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Normed Space of Fuzzy Intervals and Its Topological Structure

Hsien-Chung Wu

Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 82444, Taiwan; hsien.chung.wu@gmail.com

Abstract: The space, $\mathcal{F}_{cc}(\mathbb{R})$, of all fuzzy intervals in $\mathbb{R}$ cannot form a vector space. However, the space $\mathcal{F}_{cc}(\mathbb{R})$ maintains a vector structure by treating the addition of fuzzy intervals as a vector addition and treating the scalar multiplication of fuzzy intervals as a scalar multiplication of vectors. The only difficulty in taking care of $\mathcal{F}_{cc}(\mathbb{R})$ is missing the additive inverse element. This means that each fuzzy interval that is subtracted from itself cannot be a zero element in $\mathcal{F}_{cc}(\mathbb{R})$. Although $\mathcal{F}_{cc}(\mathbb{R})$ cannot form a vector space, we still can endow a norm on the space $\mathcal{F}_{cc}(\mathbb{R})$ by following its vector structure. Under this setting, many different types of open sets can be proposed by using the different types of open balls. The purpose of this paper is to study the topologies generated by these different types of open sets.

Keywords: normed space; open balls; open sets; pseudo-open sets; null set

MSC: 46A19; 15A03

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1. Introduction

The fuzzy set, $\tilde{x}$, in topological space, U, is defined by a membership function, $\xi_{\tilde{x}} : U \to [0,1]$. For $\lambda \in (0,1]$, the λ-level set of $\tilde{x}$ is defined by

$$\tilde{x}_\lambda = \{x \in U : \xi_{\tilde{x}}(x) \geq \lambda\}.$$

The 0-level set, $\tilde{x}_0$, is defined as the closure of the support $\{x \in U : \xi_{\tilde{x}}(x) > 0\}$, given by

$$\tilde{x}_0 = \mathrm{cl}(\{x \in U : \xi_{\tilde{x}}(x) > 0\}) = \mathrm{cl}\left(\bigcup_{\lambda\in(0,1]} \tilde{x}_\lambda\right).$$

Let $\tilde{x}$ be a fuzzy set in $\mathbb{R}$. We say that $\tilde{x}$ is a fuzzy interval when its λ-level set, $\tilde{x}_\lambda$, is a bounded closed interval for $\lambda \in [0,1]$. More precisely, we write

$$\tilde{x}_\lambda = \left[\tilde{x}_\lambda^L, \tilde{x}_\lambda^U\right],$$

where $\tilde{x}_\lambda^L$ denotes the left endpoint of the bounded closed interval, $\tilde{x}_\lambda$, and $\tilde{x}_\lambda^U$ denotes the right endpoint of the bounded closed interval, $\tilde{x}_\lambda$. We denote by $\mathcal{F}_{cc}(\mathbb{R})$ the family of all fuzzy intervals.

Let $\odot$ denote any one of the four basic arithmetic operations, $\oplus, \ominus, \otimes$, or $\oslash$, between two fuzzy intervals, $\tilde{x}$ and $\tilde{y}$. The membership function of $\tilde{x} \odot \tilde{y}$ is defined by

$$\xi_{\tilde{x}\odot\tilde{y}}(z) = \sup_{\{(x,y):z=x\circ y\}} \min\{\xi_{\tilde{x}}(x), \xi_{\tilde{y}}(y)\}$$

for all $z \in \mathbb{R}$. More precisely, the membership functions are given by

$$\xi_{\tilde{x}\oplus\tilde{y}}(z) = \sup_{\{(x,y):z=x+y\}} \min\{\xi_{\tilde{x}}(x), \xi_{\tilde{y}}(y)\};$$
$$\xi_{\tilde{x}\ominus\tilde{y}}(z) = \sup_{\{(x,y):z=x-y\}} \min\{\xi_{\tilde{x}}(x), \xi_{\tilde{y}}(y)\};$$
$$\xi_{\tilde{x}\otimes\tilde{y}}(z) = \sup_{\{(x,y):z=x*y\}} \min\{\xi_{\tilde{x}}(x), \xi_{\tilde{y}}(y)\};$$
$$\xi_{\tilde{x}\oslash\tilde{y}}(z) = \sup_{\{(x,y):z=x/y,y\neq 0\}} \min\{\xi_{\tilde{x}}(x), \xi_{\tilde{y}}(y)\}.$$

For example, given two fuzzy intervals, $\tilde{2}$ and $\tilde{3}$, with membership functions $\xi_{\tilde{2}}$ and $\xi_{\tilde{3}}$, the addition $\tilde{2}\oplus\tilde{3}$ is a new fuzzy interval, $\tilde{z} = \tilde{2}\oplus\tilde{3}$, with a membership function given by

$$\xi_{\tilde{z}}(z) = \xi_{\tilde{2}\oplus\tilde{3}}(z) = \sup_{\{(x,y):z=x+y\}} \min\{\xi_{\tilde{2}}(x), \xi_{\tilde{3}}(y)\}.$$

In particular, we have

$$\xi_{\tilde{z}}(5) = \xi_{\tilde{2}\oplus\tilde{3}}(5) = \sup_{\{(x,y):5=x+y\}} \min\{\xi_{\tilde{2}}(x), \xi_{\tilde{3}}(y)\}.$$

Each real number a can also be treated as a fuzzy interval, $\tilde{1}_{\{a\}}$, with a membership function defined by

$$\xi_{\tilde{1}_{\{a\}}}(r) = \begin{cases} 1 & \text{if } r = a \\ 0 & \text{if } r \neq a. \end{cases}$$

In this case, the fuzzy interval $\tilde{1}_{\{a\}}$ is also called a crisp number with a value of a. For convenience, we write $\kappa\tilde{x} \equiv \tilde{1}_{\kappa} \otimes \tilde{x}$.

Let $\tilde{x}$ and $\tilde{y}$ be two fuzzy intervals with $\tilde{x}_\lambda = [\tilde{x}^L_\lambda, \tilde{x}^U_\lambda]$ and $\tilde{y}_\lambda = [\tilde{y}^L_\lambda, \tilde{y}^U_\lambda]$ for $\lambda \in [0,1]$. It is well known that

$$(\tilde{x}\oplus\tilde{y})_\lambda = \left[\tilde{x}^L_\lambda + \tilde{y}^L_\lambda, \tilde{x}^U_\lambda + \tilde{y}^U_\lambda\right]$$

and, for $\kappa \in \mathbb{R}$,

$$(\kappa\tilde{x})_\lambda = \begin{cases} [\kappa\tilde{x}^L_\lambda, \kappa\tilde{x}^U_\lambda] & \text{if } \kappa \geq 0 \\ [\kappa\tilde{x}^U_\lambda, \kappa\tilde{x}^L_\lambda] & \text{if } \kappa < 0. \end{cases}$$

For any $\kappa \in \mathbb{R}$ and $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$, it is clear to see

$$\kappa(\tilde{x}\oplus\tilde{y}) = \kappa\tilde{x}\oplus\kappa\tilde{y}. \tag{1}$$

Given any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, we have

$$(\tilde{x}\ominus\tilde{x})_\lambda = \left[\tilde{x}^L_\lambda - \tilde{x}^U_\lambda, \tilde{x}^U_\lambda - \tilde{x}^L_\lambda\right] = \left[-\left(\tilde{x}^U_\lambda - \tilde{x}^L_\lambda\right), \tilde{x}^U_\lambda - \tilde{x}^L_\lambda\right]. \tag{2}$$

We see that each λ-level set $(\tilde{x}\ominus\tilde{x})_\lambda$ contains 0 as the middle value of this bounded closed interval. In this case, we can say that $\tilde{x}\ominus\tilde{x}$ is a fuzzy zero number. Now, we can collect all these fuzzy zero numbers as a set:

$$\Psi = \{\tilde{x}\ominus\tilde{x} : \tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})\}.$$

We also call Ψ the null set in $\mathcal{F}_{cc}(\mathbb{R})$. It is clear to see that the crisp number $\tilde{1}_{\{0\}}$ with a value of 0 is in the null set, Ψ.

By referring to (2), the elements in the null set Ψ can be realized as follows.

$$\psi \in \Psi \text{ if and only if } \psi^U_\lambda \geq 0 \text{ and } \psi^L_\lambda = -\psi^U_\lambda \text{ for all } \lambda \in [0,1].$$

The λ-level sets are given by

$$\psi_\lambda = \left[\psi_\lambda^L, \psi_\lambda^U\right] = \left[-\psi_\lambda^U, \psi_\lambda^U\right].$$

Given any fuzzy interval, $\tilde{x}$, and crisp number, $\tilde{1}_{\{0\}}$, it is clear to see

$$\tilde{x} \oplus \tilde{1}_{\{0\}} = \tilde{1}_{\{0\}} \oplus \tilde{x} = \tilde{x},$$

which shows that $\tilde{1}_{\{0\}}$ is the zero element of the space $\mathcal{F}_{cc}(\mathbb{R})$. Since $\tilde{x} \ominus \tilde{x}$ is in Ψ and is not a zero element of $\mathcal{F}_{cc}(\mathbb{R})$, this means that the space $\mathcal{F}_{cc}(\mathbb{R})$ of all fuzzy intervals cannot form a vector space under the above fuzzy addition and scalar multiplication. In other words, the additive inverse element of each fuzzy interval does not exist.

From the monographs [1–10], we see that the normed space must be based on the vector space. Since $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space, this means that we are able to endow a norm on the space $(\mathcal{F}_{cc}(\mathbb{R}, \| \cdot \|)$. The purpose of this paper is to overcome this difficulty. Since the space $\mathcal{F}_{cc}(\mathbb{R})$ maintains a vector structure by treating the addition of fuzzy intervals as a vector addition and treating the scalar multiplication of fuzzy intervals as a scalar multiplication of vectors, we can still endow a norm on $\mathcal{F}_{cc}(\mathbb{R})$ and study its topological structure by including the null set in $\mathcal{F}_{cc}(\mathbb{R})$ and following the similar axioms of the conventional norm.

In order to study the topological structure of the normed space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$, it is necessarily to consider the concept of open balls. Suppose that $(X, \| \cdot \|)$ is a (conventional) normed space. It is clear to see

$$\{y : \| x - y \| < \epsilon\} = \{x + z : \| z \| < \epsilon\}$$

by taking $y = x + z$. However, for the space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$, we cannot have the above equality. A detailed explanation is given below.

Given any $\tilde{x}, \tilde{y}, \tilde{z} \in \mathcal{F}_{cc}(\mathbb{R})$, by taking $\tilde{y} = \tilde{x} \oplus \tilde{z}$, we have

$$\| \tilde{x} \ominus \tilde{y} \| = \| \tilde{x} \ominus (\tilde{x} \oplus \tilde{z}) \| = \| \psi \ominus \tilde{z} \| \neq \| \tilde{z} \|,$$

where $\psi = \tilde{x} \ominus \tilde{x} \in \Psi$. This means that the following equality

$$\{\tilde{y} : \| \tilde{x} \ominus \tilde{y} \| < \epsilon\} = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\}$$

cannot hold true.

In this case, two different types of open balls will be considered in $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$. This also means that we can consider many different types of open sets. A more detailed definition will be presented in the context of this paper. Based on the different types of open sets, the topological structure of the normed space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ will be studied.

The fuzzy topology has been studied for a long time by referring to the monograph by Liu and Luo [11], in which the intersection and union of fuzzy sets are defined based on membership functions. The topological structure studied in this paper is based on the point-set topology, which is completely different from the fuzzy topology using membership functions. On the other hand, many different types of fuzzy normed spaces have also be introduced by many researchers, which are briefly described below.

(a) The concept of fuzzy normed space is referred to by Felbin [12] and Xiao and Zhu [13]. Let X be a vector space, and let $\mathcal{L}$ and $\mathcal{R}$ be two symmetric and nondecreasing functions defined by $[0,1] \times [0,1]$ in $[0,1]$, satisfying $\mathcal{L}(0,0) = 0$ and $\mathcal{R}(1,1) = 1$. Let $\mathcal{F}_{cc}^{+}$ be a family of all the nonnegative fuzzy numbers in $\mathcal{R}$, and let $\| \cdot \|$ be a function defined by X in $\mathcal{F}_{cc}^{+}$, where, given any fixed $x \in X$, the α-level set $\| x \|_\alpha$ of the nonnegative fuzzy number $\| x \|$ is a bounded closed interval given by

$$\| x \|_\alpha = \left[\| x \|_\alpha^L, \| x \|_\alpha^U\right] \text{ for } \alpha \in (0,1].$$

We say that $(X, \|\cdot\|, \mathcal{L}, \mathcal{R})$ is a fuzzy normed space when the following conditions are satisfied:

- $\| x \| = \tilde{0}$ if and only if $x = \theta$;
- $\| \lambda x \| = |\lambda| \otimes \| x \|$ for $x \in X$ and $\lambda \in \mathcal{R}$;
- Given any $x, y \in X$, for $s \leq \| x \|_1^L$, $t \leq \| y \|_1^L$ and $s + t \leq \| x + y \|_1^L$, we have
$$\| x + y \| (s + t) \geq \mathcal{L}(\| x \| (s), \| y \| (t)).$$
- Given any $x, y \in X$, for $s \geq \| x \|_1^L$, $t \geq \| y \|_1^L$ and $s + t \geq \| x + y \|_1^L$, we have
$$\| x + y \| (s + t) \leq \mathcal{R}(\| x \| (s), \| y \| (t)).$$

(b) The concept of a fuzzy norm in the vector space X is referred to by Bag and Samanta [14,15]. Let $\mathcal{N}$ be a function defined on $X \times [0, \infty)$ into $[0, 1]$. We say that the function $\mathcal{N}$ is a fuzzy norm in X when the following conditions are satisfied:

- $\mathcal{N}(x, 0) = 0$ for all $x \in X$;
- $\mathcal{N}(x, t) = 1$ for all $t > 0$ if and only if $x = 0$;
- $\mathcal{N}(\lambda x, t) = \mu(x, t/|\lambda|)$ for $(x, t) \in X \times [0, \infty)$ and $\lambda \neq 0$;
- $\mathcal{N}(x + y, s + t) \geq \min\{\mathcal{N}(x, t), \mathcal{N}(y, s)\}$ for all $(x, t), (y, s) \in X \times [0, \infty)$;
- Given any fixed $x \in X$, we have
$$\lim_{t \to \infty} \mathcal{N}(x, t) = 1.$$

(c) The concept of an intuitionistic fuzzy normed space is referred to by Saadati and Park [16]. Let $*$ be a continuous t-norm, let $\circ$ be a continuous t-conorm, and let X be a vector space. Given two functions μ and ν, defined by $X \times (0, \infty)$ in $[0, 1]$, we say that $(X, \mu, \nu, *, \circ)$ is an intuitionistic fuzzy normed space when the following conditions are satisfied:

- Given any $(x, t) \in X \times (0, \infty)$, we have $\mu(x, t) + \nu(x, t) \leq 1$;
- Given any $(x, t) \in X \times (0, \infty)$, we have $\mu(x, t) > 0$;
- For $(x, t) \in X \times (0, \infty)$, $\mu(x, t) = 1$ if and only if $x = 0$;
- Given any $(x, t) \in X \times (0, \infty)$, we have $\mu(\lambda x, t) = \mu(x, t/|\lambda|)$ for $a \neq 0$;
- Given any $(x, t), (y, s) \in X \times (0, \infty)$, we have $\mu(x, t) * \mu(y, s) \leq \mu(x + y, s + t)$;
- Given any fixed $x \in X$, the function $\mu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
- Given any fixed $x \in X$, we have
$$\lim_{t \to \infty} \mu(x, t) = 1 \text{ and } \lim_{t \to 0} \mu(x, t) = 0.$$
- Given any $(x, t) \in X \times (0, \infty)$, we have $\nu(x, t) < 1$;
- For $(x, t) \in X \times (0, \infty)$, $\nu(x, t) = 1$ if and only if $x = 0$;
- Given any $(x, t) \in X \times (0, \infty)$, we have $\nu(\lambda x, t) = \nu(x, t/|\lambda|)$ for $a \neq 0$;
- Given any $(x, t), (y, s) \in X \times (0, \infty)$, we have $\nu(x, t) \circ \nu(y, s) \geq \nu(x + y, s + t)$;
- Given any fixed $x \in X$, the function $\nu(x, \cdot) : (0, \infty) \to [0, 1]$ is continuous;
- Given any fixed $x \in X$, we have
$$\lim_{t \to \infty} \nu(x, t) = 0 \text{ and } \lim_{t \to 0} \nu(x, t) = 1.$$

A bunch of articles that studied these three kinds of fuzzy normed spaces have been published. In this paper, we endow a norm directly on the family of all fuzzy intervals, which is completely different from that of those three fuzzy normed space since the vector space is not taken into account.

In Section 2, we present many interesting properties of fuzzy intervals, which will be used to study the topology generated by the norm. In Section 3, the concept of the norm in the space of fuzzy intervals is introduced. Many useful properties are also provided in order to study the topology generated by the norm. In Section 4, two different types of open balls are introduced. In Section 5, using the different types of open balls, many types of open sets are introduced. Finally, in Section 6, the topologies generated by these different types of open sets are investigated.

2. Space of Fuzzy Intervals

Let us recall that the following set

$$\Psi = \{\tilde{x} \ominus \tilde{x} : \tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})\}$$

is called the null set in $\mathcal{F}_{cc}(\mathbb{R})$. For further discussion, we present some useful properties.

- We have
$$\Psi = \left\{\psi \in \mathcal{F}_{cc}(\mathbb{R}) : \psi_\alpha = \left[-\psi^U_\alpha, \psi^U_\alpha\right] \text{ for all } \alpha \in [0,1]\right\}.$$
- It is clear to see that $\psi \in \Psi$ implies $-\psi = \psi$.
- We have the equality $\lambda\Psi = \Psi$ for $\lambda \in \mathbb{R}$ with $\lambda \neq 0$.

Example 1. *The membership function of the trapezoidal fuzzy interval $\tilde{a}$ is given by*

$$\xi_{\tilde{a}}(r) = \begin{cases} (r - a^L)/(a_1 - a^L) & \text{if } a^L \leq r \leq a_1 \\ 1 & \text{if } a_1 < r \leq a_2 \\ (a^U - r)/(a^U - a_2) & \text{if } a_2 < r \leq a^U \\ 0 & \text{otherwise,} \end{cases}$$

which is denoted by $\tilde{a} = (a^L, a_1, a_2, a^U)$. The α-level set $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ is obtained by

$$\tilde{a}^L_\alpha = (1-\alpha)a^L + \alpha a_1 \text{ and } \tilde{a}^U_\alpha = (1-\alpha)a^U + \alpha a_2.$$

Let $\tilde{a} = (a^L, a_1, a_2, a^U)$ and $\tilde{b} = (b^L, b_1, b_2, b^U)$ be two trapezoidal fuzzy intervals. We can show that $\tilde{a} \oplus \tilde{b}$ is also a trapezoidal fuzzy interval given by

$$\tilde{a} \oplus \tilde{b} = (a^L + b^L, a_1 + b_1, a_2 + b_2, a^U + b^U).$$

Now, we have $\psi = \tilde{a} \ominus \tilde{a} \in \Psi$, with the α-level sets given by

$$\begin{aligned} \psi_\alpha = \tilde{a}_\alpha - \tilde{a}_\alpha &= \left[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha\right] - \left[\tilde{a}^L_\alpha, \tilde{a}^U_\alpha\right] = \left[\tilde{a}^L_\alpha - \tilde{a}^U_\alpha, \tilde{a}^U_\alpha - \tilde{a}^L_\alpha\right] \\ &= \left[(1-\alpha)\left(a^L - a^U\right) + \alpha(a_1 - a_2), (1-\alpha)\left(a^U - a^L\right) + \alpha(a_2 - a_1)\right] \end{aligned}$$

for $\alpha \in [0,1]$.

Proposition 1. *We have the following properties:*

(i) *Let $\mathfrak{F}$ be a subset of $\mathcal{F}_{cc}(\mathbb{R})$. We have the inclusion $\mathfrak{F} \subseteq \mathfrak{F} \oplus \Psi$.*
(ii) *Given any $\psi_1, \psi_2 \in \Psi$, we have $\psi_1 \oplus \psi_2 \in \Psi$. Moreover, we have $\Psi \oplus \Psi = \Psi$.*

Proof. To prove part (i), since $\tilde{1}_{\{0\}} \in \Psi$, given any $\tilde{x} \in \mathfrak{F}$, we have

$$\tilde{x} = \tilde{x} \oplus \tilde{1}_{\{0\}} \in \mathfrak{F} \oplus \Psi.$$

To prove part (ii), for $\psi_1, \psi_2 \in \Psi$, the definition of a null set says

$$\psi_1 = \tilde{x} \ominus \tilde{x} \text{ and } \psi_2 = \tilde{y} \ominus \tilde{y}$$

for some $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$. Using the distributive law (1), we obtain

$$\psi_1 \oplus \psi_2 = \tilde{x} \ominus \tilde{x} \oplus \tilde{y} \ominus \tilde{y} = (\tilde{x} \oplus \tilde{y}) \ominus (\tilde{x} \oplus \tilde{y}) \in \Psi.$$

This also shows the inclusion $\Psi \oplus \Psi \subseteq \Psi$. Now, we want to prove the other direction of inclusion. Given any $\psi \in \Psi$, since the crisp number $\tilde{1}_{\{0\}} \in \Psi$ is a zero element, we have

$$\psi = \psi \oplus \tilde{1}_{\{0\}} \in \Psi \oplus \Psi.$$

This shows the inclusion $\Psi \subseteq \Psi \oplus \Psi$. Therefore, we obtain the desired equality $\Psi \oplus \Psi = \Psi$, and the proof is complete. $\square$

Definition 1. *Given any $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$, we say that the fuzzy intervals $\tilde{x}$ and $\tilde{y}$ are almost identical when there exist $\psi^{(1)}, \psi^{(2)} \in \Psi$ satisfying*

$$\tilde{x} \oplus \psi^{(1)} = \tilde{y} \oplus \psi^{(2)}. \tag{3}$$

In this case, we write $\tilde{x} \stackrel{\Psi}{=} \tilde{y}$.

Suppose that the fuzzy interval $\tilde{x}$ is regarded as the "approximated real number r". This means $\xi_{\tilde{x}}(r) = 1$. Since $\xi_{\psi}(0) = 1$, we can say that $\psi \in \Psi$ is a fuzzy zero number. It is also clear to see $\xi_{\tilde{x} \oplus \psi}(r) = 1$. In this case, we can say that $\tilde{x} \oplus \psi$ is an "approximated real number r". Similarly, we can also say that $\tilde{y} \oplus \psi$ is an "approximated real number r". In other words, $\tilde{x} \stackrel{\Psi}{=} \tilde{y}$ means that $\tilde{x}$ and $\tilde{y}$ are identical, but they differ from the fuzzy zero elements $\psi^{(1)}$ and $\psi^{(2)}$, respectively, as referred to in (3).

Suppose that $\tilde{x} \ominus \tilde{y} = \tilde{z}$. We cannot obtain $\tilde{x} = \tilde{y} \oplus \tilde{z}$ as with the conventional operation in vector space. We can just obtain $\tilde{x} \stackrel{\Psi}{=} \tilde{y} \oplus \tilde{z}$. An explanation is given below. Since $\tilde{x} \ominus \tilde{y} = \tilde{z}$, by adding $\tilde{y}$ on both sides, we obtain

$$\tilde{x} \ominus \tilde{y} \oplus \tilde{y} = \tilde{z} \oplus \tilde{y}.$$

Let $\psi = \tilde{y} \ominus \tilde{y} \in \Psi$. Then, we obtain $\tilde{x} \oplus \psi = \tilde{y} \oplus \tilde{z}$, which gives $\tilde{x} \stackrel{\Psi}{=} \tilde{y} \oplus \tilde{z}$.

Proposition 2. *Given any $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$, we have the following properties:*

(i) *Suppose that $\tilde{x} \ominus \tilde{y} \in \Psi$. Then, we have $\tilde{x} \stackrel{\Psi}{=} \tilde{y}$.*

(ii) *Suppose that $\tilde{x} \stackrel{\Psi}{=} \tilde{y}$. Then, there exists $\psi \in \Psi$ satisfying $\tilde{x} \ominus \tilde{y} \oplus \psi \in \Psi$.*

Proof. To prove part (i), there exists $\psi^{(1)} \in \Psi$ satisfying $\tilde{x} \oplus (-\tilde{y}) = \psi^{(1)}$. By adding $\tilde{y}$ on both sides, we obtain

$$\tilde{x} \oplus (-\tilde{y}) \oplus \tilde{y} = \psi^{(1)} \oplus \tilde{y}.$$

Let $\tilde{y} \ominus \tilde{y} \in \Psi$. We also have $\tilde{x} \oplus \psi^{(2)} = \psi^{(1)} \oplus \tilde{y}$ for $\psi^{(1)}, \psi^{(2)} \in \Psi$. This shows $\tilde{x} \stackrel{\Psi}{=} \tilde{y}$.

To prove part (ii), the relation $\tilde{x} \stackrel{\Psi}{=} \tilde{y}$ implies that there exist $\psi^{(1)}, \psi^{(2)} \in \Psi$ satisfying

$$\tilde{x} \oplus \psi^{(2)} = \psi^{(1)} \oplus \tilde{y}.$$

By adding $-\tilde{y}$ on both sides, we obtain

$$\tilde{x} \ominus \tilde{y} \oplus \psi^{(2)} = \psi^{(1)} \oplus \tilde{y} \ominus \tilde{y}.$$

Let $\psi^{(3)} = \tilde{y} \ominus \tilde{y} \in \Psi$. Using part (ii) of Proposition 1, we obtain

$$\tilde{x} \ominus \tilde{y} \oplus \psi^{(2)} = \psi^{(1)} \oplus \psi^{(3)} \in \Psi.$$

This completes the proof. $\square$

The following interesting results will be used for studying the topological structure of the normed space of fuzzy intervals.

Proposition 3. *Given any two subsets, $\mathfrak{F}_1$ and $\mathfrak{F}_2$, of $\mathcal{F}_{cc}(\mathbb{R})$, we have the following inclusion*

$$(\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi \subseteq [(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)].$$

If we further assume

$$\mathfrak{F}_1 \oplus \Psi \subseteq \mathfrak{F}_1 \text{ and } \mathfrak{F}_2 \oplus \Psi \subseteq \mathfrak{F}_2 \tag{4}$$

then the following equality is satisfied

$$[(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)] = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi = \mathfrak{F}_1 \cap \mathfrak{F}_2.$$

Proof. Given any

$$\tilde{y} \in (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi,$$

there exist $\psi \in \Psi$ and $\tilde{x} \in \mathfrak{F}_i$ for $i = 1, 2$ satisfying $\tilde{y} = \tilde{x} \oplus \psi$. This gives

$$\tilde{y} \in (\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi).$$

Therefore, we obtain the following inclusion

$$(\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi \subseteq [(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)].$$

Under a further assumption (4), using part (i) of Proposition 1, we obtain

$$\mathfrak{F}_1 \oplus \Psi = \mathfrak{F}_1 \text{ and } \mathfrak{F}_2 \oplus \Psi = \mathfrak{F}_2,$$

which shows the following equality

$$[(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)] = \mathfrak{F}_1 \cap \mathfrak{F}_2 \subseteq (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi.$$

This completes the proof. □

3. Normed Space of Fuzzy Intervals

Although the space, $\mathcal{F}_{cc}(\mathbb{R})$, of all fuzzy intervals in $\mathbb{R}$ is not a vector space as we mention above, we still can endow a norm on $\mathcal{F}_{cc}(\mathbb{R})$ by following similar axioms.

Definition 2. *Let $\| \cdot \|$: $\mathcal{F}_{cc}(\mathbb{R}) \to \mathbb{R}_+$ be a nonnegative real-valued function. We consider the following conditions.*

- *(a)* $\| \lambda\tilde{x} \| = |\lambda| \, \| \tilde{x} \|$ *for* $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$;
- *(a′)* $\| \lambda\tilde{x} \| = |\lambda| \, \| \tilde{x} \|$ *for* $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ *and* $\lambda \in \mathbb{R}$ *with* $\lambda \neq 0$.
- *(b)* $\| \tilde{x} \oplus \tilde{y} \| \leq \| \tilde{x} \| + \| \tilde{y} \|$ *for* $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$.
- *(c)* $\| \tilde{x} \| = 0$ *implies* $\tilde{x} \in \Psi$.

When condition (c) is replaced by the following statement

$$\| \tilde{x} \| = 0 \text{ if and only if } \tilde{x} \in \Psi,$$

the norm $\| \cdot \|$ is said to satisfy the null condition.

Different kinds of normed space of fuzzy intervals are defined below.

- *The space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is said to be a pseudo-seminormed space when conditions (a′) and (b) are satisfied.*
- *The space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is said to be a seminormed space when conditions (a) and (b) are satisfied.*
- *The space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is said to be a pseudo-normed space when conditions (a′), (b), and (c) are satisfied.*
- *The space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is said to be a normed space whe conditions (a), (b), and (c) are satisfied.*

Regarding the null set Ψ, we also consider the following

- *The norm $\| \cdot \|$ is said to satisfy the null super-inequality when the inequality*

$$\| \tilde{x} \oplus \psi \| \geq \| \tilde{x} \|$$

  is satisfied for $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\psi \in \Psi$.
- *The norm $\| \cdot \|$ is said to satisfy the null sub-inequality when the inequality*

$$\| \tilde{x} \oplus \psi \| \leq \| \tilde{x} \|$$

  is satisfied for $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\psi \in \Psi$.
- *The norm $\| \cdot \|$ is said to satisfy the null equality when the equality*

$$\| \tilde{x} \oplus \psi \| = \| \tilde{x} \|$$

  is satisfied for $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\psi \in \Psi$.

Given any $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$, since $\tilde{x} \ominus \tilde{y} = -(\tilde{y} \ominus \tilde{x})$, by referring to the distributive law (1), we have the following equality

$$\| \tilde{x} \ominus \tilde{y} \| = \| \tilde{y} \ominus \tilde{x} \|, \tag{5}$$

which also confirms that the symmetric condition is satisfied.

Example 2. *Let us define a nonnegative real-valued function, $\| \cdot \|$, in $\mathcal{F}_{cc}(\mathbb{R})$ by*

$$\| \tilde{x} \| = \int_0^1 \left| \tilde{x}_\alpha^L - \tilde{x}_\alpha^U \right| d\alpha.$$

Then, we have the following properties:

$$\begin{aligned} \| \lambda\tilde{x} \| &= \int_0^1 \left| (\lambda\tilde{x})_\alpha^L - (\lambda\tilde{x})_\alpha^U \right| d\alpha = \begin{cases} \int_0^1 \left| \lambda\tilde{x}_\alpha^L - \lambda\tilde{x}_\alpha^U \right| d\alpha & \text{if } \lambda \geq 0 \\ \int_0^1 \left| \lambda\tilde{x}_\alpha^U - \lambda\tilde{x}_\alpha^L \right| d\alpha & \text{if } \lambda < 0 \end{cases} \\ &= |\lambda| \int_0^1 \left| \tilde{x}_\alpha^L - \tilde{x}_\alpha^U \right| d\alpha = |\lambda| \, \| \tilde{x} \| \end{aligned}$$

and

$$\begin{aligned} \| \tilde{x} \oplus \tilde{y} \| &= \int_0^1 \left| (\tilde{x} \oplus \tilde{y})_\alpha^L - (\tilde{x} \oplus \tilde{y})_\alpha^U \right| = \int_0^1 \left| \tilde{x}_\alpha^L + \tilde{y}_\alpha^L - \tilde{x}_\alpha^U - \tilde{y}_\alpha^U \right| \\ &\leq \int_0^1 \left| \tilde{x}_\alpha^L - \tilde{x}_\alpha^U \right| d\alpha + \int_0^1 \left| \tilde{y}_\alpha^L - \tilde{y}_\alpha^U \right| d\alpha = \| \tilde{x} \| + \| \tilde{y} \| . \end{aligned}$$

Given any ψ, we have $\psi_\alpha = [-\psi_\alpha^U, \psi_\alpha^U]$, which shows that

$$\| \psi \| = 2 \int_0^1 \left| \psi_\alpha^U \right| d\alpha \neq 0.$$

Therefore, $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is a seminormed space of fuzzy intervals such that the null condition is not satisfied.

Example 3. *Let us define a nonnegative real-valued function, $\| \cdot \|$, in $\mathcal{F}_{cc}(\mathbb{R})$ by*

$$\| \tilde{x} \| = \int_0^1 \left| \tilde{x}_\alpha^L + \tilde{x}_\alpha^U \right| d\alpha.$$

Then, using the argument of Example 2, we can still obtain

$$\| \lambda \tilde{x} \| = |\lambda| \, \| \tilde{x} \| \text{ and } \| \tilde{x} \oplus \tilde{y} \| \leq \| \tilde{x} \| + \| \tilde{y} \| .$$

For $\| \tilde{x} \| = 0$, *we must have* $\tilde{x}^L_\alpha + \tilde{x}^U_\alpha = 0$ *for all* $\alpha \in [0,1]$, *which gives* $\tilde{x}^L_\alpha = -\tilde{x}^U_\alpha$ *for all* $\alpha \in [0,1]$. *Therefore, we obtain* $\tilde{x} \in \Psi$. *It is also clear that* $\psi \in \Psi$ *implies* $\| \psi \| = 0$, *which shows that the null condition is satisfied. Finally, for* $\psi \in \Psi$, *i.e.,* $\psi^L_\alpha = -\psi^U_\alpha$, *we have*

$$\begin{aligned} \| \tilde{x} \oplus \psi \| &= \int_0^1 \left| (\tilde{x} \oplus \psi)^L_\alpha + (\tilde{x} \oplus \psi)^U_\alpha \right| = \int_0^1 \left| \tilde{x}^L_\alpha + \psi^L_\alpha + \tilde{x}^U_\alpha + \psi^U_\alpha \right| \\ &= \int_0^1 \left| \tilde{x}^L_\alpha + \tilde{x}^U_\alpha \right| d\alpha = \| \tilde{x} \| . \end{aligned}$$

Therefore, we conclude that $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ *is a normed space of fuzzy intervals such that the null condition and null equality are satisfied.*

Example 4. *For any* $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, *we define*

$$\| \tilde{x} \| = \sup_{\alpha \in [0,1]} \left| \tilde{x}^L_\alpha + \tilde{x}^U_\alpha \right| .$$

Then, we can show that $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ *is a normed space of fuzzy intervals such that the null condition and null equality are satisfied.*

Proposition 4. *Let* $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ *be a pseudo-seminormed space such that the norm* $\| \cdot \|$ *satisfies the null super-inequality. Given any* $\tilde{x}, \tilde{z}, \tilde{y}_1, \cdots, \tilde{y}_m \in \mathcal{F}_{cc}(\mathbb{R})$, *we have the following inequality*

$$\| \tilde{x} \ominus \tilde{z} \| \leq \| \tilde{x} \ominus \tilde{y}_1 \| + \| \tilde{y}_1 \ominus \tilde{y}_2 \| + \cdots + \| \tilde{y}_i \ominus \tilde{y}_{i+1} \| + \cdots + \| \tilde{y}_m \ominus \tilde{z} \| .$$

Proof. Since $\tilde{y}_i \ominus \tilde{y}_i \in \Psi$ for $i = 1, \cdots, m$, using the null super-inequality m times, we obtain

$$\| \tilde{x} \ominus \tilde{z} \| \leq \| \tilde{x} \oplus (-\tilde{z}) \oplus \tilde{y}_1 \oplus \cdots \oplus \tilde{y}_m \oplus (-\tilde{y}_1) \oplus \cdots \oplus (-\tilde{y}_m) \| .$$

After re-arranging and using the triangle inequality, we obtain

$$\begin{aligned} \| \tilde{x} \ominus \tilde{z} \| &\leq \| [\tilde{x} \oplus (-\tilde{y}_1)] \oplus [\tilde{y}_1 \oplus (-\tilde{y}_2)] + \cdots + [\tilde{y}_i \oplus (-\tilde{y}_{i+1})] + \cdots + [\tilde{y}_m \oplus (-\tilde{z})] \| \\ &\leq \| \tilde{x} \ominus \tilde{y}_1 \| + \| \tilde{y}_1 \ominus \tilde{y}_2 \| + \cdots + \| \tilde{y}_i \ominus \tilde{y}_{i+1} \| + \cdots + \| \tilde{y}_m \ominus \tilde{z} \| . \end{aligned}$$

This completes the proof. □

4. Open Balls

Let X is a vector space, and let $(X, \| \cdot \|)$ be a (conventional) normed space. It is clear to see

$$\{ y : \| x - y \| < \epsilon \} = \{ x + z : \| z \| < \epsilon \}$$

by taking $y = x + z$. However, in the pseudo-seminormed space $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$, we cannot have the following equality

$$\{ \tilde{y} : \| \tilde{x} \ominus \tilde{y} \| < \epsilon \} = \{ \tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon \}.$$

As a matter of fact, by taking $\tilde{y} = \tilde{x} \oplus \tilde{z}$ and using the distributive law (1), we can just obtain

$$\| \tilde{x} \ominus \tilde{y} \| = \| \tilde{x} \ominus (\tilde{x} \oplus \tilde{z}) \| = \| -\tilde{z} \oplus \psi \| \neq \| \tilde{z} \|,$$

where $\psi = \tilde{x} \ominus \tilde{x} \in \Psi$. Therefore, we can define two types of open balls as follows.

Definition 3. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space. Two different types of open balls with a radius of ϵ are defined by*

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\}$$

and

$$\mathfrak{O}(\tilde{x};\epsilon) = \{\tilde{y} : \| \tilde{y} \ominus \tilde{x} \| < \epsilon\},$$

where $\| \tilde{x} \ominus \tilde{y} \| = \| \tilde{y} \ominus \tilde{x} \|$, as shown in (5).

Remark 1. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space. Then, we have the following observations.*

- *Given any fuzzy interval, $\tilde{x}$, we cannot have the equality*

$$\| \tilde{x} \ominus \tilde{x} \| = \| \psi \| = 0.$$

  However, if the norm $\| \cdot \|$ satisfies the null condition, then we have $\| \psi \| = 0$ for any $\psi \in \Psi$, which also means $\| \tilde{x} \ominus \tilde{x} \| = 0$. Therefore, in general, we cannot have $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$, unless the norm $\| \cdot \|$ satisfies the null condition.
- *Since $\tilde{1}_{\{0\}}$ is the zero element of $\mathcal{F}_{cc}(\mathbb{R})$, this gives $\tilde{x} = \tilde{x} \oplus \tilde{1}_{\{0\}}$. Now, we assume $\| \tilde{1}_{\{0\}} \| = 0$. It is clear to see $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$.*

Example 5. *Continued from Example 4, we have the open ball*

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\} = \left\{\tilde{x} \oplus \tilde{z} : \sup_{\alpha \in [0,1]} \left| \tilde{z}_{\alpha}^{L} + \tilde{z}_{\alpha}^{U} \right| < \epsilon \right\}.$$

We remark that

$$\left\{\tilde{x} \oplus \tilde{z} : \sup_{\alpha \in [0,1]} \left| \tilde{z}_{\alpha}^{L} + \tilde{z}_{\alpha}^{U} \right| < \epsilon \right\} = \left\{\tilde{x} \oplus \tilde{z} : \left| \tilde{z}_{\alpha}^{L} + \tilde{z}_{\alpha}^{U} \right| < \epsilon \text{ for all } \alpha \in [0,1] \right\}$$

when the supremum

$$\sup_{\alpha \in [0,1]} \left| \tilde{z}_{\alpha}^{L} + \tilde{z}_{\alpha}^{U} \right| = \max_{\alpha \in [0,1]} \left| \tilde{z}_{\alpha}^{L} + \tilde{z}_{\alpha}^{U} \right|$$

is attained. For example, the function $\tilde{z}_{\alpha}^{L} + \tilde{z}_{\alpha}^{U}$ is upper semicontinuous with respect to α. We also have the following open ball

$$\mathfrak{O}(\tilde{x};\epsilon) = \{\tilde{y} : \| \tilde{y} \ominus \tilde{x} \| < \epsilon\} = \left\{\tilde{y} : \sup_{\alpha \in [0,1]} \left| \left(\tilde{y}_{\alpha}^{L} + \tilde{y}_{\alpha}^{U}\right) - \left(\tilde{x}_{\alpha}^{L} + \tilde{x}_{\alpha}^{U}\right) \right| < \epsilon \right\}.$$

Proposition 5. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *Given $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, let $\psi_{\tilde{x}} = \tilde{x} \ominus \tilde{x} \in \Psi$. Then, we have the following inclusion:*

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \psi_{\tilde{x}} \subseteq \mathfrak{O}^{\diamond}(\tilde{x};\epsilon).$$

(ii) *Suppose that the norm $\| \cdot \|$ satisfies the null sub-inequality. Then, we have the following inclusion:*

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon).$$

(iii) *Given $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, let $\psi_{\tilde{x}} = \tilde{x} \ominus \tilde{x} \in \Psi$. Suppose that the norm $\| \cdot \|$ satisfies the null sub-inequality. Then, we have the following inclusions:*

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \psi_{\tilde{x}} \subseteq \mathfrak{O}(\tilde{x};\epsilon) \text{ and } \mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \psi_{\tilde{x}} \subseteq \mathfrak{O}^{\diamond}(\tilde{x};\epsilon).$$

Proof. To prove part (i), given any $\tilde{y} \in \mathfrak{O}(\tilde{x};\epsilon)$, we have $\| \tilde{y} \ominus \tilde{x} \| < \epsilon$. Let $\tilde{z} = \tilde{y} \ominus \tilde{x}$. This means $\| \tilde{z} \| < \epsilon$. By adding $\tilde{x}$ on both sides, we also have $\tilde{y} \oplus \psi_{\tilde{x}} = \tilde{x} \oplus \tilde{z}$. Therefore, we obtain

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \psi_{\tilde{x}} \subseteq \{\tilde{y} \oplus \psi_{\tilde{x}} : \| \tilde{z} \| < \epsilon\} = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\} = \mathfrak{O}^{\diamond}(\tilde{x};\epsilon).$$

To prove part (ii), given any $\tilde{z} \in \mathcal{F}_{cc}(\mathbb{R})$ satisfying $\| \tilde{z} \| < \epsilon$, using the null sub-inequality, we obtain

$$\| (\tilde{x} \oplus \tilde{z}) \ominus \tilde{x} \| = \| \psi \oplus \tilde{z} \| \leq \| \tilde{z} \| < \epsilon.$$

This shows $\tilde{x} \oplus \tilde{z} \in \mathfrak{O}(\tilde{x};\epsilon)$. Therefore, we obtain the following inclusion:

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\} \subseteq \mathfrak{O}(\tilde{x};\epsilon).$$

Part (iii) follows immediately from parts (i) and (ii). This completes the proof. □

Proposition 6. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *Suppose that the norm $\| \cdot \|$ satisfies the null super-inequality. Then, we have the following inclusion*

$$\mathfrak{O}(\tilde{x} \oplus \psi;\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon)$$

for any $\psi \in \Psi$.

(ii) *Suppose that the norm $\| \cdot \|$ satisfies the null sub-inequality. Then, we have the following inclusions:*

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x} \oplus \psi;\epsilon) \text{ and } \mathfrak{O}^{\diamond}(\tilde{x} \oplus \psi;\epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$$

for any $\psi \in \Psi$.

(iii) *Suppose that the norm $\| \cdot \|$ satisfies the null equality. Then, we have the following equality*

$$\mathfrak{O}(\tilde{x} \oplus \psi;\epsilon) = \mathfrak{O}(\tilde{x};\epsilon)$$

for any $\psi \in \Psi$.

Proof. To prove part (i), given any $\tilde{y} \in \mathfrak{O}(\tilde{x} \oplus \psi;\epsilon)$, we have $\| \tilde{y} \ominus (\tilde{x} \oplus \psi) \| < \epsilon$. Therefore, we obtain

$$\begin{aligned} \| \tilde{y} \ominus \tilde{x} \| &\leq \| (\tilde{y} \ominus \tilde{x}) \oplus \psi \| \text{ (using the null super-inequality)} \\ &= \| (\tilde{y} \ominus \tilde{x}) \ominus \psi \| \text{ (since } -\psi = \psi) \\ &= \| \tilde{y} \ominus (\tilde{x} \oplus \psi) \| < \epsilon \text{ (using the distributive law (1)).} \end{aligned}$$

This shows the desired inclusion.

To prove part (ii), for $\tilde{y} \in \mathfrak{O}(\tilde{x};\epsilon)$, we have $\| \tilde{x} \ominus \tilde{y} \| < \epsilon$. Using the null sub-inequality, we obtain

$$\| (\tilde{x} \oplus \psi) \ominus \tilde{y} \| = \| (\tilde{x} \ominus \tilde{y}) \oplus \psi \| \leq \| \tilde{x} \ominus \tilde{y} \| < \epsilon.$$

This shows the following inclusion

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x} \oplus \psi;\epsilon).$$

Now, for $\tilde{y} \in \mathfrak{O}^{\diamond}(\tilde{x} \oplus \psi;\epsilon)$, we have

$$\tilde{y} = \tilde{x} \oplus \psi \oplus \tilde{z} \text{ with } \| \tilde{z} \| < \epsilon.$$

Let $\tilde{v} = \psi \oplus \tilde{z}$. Using the null sub-inequality, we obtain

$$\| \tilde{v} \| = \| \psi \oplus \tilde{z} \| \leq \| \tilde{z} \| < \epsilon. \tag{6}$$

This shows
$$\tilde{y} = \tilde{x} \oplus \tilde{v} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon).$$
Therefore, we obtain the following inclusion
$$\mathfrak{O}^{\diamond}(\tilde{x} \oplus \psi;\epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x};\epsilon).$$
Finally, part (iii) follows immediately from parts (i) and (ii). This completes the proof. □

Let X be a vector space, and let $(X, \|\cdot\|)$ be a normed space. It is clear to see
$$\mathfrak{O}(x;\epsilon) + y = \mathfrak{O}(x+y;\epsilon)$$
for $x, y \in X$. However, in the pseudo-seminormed space $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ of fuzzy intervals, we cannot have
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \tilde{y} = \mathfrak{O}(\tilde{x} \oplus \tilde{y};\epsilon)$$
for $\tilde{x}, \tilde{y} \in \mathcal{F}_{cc}(\mathbb{R})$. An interesting relationship is presented below.

Proposition 7. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *For any $\tilde{x}, \tilde{x}^* \in \mathcal{F}_{cc}(\mathbb{R})$, we have the following equality:*
$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \tilde{x}^* = \mathfrak{O}^{\diamond}(\tilde{x} \oplus \tilde{x}^*;\epsilon).$$
By taking $\tilde{x}^* = \psi \in \Psi$, we also have the following equality:
$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \psi = \mathfrak{O}^{\diamond}(\tilde{x} \oplus \psi;\epsilon).$$

(ii) *Suppose that the norm $\|\cdot\|$ satisfies the null sub-inequality. Then, we have the following inclusion:*
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \tilde{x}^* \subseteq \mathfrak{O}(\tilde{x} \oplus \tilde{x}^*;\epsilon).$$
We further assume that $\|\cdot\|$ satisfies the null equality. Given any $\psi \in \Psi$, we also have the following inclusions:
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \psi \subseteq \mathfrak{O}(\tilde{x};\epsilon) \text{ and } \mathfrak{O}(\psi;\epsilon) \oplus \tilde{x}^* \subseteq \mathfrak{O}(\tilde{x}^*;\epsilon).$$

(iii) *Suppose that the norm $\|\cdot\|$ satisfies the null sub-inequality. Given any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, let $\psi_{\tilde{x}} = \tilde{x} \ominus \tilde{x} \in \Psi$. Then, we have the following inclusion:*
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \psi_{\tilde{x}} \subseteq \tilde{x} \oplus \mathfrak{O}(\psi_{\tilde{x}};\epsilon).$$

(iv) *Given any $\tilde{x}^* \in \mathcal{F}_{cc}(\mathbb{R})$, let $\psi_{\tilde{x}^*} = \tilde{x}^* \ominus \tilde{x}^* \in \Psi$. Then, we have the following inclusion:*
$$\mathfrak{O}(\tilde{x} \oplus \tilde{x}^*;\epsilon) \oplus \psi_{\tilde{x}^*} \subseteq \mathfrak{O}(\tilde{x};\epsilon) \oplus \tilde{x}^*.$$

Proof. To prove part (i), for $\| \tilde{z} \| < \epsilon$, we have the following equality:
$$(\tilde{x} \oplus \tilde{z}) \oplus \tilde{x}^* = (\tilde{x} \oplus \tilde{x}^*) \oplus \tilde{z},$$
which shows the desired equality.

To prove part (ii), given any $\tilde{y} \in \mathfrak{O}(\tilde{x};\epsilon) \oplus \tilde{x}^*$, there exists $\tilde{y}^* \in \mathfrak{O}(\tilde{x};\epsilon)$ satisfying $\tilde{y} = \tilde{y}^* \oplus \tilde{x}^*$. We also have $\| \tilde{x} \ominus \tilde{y}^* \| < \epsilon$. Let $\psi = \tilde{x}^* \ominus \tilde{x}^* \in \Psi$. Then, we obtain
$$\begin{aligned}
\| (\tilde{x} \oplus \tilde{x}^*) \ominus \tilde{y} \| &= \| (\tilde{x} \oplus \tilde{x}^*) \ominus (\tilde{y}^* \oplus \tilde{x}^*) \| \\
&= \| (\tilde{x} \ominus \tilde{y}^*) \oplus (\tilde{x}^* \ominus \tilde{x}^*) \| \text{ (using the distributive law (1))} \\
&= \| (\tilde{x} \ominus \tilde{y}^*) \oplus \psi \| \leq \| \tilde{x} \ominus \tilde{y}^* \| < \epsilon \text{ (using the null sub-inequality).}
\end{aligned}$$

This shows $\tilde{y} \in \mathfrak{O}(\tilde{x} \oplus \tilde{x}^*; \epsilon)$. Therefore, we obtain the following inclusion:

$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \tilde{x}^* \subseteq \mathfrak{O}(\tilde{x} \oplus \tilde{x}^*; \epsilon).$$

Now, taking $\tilde{x}^* = \psi$ and using part (iii) of Proposition 6, we obtain

$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \psi \subseteq \mathfrak{O}(\tilde{x} \oplus \psi; \epsilon) = \mathfrak{O}(\tilde{x}; \epsilon).$$

Similarly, by taking $\tilde{x} = \psi$, we also obtain

$$\mathfrak{O}(\psi; \epsilon) \oplus \tilde{x}^* \subseteq \mathfrak{O}(\psi \oplus \tilde{x}^*; \epsilon) = \mathfrak{O}(\tilde{x}^*; \epsilon).$$

To prove part (iii), for $\tilde{x}^* \in \mathfrak{O}(\tilde{x}; \epsilon)$, we have

$$\| \tilde{x}^* \ominus \tilde{x} \| < \epsilon \text{ and } \tilde{x}^* \oplus \psi_{\tilde{x}} = \tilde{x} \oplus (\tilde{x}^* \ominus \tilde{x}).$$

Using the null sub-inequality, we also have

$$\| \psi_{\tilde{x}} \ominus (\tilde{x}^* \ominus \tilde{x}) \| \leq \| \tilde{x}^* \ominus \tilde{x} \| < \epsilon.$$

This shows $\tilde{x}^* \ominus \tilde{x} \in \mathfrak{O}(\psi_{\tilde{x}}; \epsilon)$. Therefore, we obtain

$$\tilde{x}^* \oplus \psi_{\tilde{x}} = \tilde{x} \oplus (\tilde{x}^* \ominus \tilde{x}) \in \tilde{x} \oplus \mathfrak{O}(\psi_{\tilde{x}}; \epsilon),$$

which shows the desired inclusion.

To prove part (iv), for $\tilde{y} \in \mathfrak{O}(\tilde{x} \oplus \tilde{x}^*; \epsilon)$, we have $\| \tilde{y} \ominus (\tilde{x} \oplus \tilde{x}^*) \| < \epsilon$. Using the distributive law (1), we also have

$$\epsilon > \| \tilde{y} \ominus (\tilde{x} \oplus \tilde{x}^*) \| = \| (\tilde{y} \ominus \tilde{x}^*) \ominus \tilde{x} \| .$$

This shows $\tilde{y} \ominus \tilde{x}^* \in \mathfrak{O}(\tilde{x}; \epsilon)$. Since

$$\tilde{y} \oplus \psi_{\tilde{x}^*} = (\tilde{y} \ominus \tilde{x}^*) \oplus \tilde{x}^*,$$

we obtain

$$\tilde{y} \oplus \psi_{\tilde{x}^*} \in \mathfrak{O}(\tilde{x}; \epsilon) \oplus \tilde{x}^*,$$

which shows the desired inclusion. This completes the proof. □

Proposition 8. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals.*

(i) *We have the following properties:*

  (a) *Suppose that the norm $\| \cdot \|$ satisfies the null super-inequality. Given any $\psi \in \Psi$,*

$$\tilde{x} \oplus \psi \in \mathfrak{O}(\tilde{x}^*; \epsilon) \textit{ implies } \tilde{x} \in \mathfrak{O}(\tilde{x}^*; \epsilon).$$

  (b) *Suppose that the norm $\| \cdot \|$ satisfies the null sub-inequality. Given any $\psi \in \Psi$,*

$$\tilde{x} \in \mathfrak{O}(\tilde{x}^*; \epsilon) \textit{ implies } \tilde{x} \oplus \psi \in \mathfrak{O}(\tilde{x}^*; \epsilon)$$

  and

$$\tilde{x} \in \mathfrak{O}^\diamond(\tilde{x}^*; \epsilon) \textit{ implies } \tilde{x} \oplus \psi \in \mathfrak{O}^\diamond(\tilde{x}^*; \epsilon).$$

  (c) *Suppose that the norm $\| \cdot \|$ satisfies the null equality. Then, given any $\psi \in \Psi$,*

$$\tilde{x} \oplus \psi \in \mathfrak{O}(\tilde{x}^*; \epsilon) \textit{ if and only if } \tilde{x} \in \mathfrak{O}(\tilde{x}^*; \epsilon).$$

(ii) *We have the following inclusions:*

$$\mathfrak{O}(\tilde{x}; \epsilon) \subseteq \mathfrak{O}(\tilde{x}; \epsilon) \oplus \Psi \textit{ and } \mathfrak{O}^\diamond(\tilde{x}; \epsilon) \subseteq \mathfrak{O}^\diamond(\tilde{x}; \epsilon) \oplus \Psi.$$

We further assume that the norm $\| \cdot \|$ satisfies the null sub-inequality. Then, we have the following equalities:

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi = \mathfrak{O}(\tilde{x};\epsilon) \text{ and } \mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \Psi = \mathfrak{O}^{\diamond}(\tilde{x};\epsilon).$$

(iii) *Suppose that the norm $\| \cdot \|$ satisfies the null condition. Given a fixed $\psi \in \Psi$, we have the following inclusions:*

$$\Psi \oplus \psi \subseteq \mathfrak{O}^{\diamond}(\psi;\epsilon) \text{ and } \Psi \subseteq \mathfrak{O}(\psi;\epsilon).$$

Proof. To prove case (a) of part (i), using the null super-inequality, we have

$$\| \tilde{x} \ominus \tilde{x}^* \| \leq \| (\tilde{x} \oplus \psi) \ominus \tilde{x}^* \| < \epsilon,$$

which shows the desired implication.

To prove case (b) of part (i), given any $\tilde{x} \in \mathfrak{O}(\tilde{x}^*;\epsilon)$, using the null sub-inequality, we have

$$\| (\tilde{x} \oplus \psi) \ominus \tilde{x}^* \| \leq \| \tilde{x} \ominus \tilde{x}^* \| < \epsilon.$$

This shows

$$\tilde{x} \oplus \psi \in \mathfrak{O}(\tilde{x}^*;\epsilon).$$

Now, given any $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon)$, we have $\tilde{x} = \tilde{x}^* \oplus \tilde{z}$ and $\| \tilde{z} \| < \epsilon$. Given an $\psi \in \Psi$, let $\tilde{z}^* = \tilde{z} \oplus \psi$. Then, we have

$$\tilde{x} \oplus \psi = \tilde{x}^* \oplus \tilde{z}^*,$$

where

$$\| \tilde{z}^* \| = \| \tilde{z} \oplus \psi \| \leq \| \tilde{z} \| < \epsilon.$$

This shows

$$\tilde{x} \oplus \psi \in \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon).$$

It is clear to see that case (c) of part (i) follows immediately from the previous cases (a) and (b).

To prove part (ii), since $\tilde{1}_{\{0\}} \in \Psi$ is a zero element, we have $\tilde{y} = \tilde{y} \oplus \tilde{1}_{\{0\}}$, which shows the following inclusions:

$$\mathfrak{O}(\tilde{x}^*;\epsilon) \subseteq \mathfrak{O}(\tilde{x}^*;\epsilon) \oplus \Psi \text{ and } \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon) \oplus \Psi.$$

We further assume that the norm $\| \cdot \|$ satisfies the null sub-inequality. Given any $\tilde{x} \in \mathfrak{O}(\tilde{x}^*;\epsilon)$ and $\psi \in \Psi$, using case (b) of part (i), we have $\tilde{x} \oplus \psi \in \mathfrak{O}(\tilde{x}^*;\epsilon)$. This shows the following inclusion:

$$\mathfrak{O}(\tilde{x}^*;\epsilon) \oplus \Psi \subseteq \mathfrak{O}(\tilde{x}^*;\epsilon).$$

Given any $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon)$ and $\psi \in \Psi$, using case (b) of part (i), we have $\tilde{x} \oplus \psi \in \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon)$. This shows the following inclusion:

$$\mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon) \oplus \Psi \subseteq \mathfrak{O}^{\diamond}(\tilde{x}^*;\epsilon).$$

Therefore, we obtain the desired equalities.

To prove part (iii), for any $\psi' \in \Psi$, we have $\| \psi' \| = 0$. This shows

$$\psi \oplus \psi' \in \mathfrak{O}^{\diamond}(\psi;\epsilon),$$

which shows the inclusion $\Psi \oplus \psi \subseteq \mathfrak{O}^{\diamond}(\psi;\epsilon)$. On the other hand, we also have

$$\| \psi' \ominus \psi \| = \| \psi' \oplus (-\psi) \| \leq \| \psi' \| + \| -\psi \| = \| \psi' \| + \| \psi \| = 0.$$

This shows $\psi' \in \mathfrak{O}(\psi;\epsilon)$. Therefore, we obtain $\Psi \subseteq \mathfrak{O}(\psi;\epsilon)$. This completes the proof. □

Proposition 9. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals.*

(i) *Given any fixed $\psi \in \Psi$ and $\lambda \neq 0$, we have the following properties:*
   (a) *Suppose that the norm $\| \cdot \|$ satisfies the null equality. Then, we have the following inclusion:*
   $$\lambda \mathfrak{O}(\psi;\epsilon) \subseteq \mathfrak{O}(\psi;|\lambda|\epsilon).$$
   (b) *Suppose that the norm $\| \cdot \|$ satisfies the null super-inequality and $\lambda < 1$. Then, we have the following inclusion:*
   $$\mathfrak{O}(\psi;|\lambda|\epsilon) \subseteq \lambda \mathfrak{O}(\psi;\epsilon).$$

(ii) *Given any fixed $\psi \in \Psi$ and $\lambda \neq 0$, we have the following equality:*
$$\lambda \mathfrak{O}^{\diamond}(\psi;\epsilon) = \mathfrak{O}^{\diamond}(\lambda\psi;|\lambda|\epsilon).$$

Proof. To prove case (a) of part (i), given any $\tilde{x} \in \mathfrak{O}(\psi;\epsilon)$, since $\lambda\psi \in \Psi$, using the null equality, we have
$$\| \psi \ominus \lambda\tilde{x} \| = \| (\psi \oplus \lambda\psi) \ominus \lambda\tilde{x} \| = \| \lambda\psi \ominus \lambda\tilde{x} \| = \| \lambda(\tilde{x} \ominus \psi) \| = |\lambda| \; \| \tilde{x} \ominus \psi \| < |\lambda|\epsilon.$$
This gives $\lambda\tilde{x} \in \mathfrak{O}(\psi;|\lambda|\epsilon)$. Therefore, we obtain the following inclusion:
$$\lambda \mathfrak{O}(\psi;\epsilon) \subseteq \mathfrak{O}(\psi;|\lambda|\epsilon).$$

To prove case (b) of part (i), given any $\tilde{x} \in \mathfrak{O}(\psi;|\lambda|\epsilon)$, we have $\| \psi \ominus \tilde{x} \| < |\lambda|\epsilon$. This gives
$$\| (\psi/\lambda) \ominus (\tilde{x}/\lambda) = \| (\psi \ominus \tilde{x})/\lambda \| = 1/|\lambda| \cdot \| \psi \ominus \tilde{x} \| < \epsilon.$$
Since $\lambda < 1$, we have
$$\frac{\psi}{\lambda} = \psi \oplus \left(\frac{1}{\lambda} - 1\right)\psi \text{ and } \left(\frac{1}{\lambda} - 1\right)\psi \in \Psi.$$
Therefore, there exists $\psi^* \in \Psi$ satisfying $\psi/\lambda = \psi \oplus \psi^*$. Now, we obtain
$$\| \psi \ominus (\tilde{x}/\lambda) \| \leq \| (\psi \oplus \psi^*) \ominus (\tilde{x}/\lambda) \| = \| (\psi/\lambda) \ominus (\tilde{x}/\lambda) \| < \epsilon.$$
This gives $\tilde{x}/\lambda \in \mathfrak{O}(\psi;\epsilon)$, which also gives $\tilde{x} \in \lambda\mathfrak{O}(\psi;\epsilon)$. Therefore, we obtain the following inclusion:
$$\mathfrak{O}(\psi;|\lambda|\epsilon) \subseteq \lambda \mathfrak{O}(\psi;\epsilon).$$

To prove part (ii), given any $\tilde{x} \in \mathfrak{O}^{\diamond}(\psi;\epsilon)$, we have $\tilde{x} = \psi \oplus \tilde{z}$ and $\| \tilde{z} \| < \epsilon$. Therefore, we obtain $\lambda\tilde{x} = \lambda\psi \oplus \lambda\tilde{z}$. Let $\tilde{z}^* = \lambda\tilde{z}$. This gives $\| \tilde{z}^* \| < |\lambda|\epsilon$. Therefore, we obtain the following inclusion:
$$\lambda \mathfrak{O}^{\diamond}(\psi;\epsilon) \subseteq \mathfrak{O}^{\diamond}(\lambda\psi;|\lambda|\epsilon).$$
Now, given any $\tilde{x} \in \mathfrak{O}^{\diamond}(\lambda\psi;|\lambda|\epsilon)$, we have $\tilde{x} = \lambda\psi \oplus \tilde{z}$ and$\| \tilde{z} \| < |\lambda|\epsilon$. Let $\tilde{z}^* = \tilde{z}/\lambda$. We obtain
$$\tilde{x} = \lambda\psi \oplus \tilde{z} = \lambda\psi \oplus \lambda(\tilde{z}/\lambda) = \lambda\psi \oplus \lambda\tilde{z}^* = \lambda(\psi \oplus \tilde{z}^*) \text{ with } \| \tilde{z}^* \| < \epsilon.$$
This gives $\tilde{x} \in \lambda\mathfrak{O}^{\diamond}(\psi;\epsilon)$. Therefore, we obtain the following inclusion:
$$\mathfrak{O}^{\diamond}(\lambda\psi;|\lambda|\epsilon) \subseteq \lambda \mathfrak{O}^{\diamond}(\psi;\epsilon).$$
This completes the proof. □

5. Open Sets

Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals. The concept of open set in $\mathcal{F}_{cc}(\mathbb{R})$ is considered below by using the types of open balls studied above.

Definition 4. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$.*

- *A fuzzy interval, $\tilde{x} \in \mathfrak{F}$, is called an interior point of $\mathfrak{F}$ when there exists an open ball $\mathfrak{O}(\tilde{x};\epsilon)$ satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F}.$$
The collection of all interior points of $\mathfrak{F}$ is called the interior of $\mathfrak{F}$ and is denoted by $int(\mathfrak{F})$.
- *A fuzzy interval, $\tilde{x} \in \mathfrak{F}$, is called an α-interior point of $\mathfrak{F}$ when there exists an open ball $\mathfrak{O}(\tilde{x};\epsilon)$ satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F}.$$
The collection of all α-interior points of $\mathfrak{F}$ is called the α-interior of $\mathfrak{F}$ and is denoted by $int^{(\alpha)}(\mathfrak{F})$.
- *A fuzzy interval, $\tilde{x} \in \mathfrak{F}$, is called a β-interior point of $\mathfrak{F}$ when there exists an open ball $\mathfrak{O}(\tilde{x};\epsilon)$ satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F} \oplus \Psi.$$
The collection of all β-interior points of $\mathfrak{F}$ is called the β-interior of $\mathfrak{F}$ and is denoted by $int^{(\beta)}(\mathfrak{F})$.
- *A fuzzy interval, $\tilde{x} \in \mathfrak{F}$, is called a γ-interior point of $\mathfrak{F}$ when there exists an open ball $\mathfrak{O}(\tilde{x};\epsilon)$ satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$
The collection of all γ-interior points of $\mathfrak{F}$ is called the γ-interior of $\mathfrak{F}$ and is denoted by $int^{(\gamma)}(\mathfrak{F})$.

The different types of $\diamond$-interior points using the open ball, $\mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$, can be similarly defined. For example, a fuzzy interval, $\tilde{x} \in \mathfrak{F}$, is called a $\diamond\gamma$-interior point of $\mathfrak{F}$ when there exists an open ball, $\mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$, satisfying

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$

The collection of all $\diamond\gamma$-interior points of $\mathfrak{F}$ is called the $\diamond\gamma$-interior of $\mathfrak{F}$ and is denoted by $\mathrm{int}^{(\diamond\gamma)}(\mathfrak{F})$.

Example 6. *Continued from Example 1, given a trapezoidal fuzzy interval, $\tilde{a} = (a^L, a_1, a_2, a^U)$, the α-level set $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$ is obtained by*

$$\tilde{a}^L_\alpha = (1-\alpha)a^L + \alpha a_1 \text{ and } \tilde{a}^U_\alpha = (1-\alpha)a^U + \alpha a_2.$$

We consider a nonempty subset, $\mathfrak{F}$, of $\mathcal{F}_{cc}(\mathbb{R})$ by

$$\begin{aligned}\mathfrak{F} &= \left\{\tilde{y} : \int_0^1 \left| \left(\tilde{y}^L_\alpha + \tilde{y}^U_\alpha\right) - \left(\tilde{a}^L_\alpha + \tilde{a}^U_\alpha\right)\right| d\alpha \leq 1\right\} \\ &= \left\{\tilde{y} : \int_0^1 \left| \left(\tilde{y}^L_\alpha + \tilde{y}^U_\alpha\right) - (1-\alpha)(a^L + a^U) - \alpha(a_1 + a_2)\right| d\alpha \leq 1\right\}.\end{aligned}$$

Considering the norm given in Example 3, we have the open ball

$$\mathfrak{O}(\tilde{x};\epsilon) = \{\tilde{y} : \| \tilde{y} \ominus \tilde{x} \| < \epsilon\} = \left\{\tilde{y} : \int_0^1 \left| \left(\tilde{y}^L_\alpha + \tilde{y}^U_\alpha\right) - \left(\tilde{x}^L_\alpha + \tilde{x}^U_\alpha\right)\right| d\alpha < \epsilon\right\}.$$

The definition says that $\tilde{x}$ is an interior point of $\mathfrak{F}$ when there exists an $\epsilon > 0$ satisfying the following inclusion

$$\left\{\tilde{y} : \int_0^1 \left|\left(\tilde{y}_\alpha^L + \tilde{y}_\alpha^U\right) - \left(\tilde{x}_\alpha^L + \tilde{x}_\alpha^U\right)\right| d\alpha < \epsilon\right\}$$
$$\subseteq \left\{\tilde{y} : \int_0^1 \left|\left(\tilde{y}_\alpha^L + \tilde{y}_\alpha^U\right) - (1-\alpha)(a^L + a^U) - \alpha(a_1 + a_2)\right| d\alpha \leq 1\right\}.$$

In particular, if we take $\tilde{x}$ to be a trapezoidal fuzzy interval $\tilde{b} = (b^L, b_1, b_2, b^U)$, then $\tilde{b}$ is an interior point of $\mathfrak{F}$ when there exists an $\epsilon > 0$ satisfying the following inclusion

$$\left\{\tilde{y} : \int_0^1 \left|\left(\tilde{y}_\alpha^L + \tilde{y}_\alpha^U\right) - (1-\alpha)(b^L + b^U) - \alpha(b_1 + b_2)\right| d\alpha < \epsilon\right\}$$
$$\subseteq \left\{\tilde{y} : \int_0^1 \left|\left(\tilde{y}_\alpha^L + \tilde{y}_\alpha^U\right) - (1-\alpha)(a^L + a^U) - \alpha(a_1 + a_2)\right| d\alpha \leq 1\right\}.$$

Remark 2. *Let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. Remark 1 says $\tilde{x} \notin \mathfrak{O}(\tilde{x};\epsilon)$ in general, unless the norm $\|\cdot\|$ satisfies the null condition. For a fuzzy interval, $\tilde{x}$, satisfying $\| \tilde{x} \ominus \tilde{x} \| \neq 0$ and $\epsilon^* < \| \tilde{x} \ominus \tilde{x} \|$, it follows that $\tilde{x} \notin \mathfrak{O}(\tilde{x};\epsilon^*)$. Given any $\epsilon < \epsilon^*$, it is clear to see $\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon^*)$. When we take $\mathfrak{F} = \mathfrak{O}(\tilde{x};\epsilon^*)$, we see that the open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, is contained in $\mathfrak{F}$, even though the center $\tilde{x}$ is not in $\mathfrak{F}$.*

Remark 3. *Let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. Remark 2 says that it is possible that there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying $\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F}$ and $\tilde{x} \notin \mathfrak{F}$. In this situation, $\tilde{x}$ is not an interior point since $\tilde{x} \notin \mathfrak{F}$. Similarly, the set $\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi$ does not necessarily contain the center $\tilde{x}$. Therefore, it is possible that there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying $\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F}$ and $\tilde{x} \notin \mathfrak{F}$. In this situation, $\tilde{x}$ is not an α-interior point, since $\tilde{x} \notin \mathfrak{F}$. However, we make the following observations.*

- *Suppose that the norm $\|\cdot\|$ satisfies the null condition. Remark 1 says $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$. Since $\tilde{1}_{\{0\}} \in \Psi$ is a zero element, we have $\tilde{x} = \tilde{x} \oplus \tilde{1}_{\{0\}}$, which also means $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi$.*
- *Suppose that $\| \tilde{1}_{\{0\}} \| = 0$. Then, we have $\tilde{x} \in \mathfrak{O}^\diamond(\tilde{x};\epsilon)$ by the second observation of Remark 1. Since $\tilde{x} = \tilde{x} \oplus \tilde{1}_{\{0\}}$ and $\tilde{1}_{\{0\}} \in \Psi$, we also have $\tilde{x} \in \mathfrak{O}^\diamond(\tilde{x};\epsilon) \oplus \Psi$.*

According to Remark 3, the different types of so-called pseudo-interior points are defined below.

Definition 5. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$.*

- *A fuzzy interval, $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, is called a pseudo-interior point of $\mathfrak{F}$ when there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F}.$$
The collection of all pseudo-interior points of $\mathfrak{F}$ is called the pseudo-interior of $\mathfrak{F}$ and is denoted by $\mathrm{pint}(\mathfrak{F})$.
- *A fuzzy interval, $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, is called an α-pseudo-interior point of $\mathfrak{F}$ when there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F}.$$
The collection of all α-pseudo-interior points of $\mathfrak{F}$ is called the α-pseudo-interior of $\mathfrak{F}$ and is denoted by $\mathrm{pint}^{(\alpha)}(\mathfrak{F})$.
- *A fuzzy interval, $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, is called a β-pseudo-interior point of $\mathfrak{F}$ when there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F} \oplus \Psi.$$
The collection of all β-pseudo-interior points of $\mathfrak{F}$ is called the β-pseudo-interior of $\mathfrak{F}$ and is denoted by $\mathrm{pint}^{(\beta)}(\mathfrak{F})$.

- *A fuzzy interval, $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, is called a γ-pseudo-interior point of $\mathfrak{F}$ when there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying*
$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$
The collection of all γ-pseudo-interior points of $\mathfrak{F}$ is called the γ-pseudo-interior of $\mathfrak{F}$ and is denoted by $pint^{(\gamma)}(\mathfrak{F})$.

The different types of $\diamond$-pseudo-interior points using the open ball, $\mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$, can be similarly defined. The difference between Definitions 4 and 5 is that $\tilde{x} \in \mathfrak{F}$ is considered in Definition 4, and $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ is considered in Definition 5.

Remark 4. *Let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. Then, we have the following observations.*

- *The following inclusions are obvious:*
$$int(\mathfrak{F}) \subseteq pint(\mathfrak{F}) \text{ and } int^{(\diamond)}(\mathfrak{F}) \subseteq pint^{(\diamond)}(\mathfrak{F}).$$
We also have
$$\begin{aligned} &int^{(\alpha)}(\mathfrak{F}) \subseteq pint^{(\alpha)}(\mathfrak{F}) \text{ and } int^{(\diamond\alpha)}(\mathfrak{F}) \subseteq pint^{(\diamond\alpha)}(\mathfrak{F}) \\ &int^{(\beta)}(\mathfrak{F}) \subseteq pint^{(\beta)}(\mathfrak{F}) \text{ and } int^{(\diamond\beta)}(\mathfrak{F}) \subseteq pint^{(\diamond\beta)}(\mathfrak{F}) \\ &int^{(\gamma)}(\mathfrak{F}) \subseteq pint^{(\gamma)}(\mathfrak{F}) \text{ and } int^{(\diamond\gamma)}(\mathfrak{F}) \subseteq pint^{(\diamond\gamma)}(\mathfrak{F}). \end{aligned}$$
- *The following inclusions are obvious:*
$$int(\mathfrak{F}) \subseteq \mathfrak{F}, \quad int^{(\alpha)}(\mathfrak{F}) \subseteq \mathfrak{F}, \quad int^{(\beta)}(\mathfrak{F}) \subseteq \mathfrak{F} \text{ and } int^{(\gamma)}(\mathfrak{F}) \subseteq \mathfrak{F}.$$
However, the above inclusions may not hold true for pseudo-interior points.
- *Using Remark 1, we also have the following observations.*
  - *Suppose that the norm $\|\cdot\|$ satisfies the null condition. Since $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$, the concepts of an interior point and a pseudo-interior point are equivalent.*
  - *Suppose that $\| \theta \| = 0$. Since $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$, the concepts of a $\diamond$-interior point and a $\diamond$-pseudo-interior point are equivalent.*

Remark 5. *Suppose that the norm $\|\cdot\|$ satisfies the null sub-inequality. Using part (ii) of Proposition 8, we make the following observations.*

- *The concept of an interior point is equivalent to the concept of an α-interior point.*
- *The concept of a β-interior point is equivalent to the concept of a γ-interior point.*
- *The concept of a pseudo-interior point is equivalent to the concept of an α-pseudo-interior point.*
- *The concept of a β-pseudo-interior point is equivalent to the concept of a γ-pseudo-interior point.*

The equivalence between the concepts of $\diamond$-interior points and $\diamond$-pseudo-interior points can be similarly realized.

Remark 6. *Suppose that the norm $\|\cdot\|$ satisfies the null sub-inequality. Using part (ii) of Proposition 5, we make the following observations:*

- *If $\tilde{x}$ is an interior point, then it is also a $\diamond$-interior point.*
- *If $\tilde{x}$ is an α-interior point, then it is also a $\diamond\alpha$-interior point.*
- *If $\tilde{x}$ is a β-interior point, then it is also a $\diamond\beta$-interior point.*
- *If $\tilde{x}$ is a γ-interior point, then it is also a $\diamond\gamma$-interior point.*

Let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. Using Remark 5, we obtain the following relationships:
$$int(\mathfrak{F}) = int^{(\alpha)}(\mathfrak{F}) \subseteq int^{(\diamond\alpha)}(\mathfrak{F}) = int^{\diamond}(\mathfrak{F})$$

and

$$int^{(\beta)}(\mathfrak{F}) = int^{(\gamma)}(\mathfrak{F}) \subseteq int^{(\diamond\gamma)}(\mathfrak{F}) = int^{(\diamond\beta)}(\mathfrak{F}).$$

Similarly, for the pseudo-interior point, we also have the following relationships:

$$pint(\mathfrak{F}) = pint^{(\alpha)}(\mathfrak{F}) \subseteq pint^{(\diamond\alpha)}(\mathfrak{F}) = pint^{\diamond}(\mathfrak{F})$$

and

$$pint^{(\beta)}(\mathfrak{F}) = pint^{(\gamma)}(\mathfrak{F}) \subseteq pint^{(\diamond\gamma)}(\mathfrak{F}) = pint^{(\diamond\beta)}(\mathfrak{F}).$$

Remark 7. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. Suppose that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Then, the concept of an interior point is equivalent to the concept of a pseudo-interior point, which shows*

$$pint(\mathfrak{F}) = int(\mathfrak{F}) \subseteq \mathfrak{F}.$$

Similarly, if $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$, then we have

$$pint^{\diamond}(\mathfrak{F}) = int^{\diamond}(\mathfrak{F}) \subseteq \mathfrak{F}.$$

Remark 8. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. From part (ii) of Proposition 8, we have the following inclusions:*

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \text{ and } \mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \Psi.$$

Suppose that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, and $\epsilon > 0$. Given any $\tilde{x} \in pint^{(\alpha)}(\mathfrak{F})$, we have

$$\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F}.$$

This gives

$$\tilde{x} \in int^{(\alpha)}(\mathfrak{F}) \text{ and } \tilde{x} \in int(\mathfrak{F}).$$

Using Remark 4, we obtain the following inclusions:

$$pint^{(\alpha)}(\mathfrak{F}) \subseteq int(\mathfrak{F}) \subseteq \mathfrak{F} \text{ and } pint^{(\alpha)}(\mathfrak{F}) \subseteq int^{(\alpha)}(\mathfrak{F}) \subseteq pint^{(\alpha)}(\mathfrak{F}).$$

This shows $pint^{(\alpha)}(\mathfrak{F}) = int^{(\alpha)}(\mathfrak{F})$. Similarly, if $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$, then we can also obtain the following relationship

$$pint^{(\diamond\alpha)}(\mathfrak{F}) \subseteq int^{(\diamond)}(\mathfrak{F}) \subseteq \mathfrak{F} \text{ and } pint^{(\diamond\alpha)}(\mathfrak{F}) = int^{(\diamond\alpha)}(\mathfrak{F}).$$

Remark 9. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$ satisfying $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$. Suppose that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Given any $\tilde{x} \in pint^{(\beta)}(\mathfrak{F})$, we have the following inclusions:*

$$\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}.$$

This gives

$$\tilde{x} \in int^{(\beta)}(\mathfrak{F}) \text{ and } \tilde{x} \in int(\mathfrak{F}).$$

Using Remark 4, we obtain the following inclusions:

$$pint^{(\beta)}(\mathfrak{F}) \subseteq int(\mathfrak{F}) \subseteq \mathfrak{F} \text{ and } pint^{(\beta)}(\mathfrak{F}) \subseteq int^{(\beta)}(\mathfrak{F}) \subseteq pint^{(\beta)}(\mathfrak{F}).$$

This shows $pint^{(\beta)}(\mathfrak{F}) = int^{(\beta)}(\mathfrak{F})$. Similarly, if $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$, then we can also obtain the following relationship

$$pint^{(\diamond\beta)}(\mathfrak{F}) \subseteq int^{(\diamond)}(\mathfrak{F}) \subseteq \mathfrak{F} \text{ and } pint^{(\diamond\beta)}(\mathfrak{F}) = int^{(\diamond\beta)}(\mathfrak{F}).$$

Remark 10. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$ satisfying $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$. Suppose that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, and $\epsilon > 0$. Using part (ii) of Proposition 8, we have the following inclusions:*

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \text{ and } \mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x};\epsilon) \oplus \Psi.$$

Given any $\tilde{x} \in pint^{(\gamma)}(\mathfrak{F})$, we also have the following inclusions:

$$\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}.$$

This gives

$$\tilde{x} \in int^{(\alpha)}(\mathfrak{F}), \quad \tilde{x} \in int^{(\beta)}(\mathfrak{F}), \quad \tilde{x} \in int^{(\gamma)}(\mathfrak{F}) \text{ and } \tilde{x} \in int(\mathfrak{F}).$$

Therefore, we obtain the following inclusions:

$$pint^{(\gamma)}(\mathfrak{F}) \subseteq int(\mathfrak{F}) \subseteq \mathfrak{F}, \quad pint^{(\gamma)}(\mathfrak{F}) \subseteq int^{(\alpha)}(\mathfrak{F}) \subseteq \mathfrak{F}, \quad pint^{(\gamma)}(\mathfrak{F}) \subseteq int^{(\beta)}(\mathfrak{F}) \subseteq \mathfrak{F}$$

and

$$pint^{(\gamma)}(\mathfrak{F}) \subseteq int^{(\gamma)}(\mathfrak{F}) \subseteq \mathfrak{F}.$$

Using Remark 4, we can obtain the following inclusions:

$$pint^{(\gamma)}(\mathfrak{F}) \subseteq int^{(\gamma)}(\mathfrak{F}) \subseteq pint^{(\gamma)}(\mathfrak{F}).$$

This shows $pint^{(\gamma)}(\mathfrak{F}) = int^{(\gamma)}(\mathfrak{F})$. Similarly, if $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$, then can also obtain

$$pint^{(\diamond\gamma)}(\mathfrak{F}) \subseteq int^{(\diamond)}(\mathfrak{F}) \subseteq \mathfrak{F}, \quad pint^{(\diamond\gamma)}(\mathfrak{F}) \subseteq int^{(\diamond\alpha)}(\mathfrak{F}) \subseteq \mathfrak{F}, \quad pint^{(\diamond\gamma)}(\mathfrak{F}) \subseteq int^{(\diamond\beta)}(\mathfrak{F}) \subseteq \mathfrak{F}$$

and

$$pint^{(\diamond\gamma)}(\mathfrak{F}) = int^{(\diamond\gamma)}(\mathfrak{F}).$$

Definition 6. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$.*

- *We say that the set $\mathfrak{F}$ is open when $\mathfrak{F} = int(\mathfrak{F})$, and we say that the set $\mathfrak{F}$ is pseudo-open when $\mathfrak{F} = pint(\mathfrak{F})$.*
- *We say that the set $\mathfrak{F}$ is α-open when $\mathfrak{F} = int^{(\alpha)}(\mathfrak{F})$, and we say that the set $\mathfrak{F}$ is α-pseudo-open when $\mathfrak{F} = pint^{(\alpha)}(\mathfrak{F})$.*
- *We say that the set $\mathfrak{F}$ is β-open when $\mathfrak{F} = int^{(\beta)}(\mathfrak{F})$, and we say that the set $\mathfrak{F}$ is β-pseudo-open when $\mathfrak{F} = pint^{(\beta)}(\mathfrak{F})$.*
- *We say that the set $\mathfrak{F}$ is γ-open when $\mathfrak{F} = int^{(\gamma)}(\mathfrak{F})$, and we say that the set $\mathfrak{F}$ is γ-pseudo-open when $\mathfrak{F} = pint^{(\gamma)}(\mathfrak{F})$.*

Using the different types of $\diamond$-interior and $\diamond$-pseudo-interior points, we can similarly define the corresponding types of $\diamond$-open sets.

We adopt the convention $\emptyset \oplus \Psi = \emptyset$.

Remark 11. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$.*

- *Since the empty set, $\emptyset$, contains no elements, we can treat the empty set as an open ball, which also means that the empty set $\emptyset$ is open and pseudo-open. Since*

$$\tilde{x} \in \mathfrak{O} \subseteq X$$

for any open ball $\mathfrak{O}$, it follows that

$$\mathcal{F}_{cc}(\mathbb{R}) \subseteq int(\mathcal{F}_{cc}(\mathbb{R})) \text{ and } \mathcal{F}_{cc}(\mathbb{R}) \subseteq pint(\mathcal{F}_{cc}(\mathbb{R})).$$

This shows that $\mathcal{F}_{cc}(\mathbb{R})$ is open and pseudo-open.

- *Since $\emptyset \oplus \Psi = \emptyset \subseteq \emptyset$, this means that that the empty set $\emptyset$ is α-open and α-pseudo-open. Since*

$$\tilde{x} \in \mathfrak{O} \oplus \Psi \subseteq X$$

for any open ball $\mathfrak{O}$, it follows that

$$\mathcal{F}_{cc}(\mathbb{R}) \subseteq int^{(\alpha)}(\mathcal{F}_{cc}(\mathbb{R})) \text{ and } \mathcal{F}_{cc}(\mathbb{R}) \subseteq pint^{(\alpha)}(\mathcal{F}_{cc}(\mathbb{R})).$$

This shows that $\mathcal{F}_{cc}(\mathbb{R})$ is α-open and α-pseudo-open.

- *Since $\emptyset \subseteq \emptyset = \Psi \oplus \emptyset$, this means that the empty set $\emptyset$ is β-open and β-pseudo-open. Using part (i) of Proposition 1, given any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and any open ball, $\mathfrak{O}$, we have*

$$\tilde{x} \in \mathfrak{O} \subseteq \mathcal{F}_{cc}(\mathbb{R}) \subseteq \mathcal{F}_{cc}(\mathbb{R}) \oplus \Psi.$$

Therefore, we obtain

$$\mathcal{F}_{cc}(\mathbb{R}) \subseteq int^{(\beta)}(\mathcal{F}_{cc}(\mathbb{R})) \text{ and } \mathcal{F}_{cc}(\mathbb{R}) \subseteq pint^{(\beta)}(\mathcal{F}_{cc}(\mathbb{R})).$$

This shows that $\mathcal{F}_{cc}(\mathbb{R})$ is β-open and β-pseudo-open.

- *Since $\emptyset \oplus \Psi \subseteq \Psi \oplus \emptyset$, this means that the empty set, $\emptyset$, is γ-open and γ-pseudo-open. Given any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and any open ball, $\mathfrak{O}$, we have $\tilde{x} \in \mathfrak{O} \subseteq X$. This also means*

$$\mathfrak{O} \oplus \Psi \subseteq X \oplus \Psi.$$

Therefore, we obtain

$$\mathcal{F}_{cc}(\mathbb{R}) \subseteq int^{(\gamma)}(\mathcal{F}_{cc}(\mathbb{R})) \text{ and } \mathcal{F}_{cc}(\mathbb{R}) \subseteq pint^{(\gamma)}(\mathcal{F}_{cc}(\mathbb{R})).$$

This shows that $\mathcal{F}_{cc}(\mathbb{R})$ is γ-open and γ-pseudo-open.

Regarding the $\diamond$-open sets and $\diamond$-pseudo-open sets, we can obtain similar results.

In order to study the topological structure of $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$, we need some interesting results.

Proposition 10. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals, and let $\mathfrak{F}$ be a nonempty subset of $\mathcal{F}_{cc}(\mathbb{R})$. Then, we have the following properties:*

(i) *Suppose that $\mathfrak{F}$ is pseudo-open, i.e., $\mathfrak{F} = \mathrm{pint}(\mathfrak{F})$. Then, $\mathfrak{F}$ is also open. In other words, we have*

$$\mathfrak{F} = \mathrm{pint}(\mathfrak{F}) = \mathrm{int}(\mathfrak{F}).$$

Suppose that $\mathfrak{F} = \mathrm{pint}^{\diamond}(\mathfrak{F})$, Then, we have

$$\mathfrak{F} = \mathrm{pint}^{\diamond}(\mathfrak{F}) = \mathrm{int}^{\diamond}(\mathfrak{F}).$$

(ii) *Suppose that $\mathfrak{F} = \mathrm{pint}^{(\alpha)}(\mathfrak{F})$. Then, we have*

$$\mathfrak{F} = \mathrm{pint}^{(\alpha)}(\mathfrak{F}) = \mathrm{int}^{(\alpha)}(\mathfrak{F}).$$

Suppose that $\mathfrak{F} = \mathrm{pint}^{(\diamond\alpha)}(\mathfrak{F})$. Then, we have

$$\mathfrak{F} = \mathrm{pint}^{(\diamond\alpha)}(\mathfrak{F}) = \mathrm{int}^{(\diamond\alpha)}(\mathfrak{F}).$$

(iii) *Suppose that* $\mathfrak{F} = \mathrm{pint}^{(\beta)}(\mathfrak{F})$. *Then, we have*

$$\mathfrak{F} = \mathrm{pint}^{(\beta)}(\mathfrak{F}) = \mathrm{int}^{(\beta)}(\mathfrak{F}).$$

Suppose that $\mathfrak{F} = \mathrm{pint}^{(\diamond\beta)}(\mathfrak{F})$. *Then, we have*

$$\mathfrak{F} = \mathrm{pint}^{(\diamond\beta)}(\mathfrak{F}) = \mathrm{int}^{(\diamond\beta)}(\mathfrak{F}).$$

(iv) *Suppose that* $\mathfrak{F} = \mathrm{pint}^{(\gamma)}(\mathfrak{F})$. *Then, we have*

$$\mathfrak{F} = \mathrm{pint}^{(\gamma)}(\mathfrak{F}) = \mathrm{int}^{(\gamma)}(\mathfrak{F}).$$

Suppose that $\mathfrak{F} = \mathrm{pint}^{(\diamond\gamma)}(\mathfrak{F})$, *Then, we have*

$$\mathfrak{F} = \mathrm{pint}^{(\diamond\gamma)}(\mathfrak{F}) = \mathrm{int}^{(\diamond\gamma)}(\mathfrak{F}).$$

Proof. It suffices to prove part (i), since the other parts can be similarly realized. Given any

$$\tilde{x} \in \mathrm{pint}(\mathfrak{F}) = \mathfrak{F},$$

there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying $\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F}$. Since $\tilde{x} \in \mathfrak{F}$, it follows that $\tilde{x} \in \mathrm{int}(\mathfrak{F})$. Therefore, we obtain $\mathrm{pint}(\mathfrak{F}) \subseteq \mathrm{int}(\mathfrak{F})$. Using the first observation of Remark 4, we obtain $\mathrm{pint}(\mathfrak{F}) = \mathrm{int}(\mathfrak{F})$. This completes the proof. □

Proposition 11. *Let* $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ *be a pseudo-seminormed space of fuzzy intervals, and let* $\mathfrak{F}$ *be a nonempty subset of* $\mathcal{F}_{cc}(\mathbb{R})$.

(i) *Suppose that the norm* $\|\cdot\|$ *satisfies the null super-inequality. Then, we have the following properties:*
   (a) *Given any type of pseudo-open set,* $\mathfrak{F}$, $\tilde{x} \in \mathfrak{F}$ *implies* $\tilde{x} \oplus \psi \in \mathfrak{F}$ *for any* $\psi \in \Psi$.
   (b) *Given an open set* $\mathfrak{F}$, $\tilde{x} \in \mathfrak{F}$ *implies* $\tilde{x} \oplus \psi \in \mathrm{pint}(\mathfrak{F})$ *for any* $\psi \in \Psi$.
   (c) *Given an* α*-open set* $\mathfrak{F}$, $\tilde{x} \in \mathfrak{F}$ *implies* $\tilde{x} \oplus \psi \in \mathrm{pint}^{(\alpha)}(\mathfrak{F})$ *for any* $\psi \in \Psi$.
   (d) *Given a* β*-open set* $\mathfrak{F}$, $\tilde{x} \in \mathfrak{F}$ *implies* $\tilde{x} \oplus \psi \in \mathrm{pint}^{(\beta)}(\mathfrak{F})$ *for any* $\psi \in \Psi$.
   (e) *Given a* γ*-open set* $\mathfrak{F}$, $\tilde{x} \in \mathfrak{F}$ *implies* $\tilde{x} \oplus \psi \in \mathrm{pint}^{(\gamma)}(\mathfrak{F})$ *for any* $\psi \in \Psi$.

(ii) *Suppose that the norm* $\|\cdot\|$ *satisfies the null sub-inequality. Given any type of pseudo-open set* $\mathfrak{F}$, *we have the following properties:*
   (a) $\tilde{x} \oplus \psi \in \mathfrak{F}$ *implies* $\tilde{x} \in \mathfrak{F}$ *for any* $\psi \in \Psi$.
   (b) $\mathfrak{F} \oplus \psi \subseteq \mathfrak{F}$ *for any* $\psi \in \Psi$ *and* $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$.
   (c) $\tilde{x} \oplus \psi \in \mathfrak{F} \oplus \psi$ *implies* $\tilde{x} \in \mathfrak{F}$ *for any* $\psi \in \Psi$.
   (d) *We have the equality* $\mathfrak{F} = \mathfrak{F} \oplus \Psi$.

(iii) *Suppose that the norm* $\|\cdot\|$ *satisfies the null sub-inequality. Given any type of* $\diamond$*-pseudo-open set* $\mathfrak{F}$, $\tilde{x} \in \mathfrak{F}$ *implies* $\tilde{x} \oplus \psi \in \mathfrak{F}$ *for any* $\psi \in \Psi$.

Proof. To prove part (i), we first assume that $\mathfrak{F}$ is γ-pseudo-open. Given any

$$\tilde{x} \in \mathfrak{F} = \mathrm{pint}^{(\gamma)}(\mathfrak{F}),$$

there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$

Part (i) of Proposition 6 shows the following inclusion

$$\mathfrak{O}(\tilde{x} \oplus \psi;\epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$

Therefore, we obtain
$$\tilde{x} \oplus \psi \in \mathrm{pint}^{(\gamma)}(\mathfrak{F}) = \mathfrak{F}.$$
Now, we assume that $\mathfrak{F}$ is γ-open. Given any
$$\tilde{x} \in \mathfrak{F} = \mathrm{int}^{(\gamma)}(\mathfrak{F}) \subseteq \mathrm{pint}^{(\gamma)}(\mathfrak{F}),$$
we can similarly obtain
$$\tilde{x} \oplus \psi \in \mathrm{pint}^{(\gamma)}(\mathfrak{F}).$$
The other cases can be similarly obtained.

To prove case (a) of part (ii), it suffices to consider the case of a γ-pseudo-open set. Given any
$$\tilde{x} \oplus \psi \in \mathfrak{F} = \mathrm{pint}^{(\gamma)}(\mathfrak{F}),$$
there exists an open ball, $\mathfrak{O}(\tilde{x} \oplus \psi; \epsilon)$, satisfying
$$\mathfrak{O}(\tilde{x} \oplus \psi; \epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$
Part (ii) of Proposition 6 shows the following inclusion:
$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$
Therefore, we obtain $\tilde{x} \in \mathrm{pint}^{(\gamma)}(\mathfrak{F}) = \mathfrak{F}$.

To prove case (b) of part (ii), it suffices to consider the case of a γ-pseudo-open set. Given any $\tilde{x} \in \mathfrak{F} \oplus \psi$, there exists
$$\tilde{x}^* \in \mathfrak{F} = \mathrm{pint}^{(\gamma)}(\mathfrak{F})$$
satisfying $\tilde{x} = \tilde{x}^* \oplus \psi$. This also means that there exists an open ball, $\mathfrak{O}(\tilde{x}^*; \epsilon)$, satisfying
$$\mathfrak{O}(\tilde{x}^*; \epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$
Using part (ii) of Proposition 6, we have
$$\mathfrak{O}(\tilde{x}; \epsilon) \subseteq \mathfrak{O}(\tilde{x} \oplus \psi; \epsilon) = \mathfrak{O}(\tilde{x}^*; \epsilon).$$
Therefore, we obtain
$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi, \text{ i.e., } \tilde{x} \in \mathrm{pint}^{(\gamma)}(\mathfrak{F}) = \mathfrak{F}.$$
Now, given any $\tilde{x} \in \mathfrak{F} \oplus \Psi$, there exists a $\psi \in \Psi$ satisfying $\tilde{x} \in \mathfrak{F} \oplus \psi$. Therefore, we obtain $\tilde{x} \in \mathfrak{F}$ by using the above result, which shows the equality $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$.

To prove case (c) of part (ii), from case (b) of part (ii), we have the following relationship:
$$\tilde{x} \oplus \psi \in \mathfrak{F} \oplus \psi \subseteq \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F},$$
which shows $\tilde{x} \in \mathfrak{F}$ by using case (a) of part (ii).

To prove case (d) of part (ii), since $\tilde{1}_{\{0\}} \in \Psi$ is a zero element, we have $\tilde{x} = \tilde{x} \oplus \tilde{1}_{\{0\}}$. This shows $\mathfrak{F} \subseteq \mathfrak{F} \oplus \Psi$. Therefore, we can obtain the desired equality by using case (b) of part (ii).

To prove part (iii), part (ii) of Proposition 6 shows the following inclusion:
$$\mathfrak{O}^{\diamond}(\tilde{x} \oplus \psi; \epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x}; \epsilon).$$
Therefore, we can obtain the desired result by using a similar argument to the proof of part (i). This completes the proof. $\square$

Proposition 12. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals.*

(i) *Suppose that the norm* $\|\cdot\|$ *satisfies the null condition. Then, we have the following properties:*

(a) *We have the following inclusion:*

$$\text{int}^{(\alpha)}(\mathfrak{F}) \oplus \Psi \subseteq \mathfrak{F}.$$

When $\mathfrak{F}$ *is* α*-open, we also have* $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$.

(b) *We have the following inclusion:*

$$\text{int}^{(\beta)}(\mathfrak{F}) \subseteq \mathfrak{F} \oplus \Psi.$$

When $\mathfrak{F}$ *is* β*-open, we also have* $\mathfrak{F} \subseteq \mathfrak{F} \oplus \Psi$.

(ii) *Suppose that the norm* $\|\cdot\|$ *satisfies the null sub-inequality. Then, we have the following inclusion:*

$$(\text{pint}^{(\beta)}(\mathfrak{F}))^c \oplus \Psi \subseteq (\text{pint}^{(\beta)}(\mathfrak{F}))^c.$$

When $\mathfrak{F}$ *is* β*-pseudo-open, we also have* $\mathfrak{F}^c \oplus \Psi \subseteq \mathfrak{F}^c$.

Proof. To prove case (a) of part (i), given any $\tilde{x} \in \text{int}^{(\alpha)}(\mathfrak{F})$, there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying $\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F}$. The first observation of Remark 1 says $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$. Therefore, we have

$$\tilde{x} \oplus \Psi \subseteq \mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F},$$

which shows the inclusion $\text{int}^{(\alpha)}(\mathfrak{F}) \oplus \Psi \subseteq \mathfrak{F}$.

To prove case (b) of part (i), given any $\tilde{x} \in \text{int}^{(\beta)}(\mathfrak{F})$, there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying $\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F} \oplus \Psi$. Since $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$, we have $\tilde{x} \in \mathfrak{F} \oplus \Psi$, which shows the inclusion $\text{int}^{(\beta)}(\mathfrak{F}) \subseteq \mathfrak{F} \oplus \Psi$.

To prove part (ii), given any

$$\tilde{x} \in (\text{pint}^{(\beta)}(\mathfrak{F}))^c \oplus \Psi,$$

there exist an $\tilde{x}^* \in (\text{pint}^{(\beta)}(\mathfrak{F}))^c$ and an $\psi^* \in \Psi$ satisfying $\tilde{x} = \tilde{x}^* \oplus \psi^*$. This means $\mathfrak{O}(\tilde{x}^*;\epsilon) \not\subseteq \mathfrak{F} \oplus \Psi$ for every $\epsilon > 0$. Using part (ii) of Proposition 6, we have

$$\mathfrak{O}(\tilde{x}^*;\epsilon) \subseteq \mathfrak{O}(\tilde{x}^* \oplus \psi^*;\epsilon) = \mathfrak{O}(\tilde{x};\epsilon).$$

Therefore, we obtain $\mathfrak{O}(\tilde{x};\epsilon) \not\subseteq \mathfrak{F} \oplus \Psi$ for every $\epsilon > 0$, which means $\tilde{x} \notin \text{pint}^{(\beta)}(\mathfrak{F})$. This completes the proof. □

Proposition 13. *Let* $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ *be a pseudo-seminormed space of fuzzy intervals.*

(i) *The open ball* $\mathfrak{O}^\diamond(\tilde{x};\epsilon)$ *is simultaneously* $\diamond$*-open,* $\diamond\beta$*-open and* $\diamond\gamma$*-open. Moreover, we also have the following inclusions:*

$$\mathfrak{O}^\diamond(\tilde{x};\epsilon) \subseteq \text{pint}(\mathfrak{O}^\diamond(\tilde{x};\epsilon)), \quad \mathfrak{O}^\diamond(\tilde{x};\epsilon) \subseteq \text{pint}^{(\diamond\beta)}(\mathfrak{O}^\diamond(\tilde{x};\epsilon)) \text{ and } \mathfrak{O}^\diamond(\tilde{x};\epsilon) \subseteq \text{pint}^{(\diamond\gamma)}(\mathfrak{O}^\diamond(\tilde{x};\epsilon)).$$

(ii) *The open ball* $\mathfrak{O}(\tilde{x};\epsilon)$ *is simultaneously open,* β*-open and* γ*-open. Moreover, we also have the following inclusions:*

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \text{pint}(\mathfrak{O}(\tilde{x};\epsilon)), \quad \mathfrak{O}(\tilde{x};\epsilon) \subseteq \text{pint}^{(\beta)}(\mathfrak{O}(\tilde{x};\epsilon)) \text{ and } \mathfrak{O}(\tilde{x};\epsilon) \subseteq \text{pint}^{(\gamma)}(\mathfrak{O}(\tilde{x};\epsilon)).$$

(iii) *Suppose that the norm* $\|\cdot\|$ *satisfies the null sub-inequality. Then, the open ball* $\mathfrak{O}^\diamond(\tilde{x};\epsilon)$ *is simultaneously* $\diamond\alpha$*-open and* α*-open. Moreover, we also have the following inclusions:*

$$\mathfrak{O}^\diamond(\tilde{x};\epsilon) \subseteq \text{pint}^{(\diamond\alpha)}(\mathfrak{O}^\diamond(\tilde{x};\epsilon)) \text{ and } \mathfrak{O}(\tilde{x};\epsilon) \subseteq \text{pint}^{(\alpha)}(\mathfrak{O}(\tilde{x};\epsilon)).$$

Proof. To prove part (i), given any $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$, we have $\tilde{x} = \tilde{x}^{\circ} \oplus \tilde{z}$ and $\| \tilde{z} \| < \epsilon$. Let $\widehat{\epsilon} = \epsilon - \| \tilde{z} \| > 0$. Given any $\tilde{x}^* \in \mathfrak{O}^{\diamond}(\tilde{x};\widehat{\epsilon})$, we also have $\tilde{x}^* = \tilde{x} \oplus \tilde{v}$ and $\| \tilde{v} \| < \widehat{\epsilon}$. Therefore, we obtain

$$\tilde{x}^* = \tilde{x}^{\circ} \oplus \tilde{z} \oplus \tilde{v}$$

and

$$\| \tilde{z} \oplus \tilde{v} \| \leq \| \tilde{z} \| + \| \tilde{v} \| = \epsilon - \widehat{\epsilon} + \| \tilde{v} \| < \epsilon - \widehat{\epsilon} + \widehat{\epsilon} = \epsilon.$$

This shows $\tilde{x}^* \in \mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$. Therefore, we obtain the following inclusions:

$$\mathfrak{O}^{\diamond}(\tilde{x};\widehat{\epsilon}) \subseteq \mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon), \tag{7}$$

which also shows the following inclusion:

$$\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) \subseteq \text{int}(\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)).$$

Therefore, we obtain the following equality:

$$\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) = \text{int}(\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)).$$

Similarly, we can obtain the following inclusion:

$$\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) \subseteq \text{pint}(\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)).$$

Since $\text{pint}(\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon))$ is not necessarily contained in $\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$, we may not have the following equality:

$$\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) = \text{pint}(\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)).$$

Now, using (7), we have the following inclusion:

$$\mathfrak{O}^{\diamond}(x;\widehat{\epsilon}) \oplus \Psi \subseteq \mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) \oplus \Psi,$$

which shows that $\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$ is $\diamond\gamma$-open. Using (7) and part (ii) of Proposition 8, we obtain the following inclusions:

$$\mathfrak{O}^{\diamond}(\tilde{x};\widehat{\epsilon}) \subseteq \mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) \subseteq \mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon) \oplus \Psi,$$

which shows that $\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$ is $\diamond\beta$-open.

To prove part (ii), given any $\tilde{x} \in \mathfrak{O}(\tilde{x}^{\circ};\epsilon)$, we have $\| \tilde{x} \ominus \tilde{x}^{\circ} \| < \epsilon$. Let $\widehat{\epsilon} = \| \tilde{x} \ominus \tilde{x}^{\circ} \|$. Given any $\tilde{x}^* \in \mathfrak{O}(\tilde{x};\epsilon - \widehat{\epsilon})$, we have $\| \tilde{x}^* \ominus \tilde{x} \| < \epsilon - \widehat{\epsilon}$. Using Proposition 4, we obtain

$$\| \tilde{x}^* \ominus \tilde{x}^{\circ} \| \leq \| \tilde{x}^* \ominus \tilde{x} \| + \| \tilde{x} \ominus \tilde{x}^{\circ} \| = \widehat{\epsilon} + \| \tilde{x}^* \ominus \tilde{x} \| < \widehat{\epsilon} + \epsilon - \widehat{\epsilon} = \epsilon.$$

This means $\tilde{x}^* \in \mathfrak{O}(\tilde{x}^{\circ};\epsilon)$. Therefore, we obtain the following inclusion

$$\mathfrak{O}(\tilde{x};\epsilon - \widehat{\epsilon}) \subseteq \mathfrak{O}(\tilde{x}^{\circ};\epsilon), \tag{8}$$

which also shows the following inclusion:

$$\mathfrak{O}(\tilde{x}^{\circ};\epsilon) \subseteq \text{int}(\mathfrak{O}(\tilde{x}^{\circ};\epsilon)).$$

Therefore, we obtain the following equality:

$$\mathfrak{O}(\tilde{x}^{\circ};\epsilon) = \text{int}(\mathfrak{O}(\tilde{x}^{\circ};\epsilon)).$$

Similarly, we can obtain the following inclusion:

$$\mathfrak{O}(\tilde{x}^{\circ};\epsilon) \subseteq \text{pint}(\mathfrak{O}(\tilde{x}^{\circ};\epsilon)).$$

Now, using (8), we have the following incusion

$$\mathfrak{O}(\tilde{x};\epsilon-\widehat{\epsilon})\oplus\Psi\subseteq\mathfrak{O}(\tilde{x}^{\circ};\epsilon)\oplus\Psi,$$

which shows that $\mathfrak{O}(\tilde{x}^{\circ};\epsilon)$ is γ-open. Using (8) and part (ii) of Proposition 8, we obtain the following inclusion:

$$\mathfrak{O}(\tilde{x};\epsilon-\widehat{\epsilon})\subseteq\mathfrak{O}(\tilde{x}^{\circ};\epsilon)\subseteq\mathfrak{O}(\tilde{x}^{\circ};\epsilon)\oplus\Psi,$$

which also shows that $\mathfrak{O}(\tilde{x}^{\circ};\epsilon)$ is β-open.

To prove part (iii), using (7), (8) and part (ii) of Proposition 8, we have the following relationships:

$$\mathfrak{O}^{\diamond}(\tilde{x};\widehat{\epsilon})\oplus\Psi\subseteq\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)\oplus\Psi=\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$$

and

$$\mathfrak{O}(\tilde{x};\epsilon-\widehat{\epsilon})\oplus\Psi\subseteq\mathfrak{O}(\tilde{x}^{\circ};\epsilon)\oplus\Psi=\mathfrak{O}(\tilde{x}^{\circ};\epsilon).$$

This shows that the open ball $\mathfrak{O}^{\diamond}(\tilde{x}^{\circ};\epsilon)$ is simultaneously $\diamond\alpha$-open and α-open. This completes the proof. □

Proposition 14. *Let $(\mathcal{F}_{cc}(\mathbb{R}),\|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *Suppose that $\tilde{x}\in\mathfrak{O}(\tilde{x};\epsilon)$. Then, $\mathfrak{O}(\tilde{x};\epsilon)$ is pseudo-open. We further assume that the norm $\|\cdot\|$ satisfies the null sub-inequality. Then, $\mathfrak{O}(\tilde{x};\epsilon)$ is simultaneously α-pseudo-open, β-pseudo-open and γ-pseudo-open.*

(ii) *Suppose that $\tilde{x}\in\mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$. Then, $\mathfrak{O}(\tilde{x};\epsilon)$ is $\diamond$-pseudo-open. We further assume that the norm $\|\cdot\|$ satisfies the null sub-inequality. Then, $\mathfrak{O}(\tilde{x};\epsilon)$ is simultaneously $\diamond\alpha$-pseudo-open, $\diamond\beta$-pseudo-open and $\diamond\gamma$-pseudo-open.*

Proof. The results follow immediately from Proposition 13, Remarks 7–10 and part (ii) of Proposition 8. □

6. Topological Spaces

Using the different types of open sets presented above, we can study the topological structure generated by the pseudo-seminormed space $(\mathcal{F}_{cc}(\mathbb{R}),\|\cdot\|)$ of fuzzy intervals.

- The collection of all open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is dented by τ.
- The collection of all $\diamond$-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denoted by $\tau^{(\diamond)}$.
- The collection of all pseudo-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denote by $\mathrm{p}\tau$.
- The collection of all $\diamond$-pseudo-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denoted by $\mathrm{p}\tau^{(\diamond)}$.
- The collection of all α-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denote by $\tau^{(\alpha)}$.
- The collection of all $\diamond\alpha$-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denoted by $\tau^{(\diamond\alpha)}$
- The collection of all α-pseudo-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denoted by $\mathrm{p}\tau^{(\alpha)}$.
- The collection of all $\diamond\alpha$-pseudo-open sets in $\mathcal{F}_{cc}(\mathbb{R})$ is denoted by $\mathrm{p}\tau^{(\diamond\alpha)}$.

The families $\tau^{(\beta)}$, $\tau^{(\gamma)}$, $\tau^{(\diamond\beta)}$, $\tau^{(\diamond\gamma)}$, $\mathrm{p}\tau^{(\beta)}$, $\mathrm{p}\tau^{(\gamma)}$, $\mathrm{p}\tau^{(\diamond\beta)}$ and $\mathrm{p}\tau^{(\diamond\gamma)}$ can be similarly realized.

Theorem 1. *Let $(\mathcal{F}_{cc}(\mathbb{R}),\|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *$(\mathcal{F}_{cc}(\mathbb{R}),\tau^{(\alpha)})$ and $(\mathcal{F}_{cc}(\mathbb{R}),\tau^{(\diamond\alpha)})$ are topological spaces.*

(ii) *Suppose that $\tilde{x}\in\mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x}\in\mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon>0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}),p\tau^{(\alpha)})=(\mathcal{F}_{cc}(\mathbb{R}),\tau^{(\alpha)})$$

is a topological space.

(iii) *Suppose that $\tilde{x}\in\mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ for any $\tilde{x}\in\mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon>0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}),p\tau^{(\diamond\alpha)})=(\mathcal{F}_{cc}(\mathbb{R}),\tau^{(\diamond\alpha)})$$

is a topological space.

Proof. To prove part (i), the second observation of Remark 11 says $\emptyset \in \tau^{(\alpha)}$ and $X \in \tau^{(\alpha)}$. Let $\mathfrak{F} = \bigcap_{i=1}^{n} \mathfrak{F}_i$, where $\mathfrak{F}_i$ are α-open sets for $i = 1, \cdots, n$. Given any $\tilde{x} \in \mathfrak{F}$, we have $\tilde{x} \in \mathfrak{F}_i$ for $i = 1, \cdots, n$. Therefore, there exist open balls, $\mathfrak{O}(\tilde{x}; \epsilon_i)$, satisfying $\mathfrak{O}(\tilde{x}; \epsilon_i) \oplus \Psi \subseteq \mathfrak{F}_i$ for $i = 1, \cdots, n$. Let $\epsilon = \min\{\epsilon_1, \cdots, \epsilon_n\}$. Then, we have the following inclusions:

$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \Psi \subseteq \mathfrak{O}(\tilde{x}; \epsilon_i) \oplus \Psi \subseteq \mathfrak{F}_i$$

for $i = 1, \cdots, n$. This shows

$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \Psi \subseteq \bigcap_{i=1}^{n} \mathfrak{F}_i = \mathfrak{F}.$$

Therefore, we obtain $\mathfrak{F} \subseteq \mathrm{int}^{(\alpha)}(\mathfrak{F})$. Using Remark 4, we must have the equality $\mathfrak{F} = \mathrm{int}^{(\alpha)}(\mathfrak{F})$.

Let $\mathfrak{F} = \bigcup_{\gamma \in \Gamma} \mathfrak{F}_\gamma$. Given any $\tilde{x} \in \mathfrak{F}$, we have $\tilde{x} \in \mathfrak{F}_\gamma$ for some $\gamma \in \Gamma$. Therefore, there exists an open ball $\mathfrak{O}(\tilde{x}; \epsilon)$ satisfying

$$\mathfrak{O}(\tilde{x}; \epsilon) \oplus \Psi \subseteq \mathfrak{F}_\gamma \subseteq \mathfrak{F}.$$

This shows $\mathfrak{F} \subseteq \mathrm{int}^{(\alpha)}(\mathfrak{F})$. Using Remark 4, we must have the equality $\mathfrak{F} = \mathrm{int}^{(\alpha)}(\mathfrak{F})$. Therefore, we conclude that $(\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\alpha)})$ is a topological space. We can similarly show that $(\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\diamond\alpha)})$ is a topological space.

Finally, using Remark 8, parts (ii) and (iii) can be obtained immediately from part (i). This completes the proof. □

By looking at parts (ii) and (iii) of Proposition 1, we need the assumption $\tilde{x} \in \mathfrak{O}(\tilde{x}; \epsilon)$. We note that Remark 1 has shown sufficient evidence to guarantee $\tilde{x} \in \mathfrak{O}(\tilde{x}; \epsilon)$.

Example 7. *Continuing from Example 3, let us define a nonnegative real-valued function, $\| \cdot \|$, in $\mathcal{F}_{cc}(\mathbb{R})$ by*

$$\| \tilde{x} \| = \int_0^1 \left| \tilde{x}_\alpha^L + \tilde{x}_\alpha^U \right| d\alpha.$$

Then, $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ is a normed space of fuzzy intervals such that the null condition and null equality are satisfied. The open ball is given by

$$\mathfrak{O}(\tilde{x}; \epsilon) = \{\tilde{y} : \| \tilde{y} \ominus \tilde{x} \| < \epsilon\} = \left\{\tilde{y} : \int_0^1 \left| \left(\tilde{y}_\alpha^L + \tilde{y}_\alpha^U\right) - \left(\tilde{x}_\alpha^L + \tilde{x}_\alpha^U\right) \right| d\alpha < \epsilon \right\}, \tag{9}$$

which can define the α-open set and $\diamond\alpha$-open set $\mathfrak{F}$, satisfying

$$\mathfrak{F} = int^{(\alpha)}(\mathfrak{F}) \text{ and } \mathfrak{F} = int^{(\diamond\alpha)}(\mathfrak{F}),$$

respectively. Theorem 1 shows that the collection of all such kind of α-open sets and $\diamond\alpha$-open sets form the topological spaces $(\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\alpha)})$ and $(\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\diamond\alpha)})$.

From (9), it is clear to see $\tilde{x} \in \mathfrak{O}(\tilde{x}; \epsilon)$ for any $\epsilon > 0$, since

$$\| \tilde{x} \ominus \tilde{x} \| = \int_0^1 \left| \left(\tilde{x}_\alpha^L + \tilde{x}_\alpha^U\right) - \left(\tilde{x}_\alpha^L + \tilde{x}_\alpha^U\right) \right| d\alpha = 0 < \epsilon.$$

Therefore, part (ii) of Theorem 1 says that

$$(\mathcal{F}_{cc}(\mathbb{R}), p\tau^{(\alpha)}) = (\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\alpha)})$$

is a topological space.

On the other hand, regarding the open ball:

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\} = \left\{\tilde{x} \oplus \tilde{z} : \int_0^1 \left|\tilde{z}_\alpha^L + \tilde{z}_\alpha^U\right| d\alpha < \epsilon\right\}.$$

We take $\tilde{z} = \tilde{1}_{\{0\}}$. *Then, we have*

$$\tilde{z}_\alpha^L = \tilde{z}_\alpha^U = 0 \text{ for all } \alpha \in [0,1]$$

and

$$\tilde{x} \oplus \tilde{z} = \tilde{x} \oplus \tilde{1}_{\{0\}} = \tilde{x},$$

which shows $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ *for any* $\epsilon > 0$. *Therefore, part (iii) of Theorem 1 says that*

$$(\mathcal{F}_{cc}(\mathbb{R}), p\tau^{(\diamond\alpha)}) = (\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\diamond\alpha)})$$

is a topological space.

Theorem 2. *Let* $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ *be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) $(\mathcal{F}_{cc}(\mathbb{R}), \tau)$ *and* $(\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\diamond)})$ *are topological spaces.*

(ii) *Suppose that* $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ *for any* $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ *and* $\epsilon > 0$. *Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}), \tau) = (\mathcal{F}_{cc}(\mathbb{R}), p\tau)$$

is a topological space.

(iii) *Suppose that* $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ *for any* $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ *and* $\epsilon > 0$. *Then, we have that*

$$(\mathcal{F}_{cc}(\mathbb{R}), \tau^{(\diamond)}) = (\mathcal{F}_{cc}(\mathbb{R}), p\tau^{(\diamond)})$$

is a topological space.

Proof. The first observation of Remark 11 says that $\emptyset$ and $\mathcal{F}_{cc}(\mathbb{R})$ are open sets. Therefore, we can use the similar argument of Proposition 1 without considering the null set, Ψ. □

Let $(\mathcal{F}_{cc}(\mathbb{R}), \| \cdot \|)$ be a pseudo-seminormed space of fuzzy intervals. We consider four new families as follows:

$$\begin{aligned}
\widehat{\tau}^{(\beta)} &= \left\{\mathfrak{F} \in \tau^{(\beta)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\} \\
\widehat{\tau}^{(\gamma)} &= \left\{\mathfrak{F} \in \tau^{(\gamma)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\} \\
\widehat{\tau}^{(\diamond\beta)} &= \left\{\mathfrak{F} \in \tau^{(\diamond\beta)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\} \\
\widehat{\tau}^{(\diamond\gamma)} &= \left\{\mathfrak{F} \in \tau^{(\diamond\gamma)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\}
\end{aligned}$$

Then, we have the following inclusions:

$$\widehat{\tau}^{(\beta)} \subseteq \tau^{(\beta)}, \quad \widehat{\tau}^{(\gamma)} \subseteq \tau^{(\gamma)}, \quad \widehat{\tau}^{(\diamond\beta)} \subseteq \tau^{(\diamond\beta)} \text{ and } \widehat{\tau}^{(\diamond\gamma)} \subseteq \tau^{(\diamond\gamma)}.$$

We also consider the following new families:

$$\widehat{\mathrm{p}\tau}^{(\beta)} = \left\{\mathfrak{F} \in \mathrm{p}\tau^{(\beta)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\}$$
$$\widehat{\mathrm{p}\tau}^{(\gamma)} = \left\{\mathfrak{F} \in \mathrm{p}\tau^{(\gamma)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\}$$
$$\widehat{\mathrm{p}\tau}^{(\diamond\beta)} = \left\{\mathfrak{F} \in \mathrm{p}\tau^{(\diamond\beta)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\}$$
$$\widehat{\mathrm{p}\tau}^{(\diamond\gamma)} = \left\{\mathfrak{F} \in \mathrm{p}\tau^{(\diamond\gamma)} : \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}\right\}$$

Then, we have the following inclusions:

$$\widehat{\mathrm{p}\tau}^{(\beta)} \subseteq \mathrm{p}\tau^{(\beta)}, \quad \widehat{\mathrm{p}\tau}^{(\gamma)} \subseteq \mathrm{p}\tau^{(\gamma)}, \quad \widehat{\mathrm{p}\tau}^{(\diamond\beta)} \subseteq \mathrm{p}\tau^{(\diamond\beta)} \text{ and } \widehat{\mathrm{p}\tau}^{(\diamond\gamma)} \subseteq \mathrm{p}\tau^{(\diamond\gamma)}.$$

Theorem 3. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Suppose that the norm $\|\cdot\|$ satisfies the null sub-inequality. Then, we have the following identical families*

$$\widehat{\mathrm{p}\tau}^{(\beta)} = \mathrm{p}\tau^{(\beta)} = \mathrm{p}\tau^{(\gamma)} = \widehat{\mathrm{p}\tau}^{(\gamma)} \text{ and } \widehat{\tau}^{(\beta)} = \tau^{(\beta)} = \tau^{(\gamma)} = \widehat{\tau}^{(\gamma)}.$$

Proof. Using Remark 5 and case (d) of part (ii) of Proposition 11, we can obtain the desired results. This completes the proof. □

Theorem 4. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *$(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\beta)})$ and $(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\diamond\beta)})$ are topological spaces.*

(ii) *Suppose that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}), \widehat{p\tau}^{(\beta)}) = (\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\beta)})$$

is a topological space.

(iii) *Suppose that $\tilde{x} \in \mathfrak{O}^{\diamond}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}), \widehat{p\tau}^{(\diamond\beta)}) = (\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\diamond\beta)})$$

is a topological space.

Proof. To prove part (i), given any $\mathfrak{F}_1, \mathfrak{F}_2 \in \widehat{\tau}^{(\beta)}$, let $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Given any $\tilde{x} \in \mathfrak{F}$, we also have $\tilde{x} \in \mathfrak{F}_i$ for $i = 1,2$. Therefore, there exist open balls, $\mathfrak{O}(\tilde{x};\epsilon_i)$, satisfying $\mathfrak{O}(\tilde{x};\epsilon_i) \subseteq \mathfrak{F}_i \oplus \Psi$ for $i = 1,2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then, we have the following inclusions:

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon_i) \subseteq \mathfrak{F}_i \oplus \Psi$$

for $i = 1,2$. Using Proposition 3, we can obtain

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq [(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)] = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi = \mathfrak{F} \oplus \Psi.$$

This shows that the intersection $\mathfrak{F}$ is β-open. Now, given any $\tilde{x} \in \mathfrak{F} \oplus \Psi$, there exist $\tilde{x}^* \in \mathfrak{F}$ and $\psi \in \Psi$ satisfying $\tilde{x} = \tilde{x}^* \oplus \psi$. Since $\tilde{x}^* \in \mathfrak{F}_1 \cap \mathfrak{F}_2$, it follows that

$$\tilde{x} \in \mathfrak{F}_1 \oplus \Psi \subseteq \mathfrak{F}_1 \text{ and } \tilde{x} \in \mathfrak{F}_2 \oplus \Psi \subseteq \mathfrak{F}_2.$$

Therefore, we obtain

$$\tilde{x} \in \mathfrak{F}_1 \cap \mathfrak{F}_2 = \mathfrak{F}, \text{ i.e., } \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}.$$

This shows that the intersection $\mathfrak{F}$ is indeed in $\widehat{\tau}^{(\beta)}$.

Given a family, $\{\mathfrak{F}_\gamma\}_{\gamma\in\Gamma}$, in $\widehat{\tau}^{(\beta)}$, let $\mathfrak{F} = \bigcup_{\gamma\in\Gamma} \mathfrak{F}_\gamma$. Given any $\tilde{x} \in \mathfrak{F}$, we have $\tilde{x} \in \mathfrak{F}_\gamma$ for some $\gamma \in \Gamma$. Therefore, there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F}_\gamma \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$

This shows that the union $\mathfrak{F}$ is β-open. Now, given any $\tilde{x} \in \mathfrak{F} \oplus \Psi$, there exist $\tilde{x}^* \in \mathfrak{F}$ and $\psi \in \Psi$ satisfying $\tilde{x} = \tilde{x}^* \oplus \psi$, which also means $\tilde{x}^* \in \mathfrak{F}_\gamma$ for some $\gamma \in \Gamma$. Therefore, we obtain the following inclusions:

$$\tilde{x} \in \mathfrak{F}_\gamma \oplus \Psi \subseteq \mathfrak{F}_\gamma \subseteq \mathfrak{F}, \text{ i.e., } \mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}.$$

This shows that the union $\mathfrak{F}$ is indeed in $\widehat{\tau}^{(\beta)}$.

The third observation of Remark 11 says that $\emptyset$ and $\mathcal{F}_{cc}(\mathbb{R})$ are β-open sets. It is clear to see that

$$\emptyset \oplus \Psi = \emptyset \text{ and } \mathcal{F}_{cc}(\mathbb{R}) \oplus \Psi \subseteq \mathcal{F}_{cc}(\mathbb{R}).$$

Therefore, we have $\emptyset, X \in \widehat{\tau}^{(\beta)}$. This shows that the family $(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\beta)})$ is indeed a topological space. The above arguments are also valid to show that $(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\diamond\beta)})$ is a topological space.

Finally, using Remark 9, parts (ii) and (iii) can be obtained immediately from part (i). This completes the proof. □

Theorem 5. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Then, we have the following properties:*

(i) *$(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\gamma)})$ and $(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\diamond\gamma)})$ are topological spaces.*

(ii) *Suppose that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}), \widehat{p\tau}^{(\gamma)}) = (\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\gamma)})$$

is a topological space.

(iii) *Suppose that $\tilde{x} \in \mathfrak{O}^\diamond(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}), \widehat{p\tau}^{(\diamond\gamma)}) = (\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\diamond\gamma)})$$

is a topological space.

Proof. To prove part (i), the fourth observation of Remark 11 says $\emptyset, \mathcal{F}_{cc}(\mathbb{R}) \in \tau^{(\gamma)}$. It is clear to see that

$$\emptyset \oplus \Psi = \emptyset \text{ and } \mathcal{F}_{cc}(\mathbb{R}) \oplus \Psi \subseteq \mathcal{F}_{cc}(\mathbb{R}).$$

Therefore, we obtain $\emptyset, \mathcal{F}_{cc}(\mathbb{R}) \in \widehat{\tau}^{(\gamma)}$.

Given any $\mathfrak{F}_1, \mathfrak{F}_2 \in \widehat{\tau}^{(\gamma)}$, let $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Given any $\tilde{x} \in \mathfrak{F}$, there exist open balls, $\mathfrak{O}(\tilde{x};\epsilon_i)$, satisfying

$$\mathfrak{O}(\tilde{x};\epsilon_i) \oplus \Psi \subseteq \mathfrak{F}_i \oplus \Psi$$

for $i = 1,2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then, we have the following inclusions:

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{O}(\tilde{x};\epsilon_i) \oplus \Psi \subseteq \mathfrak{F}_i \oplus \Psi$$

for $i = 1,2$. Using Proposition 3, we obtain

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq [(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)] = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi = \mathfrak{F} \oplus \Psi.$$

This shows that the intersection $\mathfrak{F}$ is γ-open. From the proof of Proposition 4, we can similarly obtain $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$. This shows that the intersection $\mathfrak{F}$ is indeed in $\widehat{\tau}^{(\gamma)}$.

Given a family, $\{\mathfrak{F}_\gamma\}_{\gamma\in\Gamma}$, in $\widehat{\tau}^{(\gamma)}$, let $\mathfrak{F} = \bigcup_{\gamma\in\Gamma} \mathfrak{F}_\gamma$. Given any $\tilde{x} \in \mathfrak{F}$, we have $\tilde{x} \in \mathfrak{F}_\gamma$ for some γ. Therefore, there exists an open ball, $\mathfrak{O}(\tilde{x};\epsilon)$, satisfying

$$\mathfrak{O}(\tilde{x};\epsilon) \oplus \Psi \subseteq \mathfrak{F}_\gamma \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$

This shows that the union $\mathfrak{F}$ is γ-open. From the proof of Proposition 4, we can similarly obtain $\mathfrak{F} \oplus \Psi \subseteq \mathfrak{F}$. This shows $\mathfrak{F} \in \widehat{\tau}^{(\gamma)}$. Therefore, we conclude that the family $(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\gamma)})$ is indeed a topological space. We can similarly show that $(\mathcal{F}_{cc}(\mathbb{R}), \widehat{\tau}^{(\diamond\gamma)})$ is a topological space.

Finally, using Remark 10, parts (ii) and (iii) can be obtained immediately from part (i). This completes the proof. $\square$

Theorem 6. *Let $(\mathcal{F}_{cc}(\mathbb{R}), \|\cdot\|)$ be a pseudo-seminormed space of fuzzy intervals. Suppose that the norm $\|\cdot\|$ satisfies the null sub-inequality and that $\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon)$ for any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$ and $\epsilon > 0$. Then,*

$$(\mathcal{F}_{cc}(\mathbb{R}), \mathrm{p}\tau^{(\beta)}) = (\mathcal{F}_{cc}(\mathbb{R}), \mathrm{p}\tau^{(\gamma)})$$

is a topological space.

Proof. The third observation of Remark 11 says $\emptyset, \mathcal{F}_{cc}(\mathbb{R}) \in \mathrm{p}\tau^{(\beta)}$. Given any $\mathfrak{F}_1, \mathfrak{F}_2 \in \mathrm{p}\tau^{(\beta)}$, let $\mathfrak{F} = \mathfrak{F}_1 \cap \mathfrak{F}_2$. Given any $\tilde{x} \in \mathfrak{F}$, we have $\tilde{x} \in \mathfrak{F}_i$ for $i = 1,2$. Therefore, there exist open balls, $\mathfrak{O}(\tilde{x};\epsilon_i)$, satisfying

$$\mathfrak{O}(\tilde{x};\epsilon_i) \subseteq \mathfrak{F}_i \oplus \Psi$$

for $i = 1,2$. Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$. Then, we have the following inclusions:

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{O}(\tilde{x};\epsilon_i) \subseteq \mathfrak{F}_i \oplus \Psi$$

for $i = 1,2$. Using case (d) of part (ii) of Proposition 11, we obtain

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq [(\mathfrak{F}_1 \oplus \Psi) \cap (\mathfrak{F}_2 \oplus \Psi)] = \mathfrak{F}_1 \cap \mathfrak{F}_2 = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi = \mathfrak{F} \oplus \Psi,$$

which shows $\tilde{x} \in \mathrm{int}^{(\beta)}(\mathfrak{F})$. Using Remark 4, we also obtain the following inclusions:

$$\mathfrak{F} \subseteq \mathrm{int}^{(\beta)}(\mathfrak{F}) \subseteq \mathrm{pint}^{(\beta)}(\mathfrak{F}).$$

Given any $\tilde{x} \in \mathrm{pint}^{(\beta)}(\mathfrak{F})$, part (ii) of Proposition 11 says

$$\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F} \oplus \Psi = (\mathfrak{F}_1 \cap \mathfrak{F}_2) \oplus \Psi \subseteq \mathfrak{F}_1 \oplus \Psi = \mathfrak{F}_1.$$

We can similarly show $\tilde{x} \in \mathfrak{F}_2$, which implies $\tilde{x} \in \mathfrak{F}_1 \cap \mathfrak{F}_2 = \mathfrak{F}$. Therefore, we obtain the inclusion $\mathrm{pint}^{(\beta)}(\mathfrak{F}) \subseteq \mathfrak{F}$. This shows the equality $\mathrm{pint}^{(\beta)}(\mathfrak{F}) = \mathfrak{F}$.

Given a family, $\{\mathfrak{F}_\gamma\}_{\gamma\in\Gamma}$, in $\mathrm{p}\tau^{(\beta)}$, let $\mathfrak{F} = \bigcup_{\gamma\in\Gamma} \mathfrak{F}_\gamma$. Given any $\tilde{x} \in \mathfrak{F}$, we have $\tilde{x} \in \mathfrak{F}_\gamma$ for some $\gamma \in \Gamma$. Therefore, there exists an open ball $\mathfrak{O}(\tilde{x};\epsilon)$ satisfying

$$\mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F}_\gamma \oplus \Psi \subseteq \mathfrak{F} \oplus \Psi.$$

This shows the following inclusions:

$$\mathfrak{F} \subseteq \mathrm{int}^{(\beta)}(\mathfrak{F}) \subseteq \mathrm{pint}^{(\beta)}(\mathfrak{F}).$$

Given any $\tilde{x} \in \mathrm{pint}^{(\beta)}(\mathfrak{F})$, part (ii) of Proposition 11 says

$$\tilde{x} \in \mathfrak{O}(\tilde{x};\epsilon) \subseteq \mathfrak{F} \oplus \Psi = \mathfrak{F},$$

which shows the inclusion $\mathrm{pint}^{(\beta)}(\mathfrak{F}) \subseteq \mathfrak{F}$. Therefore, we obtain the equality $\mathfrak{F} = \mathrm{pint}^{(\beta)}(\mathfrak{F})$. This shows that the union $\mathfrak{F}$ is a member of $\mathrm{p}\tau^{(\beta)}$. Finally, using Remark 5, we conclude that

$$(\mathcal{F}_{cc}(\mathbb{R}), \mathrm{p}\tau^{(\beta)}) = (\mathcal{F}_{cc}(\mathbb{R}), \mathrm{p}\tau^{(\gamma)})$$

is a topological space. This completes the proof. □

7. Conclusions

As we mentioned above, the space, $\mathcal{F}_{cc}(\mathbb{R})$, of all fuzzy intervals in $\mathbb{R}$ cannot be a (conventional) vector space. The main reason is that each fuzzy interval cannot have an additive inverse element. It also means that each fuzzy interval that is subtracted from itself cannot be a zero element in $\mathcal{F}_{cc}(\mathbb{R})$. Although $\mathcal{F}_{cc}(\mathbb{R})$ is not a vector space, it maintains a vector structure by treating the addition of fuzzy intervals as a vector addition and treating the scalar multiplication of fuzzy intervals as a scalar multiplication of vectors.

In this case, we still can endow a norm on the space $\mathcal{F}_{cc}(\mathbb{R})$. For example, given any $\tilde{x} \in \mathcal{F}_{cc}(\mathbb{R})$, we can define three norms as follows:

$$\begin{aligned}
\| \tilde{x} \| &= \int_0^1 \left| \tilde{x}_\alpha^L - \tilde{x}_\alpha^U \right| d\alpha \\
\| \tilde{x} \| &= \int_0^1 \left| \tilde{x}_\alpha^L + \tilde{x}_\alpha^U \right| d\alpha \\
\| \tilde{x} \| &= \sup_{\alpha \in [0,1]} \left| \tilde{x}_\alpha^L + \tilde{x}_\alpha^U \right| .
\end{aligned}$$

Two different types of open balls with a radius of ϵ are defined by

$$\mathfrak{O}^{\diamond}(\tilde{x};\epsilon) = \{\tilde{x} \oplus \tilde{z} : \| \tilde{z} \| < \epsilon\}$$

and

$$\mathfrak{O}(\tilde{x};\epsilon) = \{\tilde{y} : \| \tilde{y} \ominus \tilde{x} \| < \epsilon\},$$

Using these two different types of open balls, many types of open sets and pseudo-open sets are defined. Theorems 1 and 2 show the topological structure regarding the open sets, and Theorems 4 and 5 show the topological structure regarding the pseudo-open sets.

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