$ and $\chi_2(e) > \chi_1(e)$ hold. Similarly, $\omega_2(e) > \omega_1(e)$ holds. □

5. Case Study

The contemporary world has undergone a revolutionary transformation through the integration of AI and its diverse applications. This advancement has led to the development of numerous robotic devices that effectively alleviate human challenges. Despite these strides, several aspects of our lives still necessitate substantial advancement. Particularly, individuals with special needs continue to encounter significant obstacles.

In response to this, a team of skilled engineers and mathematicians has crafted a specialized device aimed at addressing this issue, thereby enhancing the quality of life for these individuals. This innovative gadget is designed to empower individuals with special needs through a sophisticated mechanism. To achieve this, the device undergoes a training process wherein it learns to identify and interpret specific gestures from a predetermined list of gesture qualities. The decision-making process within the device is executed through a binary input of "Yes" or "No," corresponding to the recognized or unrecognized nature of a particular gesture. For instance, consider a scenario involving six distinct gestures denoted as "a" to "f." Each of these gestures possesses four attributes: finger movement, gesture speed, hand movement, and the degree of hand circulation. These attributes, represented as $A_t = \{A_1, A_2, A_3, A_4\}$, play a crucial role in distinguishing between the gestures. The significance of these attributes is quantified through a membership and non-membership degree framework. For example, a representation such as (0.70, 0.25) indicates that the membership and non-membership degrees for the attribute A_1 pertaining to a specific gesture are 0.70 and 0.25, respectively.

Furthermore, the device's decision-making process is associated with three distinct actions: recognizing the indicated gesture and responding with a spoken confirmation ("gesture" a_P), failing to recognize the indicated gesture and responding negatively ("No to the gesture" a_N), and an indeterminate scenario leading to silence ("a_B"). Additionally, the outcomes of these actions are tied to two states: one where the attributed results are satisfied (Ω) and the other where they are not $(-\Omega)$. These intricate interactions are encapsulated within an IFNs parameter matrix, as illustrated in Table 4.

Table 4. Intuitionistic fuzzy information table.

Alternatives/Attributes	A_1	A_2	A_3	A_4	d
a	$(0.70, 0.25)$	$(0.50, 0.42)$	$(0.39, 0.60)$	$(0.80, 0.20)$	Yes
b	$(0.51, 0.49)$	$(0.43, 0.51)$	$(0.64, 0.16)$	$(0.21, 0.73)$	No
c	$(0.87, 0.11)$	$(0.59, 0.40)$	$(0.76, 0.21)$	$(0.72, 0.22)$	No
d	$(0.32, 0.67)$	$(0.73, 0.26)$	$(0.81, 0.18)$	$(0.56, 0.32)$	Yes
e	$(0.71, 0.29)$	$(0.51, 0.44)$	$(0.91, 0.08)$	$(0.33, 0.67)$	Yes
f	$(0.56, 0.18)$	$(0.81, 0.1)$	$(0.32, 0.26)$	$(0.5, 0.4)$	Yes

To be precise with the IFNs given in the information table, the cosine similarity measure is applied, which is defined in Definition 6. We get the information and present it in Table 5.

Table 5. Information of alternative based on Cosine Similarity measure [51].

Similarity of Alternatives	a	b	c	d	e	f
a	1	0.782	0.929	0.830	0.815	0.924
b	0.782	1	0.827	0.879	0.972	0.838
c	0.929	0.827	1	0.870	0.906	0.929
d	0.830	0.879	0.870	1	0.873	0.886
e	0.815	0.972	0.906	0.873	1	0.890
f	0.924	0.838	0.929	0.886	0.890	1

To get the similarity classes based on the data collected using Equation (10), the threshold $\rho = 0.85$, is used to develop all the alternatives.

Based on the above threshold, the similarity classes are received as below.

$$\mathbb{S}_a = \{a,\ c, f\},\ \mathbb{S}_b = \{\ b, d, e\},\ \mathbb{S}_c = \{a,\ c, d, e, f\},\ \mathbb{S}_d = \{a, c, d, e, f\},$$
$$\mathbb{S}_e = \{b, c, d, e, f\},\ \mathbb{S}_f = \{a,\ c, d, e, f\}$$

According to DTRS [22], the set of states is given, and by considering this, a probabilistic decision is designed.

Here, $\Omega = \{a,\ d,\ e,\ f\}\ \Omega' = \{b,\ c\}$

Table 6 shows the values of the objects.

Table 6. Corresponding values of objects.

Alternatives	Probability Values	Complement of Probability	Error Values
$\boldsymbol{\alpha}(\mathbb{S}_a)$	$\mathscr{P}r(U\vert\mathbb{S}_a) = 0.377$	$1 - \mathscr{P}r(U\vert\mathbb{S}_a) = 0.122$	$\Delta(a) = 0.5$
$\boldsymbol{\alpha}(\mathbb{S}_b)$	$\mathscr{P}r(U\vert\mathbb{S}_b) = 0.311$	$1 - \mathscr{P}r(U\vert\mathbb{S}_b) = 0.277$	$\Delta(b) = 0.411$
$\boldsymbol{\alpha}(\mathbb{S}_c)$	$\mathscr{P}r(U\vert\mathbb{S}_c) = 0.577$	$1 - \mathscr{P}r(U\vert\mathbb{S}_c) = 0.255$	$\Delta(c) = 0.166$
$\boldsymbol{\alpha}(\mathbb{S}_d)$	$\mathscr{P}r(U\vert\mathbb{S}_d) = 0.544$	$1 - \mathscr{P}r(U\vert\mathbb{S}_d) = 0.333$	$\Delta(d) = 0.122$
$\boldsymbol{\alpha}(\mathbb{S}_e)$	$\mathscr{P}r(U\vert\mathbb{S}_e) = 0.577$	$1 - \mathscr{P}r(U\vert\mathbb{S}_e) = 0.255$	$\Delta(e) = 0.166$
$\boldsymbol{\alpha}(\mathbb{S}_f)$	$\mathscr{P}r(U\vert\mathbb{S}_f) = 0.577$	$1 - \mathscr{P}r(U\vert\mathbb{S}_f) = 0.255$	$\Delta(f) = 0.166$

As it is described, the cost parameter helps to take the correct action. To minimize the loss of making the decision, the loss function assists and identifies the best position for an alternative. In Table 7, cost parameters are listed in IFNs; with the help of membership and non-membership values, we get thresholds, which are displayed in Table 8.

Table 7. Cost Parameter Matrix based on IFNs.

	$\boldsymbol{\Omega}$	$-\boldsymbol{\Omega}$
$\mathscr{a}_{\mathscr{P}}$	$(0, 1)$	$(0.8,\ 0.1)$
$\mathscr{a}_{\mathscr{B}}$	$(0.3, 0.7)$	$(0.5,\ 0.4)$
$\mathscr{a}_{\mathscr{N}}$	$(0.9,\ 0.1)$	$(0.05,\ 0.8)$

Table 8. Obtained values of thresholds for conditional probability.

Alternatives	$\boldsymbol{\chi_1}$ **(*e*)**	$\boldsymbol{\psi_1}$ **(*e*)**	$\boldsymbol{\omega_1}$ **(*e*)**
a	0.359	0.201	0.124
b	0.423	0.237	0.146
c	0.599	0.335	0.206
d	0.631	0.353	0.217
e	0.599	0.335	0.206
f	0.599	0.335	0.206

Using Equations (13)–(15), we get the following threshold values in Table 8.

Based on the decision rules defined in $(\boldsymbol{P3}) - (\boldsymbol{N3})$, we get the results in Table 9 that the alternative a goes to the positive region and all the other alternatives b, c, d, e, and f will go to the boundary region.

Table 9. Classification of the objects.

Classification	$\mathscr{P}$**os (*U*)**	$\mathscr{N}$**eg (*U*)**	$\mathscr{B}$**nd (*U*)**
Participants	a	$\varnothing$	b, c, d, e, f

Additionally, on the basis of Bayesian concept and the proposed approach of TWDM for IFNs, (16)–(18) for alternative a, the positive action $\mathscr{a}_{\mathscr{P}}$ is selected and for all remaining alternatives b, c, d, e, and f the action of deferment $\mathscr{a}_{\mathscr{B}}$ is considered.

Comparative Analysis

In this section, we have studied the comparative analysis of the proposed study and the existing studies in Table 10, such as Khan et al. [35], Mahmood et al. [40], Zhang et al. [50], Ejegwa et al. [52], and Ali et al. [53]. Moreover, we discussed some benefits and preferences of the proposed approach for the literature, and for the geometrical presentation of the comparative study, we added Figure 1.

i When it comes to analyzing different approaches to solving a problem, it is important to consider their effectiveness, feasibility, and scalability. In this case, we compared an established approach with an existing approach, focusing on how they satisfy results and their benefits.

ii The established approach typically refers to a well-known and widely used method for solving a problem. This approach is based on a proven methodology that has been tested and validated over time, and it often has a track record of delivering reliable results. The existing approach, on the other hand, refers to a method that has been developed but may not be as widely known or tested. For example, all the existing approaches based on similarity measures give the majority of elements of the acceptance region the same, such as $\{a\}$, similarly to the negative and boundary zones.

iii One advantage of the established approach is that it is often easier to solve. This is because the methodology has been refined and improved over time, and there are typically more resources available to help people understand and apply it.

iv Another advantage of the established approach is that it is often more general. This means that it can be applied to a wider range of problems or scenarios. For example, if we were comparing an established statistical model with a newer one, the established model may have been designed to handle a wider range of data types or distributions, making it more versatile.

v In addition, the established approach often uses well-defined similarity measures and similarity classes. These measures and classes help to ensure that the results are consistent and meaningful.

vi However, it is important to note that the existing approach may have benefits as well. For example, it may be more specialized, meaning that it is designed specifically for a particular problem or scenario. This can make it more effective than the established approach in certain contexts.

vii In conclusion, when comparing an established approach with an existing approach, it is important to consider factors such as ease of implementation, generalizability, and the use of well-defined similarity measures and similarity classes. While the established approach has advantages in these areas, the existing approach may be more effective in certain contexts due to its specialization. Ultimately, the choice of approach will depend on the specific problem being solved and the resources available.

Table 10. Comparison of proposed and existing Techniques.

Techniques	Classification		
	$\mathscr{P}os$ (U)	$\mathscr{N}eg$ (U)	$\mathscr{B}nd$ (U)
Khan et al. [35]	$\{a\}$	$\{b,c,f\}$	$\{d,e\}$
Mahmood et al. [40]	$\{a, f\}$	$\{\varnothing\}$	$\{b,c, d,e\}$
Zhang et al. [50]	$\{a, f, d\}$	$\{\varnothing\}$	$\{b,c, e\}$
Ejegwa et al. [52]	$\{a\}$	$\{\varnothing\}$	$\{b,c,d,e f\}$
Ali et al. [53]	$\{a\}$	$\{\varnothing\}$	$\{b,c,d,e f\}$
Proposed	$\{a\}$	$\{\varnothing\}$	$\{b,c,d,e f\}$

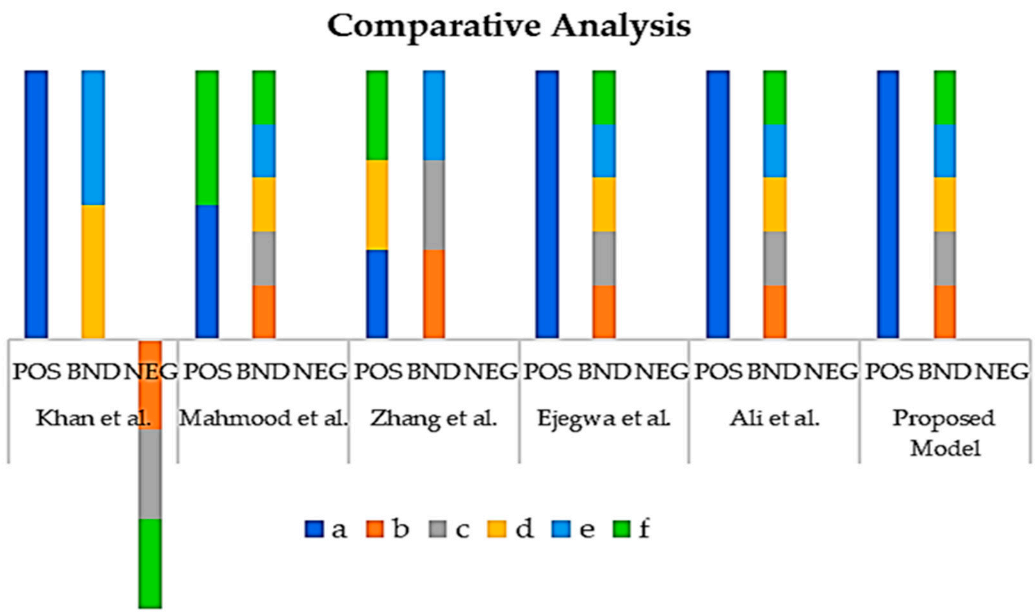

Figure 1. Geometrical representation of existing [35,40,50,52,53] and novel study.

6. Conclusions

In this era of cognitive instability, the combination of the 3WD theory with rough sets and fuzzy sets has received significant attention. While IFS, an extension of fuzzy sets, is widely used to handle uncertainty, IFNs offer a more comprehensive representation by considering both membership and non-membership degrees. Therefore, it is crucial to develop an IFN-based TWDM that aligns with human cognition. In this paper, we first reviewed the existing 3WD model in detail and then proposed a more accurate and generalized model based on IFNs, incorporating new developments that yield improved results. Specifically, we employed similarity measures to partition the information system and designed probability, complement of probability, and error functions to capture the essence of DTRS for IFNs. During this establishment, we retained the basic idea of the conditional probability, which is directed to the membership grade of the decided portion. Moreover, we have presented the effectiveness and validity of the established approach by adding a practical model of an electronic gesture device for taking decisions. We briefly discussed the benefits and limitations of our developed model and compared it with the existing models.

In the future, we will explore novel aggregation operators for aggregating the information for 3WD based on DTRSs. We will further utilize our developed approach to improve the existing literature [54–58] and design the real application in artificial intelligence and data sciences.

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