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Exponential Stability and Relative Controllability of Nonsingular Conformable Delay Systems

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Abstract: In this paper, we investigate a delayed matrix exponential and utilize it to derive a representation of solutions to a linear nonsingular delay problem with permutable matrices. To begin with, we present a novel definition of α-exponential stability for these systems. Subsequently, we put forward several adequate conditions to ensure the α-exponential stability of solutions for such delay systems. Moreover, by constructing a Grammian matrix that accounts for delays, we provide a criterion to determine the relative controllability of a linear problem. Additionally, we extend our analysis to nonlinear problems. Lastly, we offer several examples to verify the effectiveness of our theoretical findings.

Keywords: conformable delay systems; α-exponential stability; representation of solutions; relative controllability

MSC: 39B72; 93B05

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Citation: Zhou, A. Exponential Stability and Relative Controllability of Nonsingular Conformable Delay Systems. *Axioms* **2023**, *12*, 994. https://doi.org/10.3390/axioms12100994

Academic Editors: Valery Y. Glizer and Nicolae Lupa

Received: 20 September 2023
Revised: 10 October 2023
Accepted: 17 October 2023
Published: 20 October 2023

1. Introduction

Delay systems are common in various engineering applications, ranging from chemical processes to mechanical systems. The presence of time delays in these systems can cause instability and oscillatory behavior, which can be destructive to the performance and reliability of the system. Consequently, the investigation into the stability and controllability of time-delay systems has garnered substantial research attention over the years. In [1], the concept of delayed matrix exponential was introduced by the authors. They utilized this concept to derive a representation of solutions for a linear delay problem, specifically when permutable matrices are taken into account. In a subsequent work by [2], by using the concepts presented in [1], the authors extended those ideas to a discrete matrix delayed exponential function and investigated the representation of solutions to linear discrete delay systems. Motivated by the works of [1,2], numerous scholars have focused on investigating a wide range of delay systems involving permutable matrices. For a comprehensive understanding of delay systems, we refer interested readers to the following papers: [3–20]. These references delve into various aspects related to the aforementioned systems.

Exponential stability is a very important property of a control system. It is related to the behavior of the system under continuous parameter variations. As for exponential stability, it is crucial for linear control systems because it ensures that the output signals remain approximately constant in the presence of noise and parameter variations. Here, we present a new definition of exponential stability for the investigated systems [4,5,7,11,13,16,18,21–24].

Control theory has attracted a lot of research interest for many years, with the aim of designing efficient and robust control strategies for various engineering applications. Relative controllability is an important concept in control theory, which measures the ability of a control input to guide the system from one state to another. Specifically, we examine the relative controllability of a conformable delay system by employing the concept of null controllability. Null controllability is a fundamental property of linear systems that has been widely studied in the literature and provides a powerful tool for analyzing the controllability of nonlinear systems [3,8,13–15,17,25–29].

In this paper, we build upon the concepts introduced in [1,3,5] to investigate the α-exponential stability and relative controllability of investigated systems. Initially, we focus on the α-exponential stability analysis of systems

$$\begin{cases} B\mathfrak{D}_{\alpha}^{0}z(\zeta) = Fz(\zeta) + Pz(\zeta-\kappa),\ \zeta \geq 0,\ \kappa \geq 0, \\ z(\zeta) = \psi(\zeta),\ \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{1}$$

and

$$\begin{cases} B\mathfrak{D}_{\alpha}^{0}z(\zeta) = Fz(\zeta) + Pz(\zeta-\kappa) + m(\zeta, z(\zeta)),\ \zeta \geq 0,\ \kappa \geq 0, \\ z(\zeta) = \psi(\zeta),\ \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{2}$$

and

$$\begin{cases} B\mathfrak{D}_{\alpha}^{0}z(\zeta) = Fz(\zeta) + Pz(\zeta-\kappa) + l(\zeta, z(\zeta-\kappa)),\ \zeta \geq 0,\ \kappa \geq 0, \\ z(\zeta) = \psi(\zeta),\ \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{3}$$

here $\mathfrak{D}_{\alpha}^{0}(0 < \alpha < 1)$ represents the conformable derivative with lower index zero (see Definition 1 [30]), $B, F, P \in \mathbb{R}^{n\times n}$ are constant permutable matrices and B is a nonsingular matrix and $\psi \in C^{1}([-\kappa, 0], \mathbb{R}^{n})$. Together with, $m, l \in C([0, +\infty] \times \mathbb{R}^{n}, \mathbb{R}^{n})$. Also we are going to discuss the relatively controllability of

$$\begin{cases} B\mathfrak{D}_{\alpha}^{0}z(\zeta) = Fz(\zeta) + Pz(\zeta-\kappa) + y(\zeta, z(\zeta)) + Jv(\zeta),\ \zeta \in I,\ \kappa \geq 0, \\ z(\zeta) = \psi(\zeta),\ \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{4}$$

where $I := [0, \zeta_1], \zeta_1 > 0$, $J \in \mathbb{R}^{n}$, $y \in C(I \times \mathbb{R}^{n}, \mathbb{R}^{n})$ and the control function $v(\cdot)$ takes values from $L^{2}(I, \mathbb{R}^{n})$.

We investigate sufficient conditions to ensure the α-exponential stability of solutions to Equations (1)–(3) using Gronwall inequality techniques. Additionally, we explore the application of a delay Grammian matrix in establishing both sufficient and necessary conditions for relative controllability in linear delayed systems. The discussion also extends to nonlinear problems, employing Krasnoselskii's fixed point theorem.

The remainder of this paper is structured as follows. In Section 2, we provide relevant notations, concepts, and lemmas, along with a representation of solutions to investigated systems. Section 3 is dedicated to the study of α-exponential stability of solutions for Equations (1)–(3). The relative controllability of system (4) is investigated in Section 4. Finally, we present numerical examples in the Section 5 to illustrate our main findings.

2. Preliminaries

Let $\mathbb{R}^{n}$ represent the n-dimensional Euclidean space with the vector norm $\|\cdot\|$, and $\mathbb{R}^{n\times n}$ denote the $n \times n$ matrix space with real-valued elements. For $z \in \mathbb{R}^{n}$ and $F \in \mathbb{R}^{n\times n}$, we define $\|z\| = \max_{1\leq i\leq n} |z_i|$ and $\|F\| = \max_{1\leq i\leq n} \sum_{j=1}^{n} |f_{ij}|$, here, $z_i \in z$ and $f_{ij} \in F$. Let z lies in Banach space $C(I, \mathbb{R}^{n})$, $\|z\|_C = \sup_{\zeta\in I} \|z(\zeta)\|$. Furthermore, we define the norm $\|\psi\|_C = \sup_{\zeta\in[-\kappa,0]} \|\psi(\zeta)\|$. Let Y_1, Y_2 be two Banach spaces, $L_b(Y_1, Y_2)$ stands for the space of all bounded linear operators from Y_1 to Y_2. Additionally, $L^{g}(I, Y_2)$ stands for the Banach space of functions $x : I \to Y_2$ that are Bochner integrable, with the norm $\|y\|_{L^{g}(I,Y_2)}$ (where $1 < g < \infty$).

Consider the matrices Θ and $\mathcal{I}$, where Θ represents the zero matrix and $\mathcal{I}$ represents the identity matrix. Similar to Lemma 2.1 (You and Wang 2019 [21]), we introduce the following lemma without proof.

Lemma 1. *Suppose Ω lie in a constant $n \times n$ matrix. If $\|\Omega\| \leq \beta e^{\beta\kappa}, \beta \in \mathbb{R}^+$, then*

$$\|e_{\kappa,\alpha}^{\Omega(\zeta-\kappa)}\| \leq e^{\frac{\beta}{\alpha}\zeta^\alpha}, \zeta \in \mathbb{R}, \tag{5}$$

where

$$e_{\kappa,\alpha}^{\Omega\zeta} = \begin{cases} \Theta, \ -\infty < \zeta < -\kappa, \kappa > 0 \\ \mathcal{I}, \ -\kappa \leq \zeta \leq 0, \\ \mathcal{I} + \Omega\frac{\zeta^\alpha}{\alpha} + \Omega^2\frac{1}{2!}\left(\frac{(\zeta-\kappa)^\alpha}{\alpha}\right)^2 + \Omega^3\frac{1}{3!}\left(\frac{(\zeta-2\kappa)^\alpha}{\alpha}\right)^3 \\ + \cdots + \Omega^k\frac{1}{k!}\left(\frac{(\zeta-(k-1)\kappa)^\alpha}{\alpha}\right)^k, (k-1)\kappa \leq \zeta < k\kappa, k = 1, 2, \cdots, \end{cases} \tag{6}$$

(this is referred to as the delayed matrix exponential (see [1])).

Remark 1. *Clearly, (5) is reflated to α-time power function, which is different from 1-time power function in Lemma 2.1 (You and Wang 2019 [21]).*

From (6), the uniform continuity of $e_{\kappa,\alpha}^{\Omega\cdot}$ on a compact interval can be observed easily. By employing the transformation $x = Bz$ in Equation (1), we obtain

$$\begin{cases} \mathfrak{D}_\alpha^0 x(\zeta) = FB^{-1}x(\zeta) + PB^{-1}x(\zeta-\kappa), \ \zeta \geq 0, \ \kappa \geq 0, \\ x(\zeta) = B\psi(\zeta), \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{7}$$

Using Theorem 3.4 [31], for (7), we have

$$x(\zeta) \;=\; \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}B\psi(-\kappa) + \int_{-\kappa}^0 \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds, \tag{8}$$

where $\mathcal{P}(\zeta) = e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}e_{\kappa,\alpha}^{P_1B^{-1}\zeta}$ and $P_1 = e^{FB^{-1}\frac{(\zeta-\kappa)^\alpha-\zeta^\alpha}{\alpha}}P$ satisfying initial condition ([31]) $\mathcal{P}(\zeta) = e^{-A\frac{(-\zeta)^\alpha}{\alpha}}, -\kappa \leq \zeta < 0$, where

$$e^{A\frac{\zeta^\alpha}{\zeta}} := \begin{cases} e^{A\frac{\zeta^\alpha}{\alpha}}, \ \zeta \geq 0, \\ e^{-A\frac{(-\zeta)^\alpha}{\alpha}}, \ -\kappa \leq \zeta \leq 0. \end{cases} \tag{9}$$

Now substituting $x = Bz$ into (8), by solving it, we can observe that any solution of Equation (1) takes the form

$$z(\zeta) \;=\; \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^0 \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds, \tag{10}$$

where $FB = BF$, $FP = PF$, $PB = BP$ are employed.

Likewise, by utilizing the transformation $x = Bz$ in

$$\begin{cases} B\mathfrak{D}_\alpha^0 z(\zeta) = Fz(\zeta) + Pz(\zeta-\kappa) + \varpi(\zeta), \ \zeta \geq 0, \ \kappa \geq 0, \\ z(\zeta) = \psi(\zeta), \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{11}$$

one has

$$\begin{cases} \mathfrak{D}_\alpha^0 x(\zeta) = FB^{-1}x(\zeta) + PB^{-1}x(\zeta-\kappa) + \varpi(\zeta), \ \zeta \geq 0, \ \kappa \geq 0, \\ x(\zeta) = B\psi(\zeta), \ -\kappa \leq \zeta \leq 0, \end{cases} \tag{12}$$

(here $x : [0, +\infty) \to \mathbb{R}^n$ is continuous).

Using Theorem 3.5 [31], for (12), we have

$$x(\zeta) = \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}B\psi(-\kappa) + \int_{-\kappa}^{0}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds + \int_0^\zeta \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\varpi(s)s^{\alpha-1}ds. \tag{13}$$

Now substituting $x = Bz$ into (13), upon solving it, it becomes evident that the form of any solution to Equation (11) is given by

$$z(\zeta) = \mathcal{P}(\zeta)e^{GB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds + B^{-1}\int_0^\zeta \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\varpi(s)s^{\alpha-1}ds.$$

Likewise, the solution to Equation (2) is given by

$$z(\zeta) = \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds + B^{-1}\int_0^\zeta \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}m(s,z(s))s^{\alpha-1}ds, \tag{14}$$

and the form of any solution to Equation (3) is given by

$$z(\zeta) = \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds + B^{-1}\int_0^\zeta \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}l(s,z(s-\kappa))s^{\alpha-1}ds. \tag{15}$$

Based on the representation (15) since $D_\alpha^0\psi$ exists (i.e., $\psi \in D_\alpha^0([-\kappa,0],\mathbb{R}^n)$), here is the presented definition.

Definition 1. *The α-exponential stability is attributed to the trivial solution of (1) (or (2), (3)), provided that positive constants c_1, c_2, δ exist, depending on B, F, P and $\|\psi\|_1$, where $\|\psi\|_1 := \max\limits_{[-\kappa,0]}\|\psi\| + \max\limits_{[-\kappa,0]}\|D_\alpha^0\psi\|$, catering to $\|z(\zeta)\| \le c_1e^{-c_2\zeta^\alpha}$, $\zeta \ge 0$, for $\|\psi\|_1 < \delta$, here, z represents any solution to (1) (or (2), (3)).*

Definition 2 (see [3])**.** *Equation (4) to be relatively controllable, suppose there exists an arbitrary initial vector function $\psi \in C^1([-\kappa,0],\mathbb{R}^n)$, and a final state of the vector $z_1 \in \mathbb{R}^n$ at ζ_1. If there is a control $v \in L^2(I,\mathbb{R}^n)$ such that Equation (4) has a solution $z \in C([-\kappa,\zeta_1],\mathbb{R}^n)$ satisfying the boundary condition $z(\zeta_1) = z_1$.*

3. α-Exponential Stability

This section focuses on examining the α-exponential stability of solutions in conformable systems. We take into account the following hypotheses .

(H_1) Set $\sigma(FB^{-1}) = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$ be the eigenvalues of FB^{-1} with

$$Re\lambda_1 \le Re\lambda_2 \le \cdots \le Re\lambda_n \le -k < 0, \quad k > 0,$$

i.e., for $0 < \alpha < 1$, there exist $W, w > 0$ such that

$$\|e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}\| \le We^{-w\frac{\zeta^\alpha}{\alpha}} \quad \text{for all} \quad \zeta \ge 0.$$

(H_2) Suppose $\|P_1B^{-1}\| \le \beta e^{\beta\kappa}$ for $\beta \in \mathbb{R}^+$.

(H_3) For $\zeta \geq 0$ and $z \in \mathbb{R}^n$, there is a positive constant L such that the inequality $\|m(\zeta, z)\| \leq L\|z\|$ holds.

(H_4) For $\zeta \geq 0$ and $z \in \mathbb{R}^n$, there is a positive constant Q such that the inequality $\|l(\zeta, z)\| \leq Q\|z\|$ holds.

(H_5) Suppose $\beta - w < 0$.

(H_6) Suppose $\|B^{-1}\|W^2 L e^{w\frac{\kappa^\alpha}{\alpha}} + \beta - w < 0$.

(H_7) Suppose $\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}} + \beta - w < 0$.

Lemma 2. *Under the hypotheses (H_1) and (H_2), the inequality*

$$\|\mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\| \leq W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}, \quad \beta, \zeta \in \mathbb{R}^+. \tag{16}$$

holds.

Proof. From (H_1) and (H_2) via Lemma 1, we have

$$\begin{aligned}
\|\mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\| &\leq \|e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}\|\|e_{\kappa,\alpha}^{P_1 B^{-1}\zeta}\|\|e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\| \\
&\leq We^{-w\frac{\zeta^\alpha}{\alpha}} e^{\beta\frac{(\zeta+\kappa)^\alpha}{\alpha}} We^{-w\frac{\kappa^\alpha}{\alpha}} \\
&\leq W^2 e^{\frac{(\beta-w)\zeta^\alpha + \beta|(\zeta+\kappa)^\alpha - \zeta^\alpha|}{\alpha}} \\
&\leq W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}.
\end{aligned}$$

Thus, the proof is concluded. □

At present, we are prepared to provide adequate conditions for the α-exponential stability of solutions in investigated systems.

Theorem 1. *Under the hypotheses $(H_1), (H_2)$ and (H_5), the solution of Equation (1) exhibits α-exponential stability.*

Proof. It is worth noting that the solution of (1) takes the form (10). Now from (10) we get

$$\begin{aligned}
\|z(\zeta)\| \leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}} \|\psi(-\kappa)\| \\
& + \|B^{-1}\| \int_{-\kappa}^0 W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta-s)^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\| s^{\alpha-1} ds \\
\leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}} \|\psi(-\kappa)\| \\
& + \|B^{-1}\| \int_{-\kappa}^0 W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)[(\zeta-s)^\alpha - \zeta^\alpha] + (\beta-w)\zeta^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\| s^{\alpha-1} ds \\
\leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)t^\alpha}{\alpha}} \|\psi(-\kappa)\| \\
& + \|B^{-1}\| \int_{-\kappa}^0 W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)|(\zeta-s)^\alpha - \zeta^\alpha| + (\beta-w)\zeta^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\| s^{\alpha-1} ds
\end{aligned}$$

$$\begin{aligned}
& \leq W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}} \|\psi(-\kappa)\| \\
& + \|B^{-1}\| \int_{-\kappa}^0 W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(-s)^\alpha + (\beta-w)\zeta^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\| s^{\alpha-1} ds \\
\leq\ & U(\psi, \mathfrak{D}_\alpha^0\psi) e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}},
\end{aligned}$$

where

$$U(\psi, \mathfrak{D}_\alpha^0\psi) = W^2\Big(e^{\beta\frac{\kappa^\alpha}{\alpha}}\|\psi(-\kappa)\| + \|B^{-1}\| \int_{-\kappa}^0 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(-s)^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\| s^{\alpha-1} ds\Big) > 0.$$

Now if we choose $\mathfrak{D}_\alpha^0\psi$ small enough, then $0 \le M(\psi, \mathfrak{D}_\alpha^0\psi) < \delta$ by (H_5). $\square$

Theorem 2. *Under the hypotheses $(H_1), (H_2), (H_3)$ and (H_6), the solution of Equation (2) exhibits α-exponential stability.*

Proof. It is important to note that the solution of Equation (2) takes the form (14). Let $t \ge 0$. From (14), using $(H_1) - (H_3)$ via (16), we get

$$
\begin{aligned}
\|z(\zeta)\| \le\ & \|\mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|\psi(-\kappa)\| + \|B^{-1}\|\int_{-\kappa}^{0}\|\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
& + \|B^{-1}\|\int_0^\zeta \|\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|m(s,z(s))\|s^{\alpha-1}ds \\
\le\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)t^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
& + \|B^{-1}\|\int_{-\kappa}^{0} W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta-s)^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
& + \|B^{-1}\|\int_0^\zeta W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta-s)^\alpha}{\alpha}} L||z(s)||s^{\alpha-1}ds \\
\le\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
& + \|B^{-1}\|\int_{-\kappa}^{0} W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(-s)^\alpha+(\beta-w)\zeta^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
& + \|B^{-1}\|\int_0^\zeta W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(-s)^\alpha]+(\beta-w)\zeta^\alpha}{\alpha}} L||z(s)||s^{\alpha-1}ds \\
\le\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
& + \|B^{-1}\|\int_{-\kappa}^{0} W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
& + \|B^{-1}\|\int_0^\zeta W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}} L||z(s)||s^{\alpha-1}ds \\
\le\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
& + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}}\int_{-\kappa}^{0} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
& + \|B^{-1}\|LW^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}}\int_0^\zeta e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}||z(s)||s^{\alpha-1}ds,
\end{aligned}
$$

so,

$$
e^{(w-\beta)\frac{\zeta^\alpha}{\alpha}}\|z(\zeta)\| \le U(\psi, \mathfrak{D}_\alpha^0\psi) + \|B^{-1}\|LW^2 e^{w\frac{\kappa^\alpha}{\alpha}}\int_0^\zeta e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|z(s)\|s^{\alpha-1}ds.
$$

By employing the classical Gronwall inequality and Equation (9), we obtain

$$
e^{(w-\beta)\frac{\zeta^\alpha}{\alpha}}\|z(\zeta)\| \le U(\psi, \mathfrak{D}_\alpha^0\psi)e^{\|B^{-1}\|W^2Le^{w\frac{\kappa^\alpha}{\alpha}}\frac{\zeta^\alpha}{\alpha}},
$$

this results in

$$
\|z(\zeta)\| \le U(\psi, \mathfrak{D}_\alpha^0\psi)e^{(\|B^{-1}\|W^2Le^{w\frac{\kappa^\alpha}{\alpha}}+\beta-w)\frac{\zeta^\alpha}{\alpha}}.
$$

Based on (H_6), the desired outcome is valid. $\square$

Theorem 3. *Under the hypotheses $(H_1), (H_2), (H_4)$ and (H_7), the solution of Equation (3) exhibits α-exponential stability.*

Proof. It is known that the solution of Equation (3) can be expressed in the form of (15). Let $\zeta \geq 0$, from (15), utilizing (H_1), (H_2) and (H_4) via (16), we get

$$\begin{aligned}
\|z(\zeta)\| \leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&+\|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}}\int_{-\kappa}^{0} e^{\frac{(\beta-w)[(\zeta-s)^\alpha-\zeta^\alpha]+(\beta-w)\zeta^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s)-F\psi(s)\|s^{\alpha-1}ds \\
&+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\int_{0}^{\zeta} e^{\frac{(\beta-w)[(\zeta-s)^\alpha-\zeta^\alpha]+(\beta-w)\zeta^\alpha}{\alpha}}||z(s-\kappa)||s^{\alpha-1}ds \\
\leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&+\|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}}\int_{-\kappa}^{0} e^{\frac{(\beta-w)(-s)^\alpha+(\beta-w)\zeta^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s)-F\psi(s)\|s^{\alpha-1}ds \\
&+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\int_{0}^{\zeta} e^{\frac{(\beta-w)(-s)^\alpha+(\beta-w)\zeta^\alpha}{\alpha}}||z(s-\kappa)||s^{\alpha-1}ds \\
\leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&+\|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}}\int_{-\kappa}^{0} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s)-F\psi(s)\|s^{\alpha-1}ds \\
&+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\int_{0}^{\zeta} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}||z(s-\kappa)||s^{\alpha-1}ds \\
\leq\ & W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&+\|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}}\int_{-\kappa}^{0} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|B\mathfrak{D}_\alpha^0\psi(s)-F\psi(s)\|s^{\alpha-1}ds \\
&+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}} e^{(\beta-w)\frac{\zeta^\alpha}{\alpha}}\int_{0}^{\zeta} e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}||z(s-\kappa)||s^{\alpha-1}ds.
\end{aligned}$$

This suggests that

$$e^{(w-\beta)\frac{\zeta^\alpha}{\alpha}}\|z(\zeta)\| \leq U(\psi,\mathfrak{D}_\alpha^0\psi)+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\int_0^\zeta e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|z(s-\kappa)\|s^{\alpha-1}ds. \tag{17}$$

Take note $e^{(w-\beta)\frac{\zeta^\alpha}{\alpha}}\|z(\zeta)\| = e^{(w-\beta)\frac{\zeta^\alpha}{\alpha}}\|\psi(\zeta)\|, \zeta \in [-\kappa, 0]$. Applied Lemma 2.4 of [21] to (17), we can obtain

$$e^{(w-\beta)\frac{\zeta^\alpha}{\alpha}}\|z(\zeta)\| \leq \Big(U(\psi,\mathfrak{D}_\alpha^0\psi)+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\int_0^\kappa e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|\psi(s-\kappa)\|s^{\alpha-1}ds\Big)e^{\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}(\frac{\zeta^\alpha}{\alpha}-\frac{\kappa^\alpha}{\alpha})},$$

which yields (from (9))

$$\|z(\zeta)\| \leq U_*(\psi,\mathfrak{D}_\alpha^0\psi)e^{(\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}+\beta-w)\frac{\zeta^\alpha}{\alpha}}.$$

where $U_*(\psi,\mathfrak{D}_\alpha^0\psi) = [U(\psi,\mathfrak{D}_\alpha^0\psi)+\|B^{-1}\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\int_0^\kappa e^{(\beta-w)\frac{(-s)^\alpha}{\alpha}}\|\psi(s-\kappa)\|s^{\alpha-1}ds]e^{-\|B^{-1}\|}$ $\|W^2 Q e^{w\frac{\kappa^\alpha}{\alpha}}\frac{\kappa^\alpha}{\alpha} > 0$. It can be observed from (H_7) that Equation (3) exhibits α-exponential stability. Therefore, the proof is concluded. □

4. Relative Controllability

We will examine the controllability relative to (4) in this section. It should be noted that any solution to (4) can be displayed in the pattern of

$$\begin{aligned}
z(\zeta) =\ & \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa)+B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s)-F\psi(s)]s^{\alpha-1}ds \\
&+B^{-1}\int_0^\zeta \mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[y(s,z(s))+Jv(s)]s^{\alpha-1}ds.
\end{aligned} \tag{18}$$

4.1. Linear Problem

Assuming $y \equiv 0$, we can turn (4) into the system

$$\begin{cases} B\mathfrak{D}_{\alpha}^{0}z(\zeta) = Fz(\zeta) + Pz(\zeta-\kappa) + Jv(\zeta),\ \zeta \in I,\ \kappa \geq 0, \\ z(\zeta) = \psi(\zeta),\ \ -\kappa \leq \zeta \leq 0. \end{cases} \tag{19}$$

The introduction of a delay Grammian matrix is taken the form:

$$M_c[0,\zeta_1] = \int_0^{\zeta_1} B^{-1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}JJ^\top e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\mathcal{P}^\top(\zeta_1-\kappa-s){B^{-1}}^\top s^{\alpha-1}ds. \tag{20}$$

where $\cdot^\top$ represents the matrix's transpose.

Theorem 4. *The linear problem* (19) *exhibits relative controllability if and only if the matrix $M_c[0,\zeta_1]$ is nonsingular.*

Proof. Sufficiency. As the matrix $M_c[0,\zeta_1]$ is nonsingular, the existence of its inverse is guaranteed.

A control function can be chosen in the following manner:

$$v(\zeta) = J^\top e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\mathcal{P}^\top(\zeta_1-\kappa-\zeta){B^{-1}}^\top M_c^{-1}[0,\zeta_1]\eta,$$

where

$$\eta = z_1 - \mathcal{P}(\zeta_1)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) - B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds.$$

Then

$$\begin{aligned} z(\zeta_1) &= \mathcal{P}(\zeta_1)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds \\ &\quad + B^{-1}\int_0^{\zeta_1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}JJ^\top e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\mathcal{P}^\top(\zeta_1-\kappa-s){B^{-1}}^\top M_c^{-1}[0,\zeta_1]\eta s^{\alpha-1}ds \\ &= z_1. \end{aligned} \tag{21}$$

It is evident that the initial condition is satisfied as well. By applying Definition 2 and Equation (21), we can conclude that (19) exhibits relative controllability.

Necessity. We will present a proof by contradiction. Let us assume that $M_c[0,\zeta_1]$ is a singular matrix, implying the existence of at least a non-zero state $\widetilde{z} \in \mathbb{R}^n$ meeting

$$\widetilde{z}^\top M_c[0,\zeta_1]\widetilde{z} = 0.$$

Moreover, one can achieve

$$\begin{aligned} 0 &= \widetilde{z}^\top M_c[0,\zeta_1]\widetilde{z} \\ &= \int_0^{\zeta_1}\widetilde{z}^\top B^{-1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}JJ^\top e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\mathcal{P}^\top(\zeta_1-\kappa-s){B^{-1}}^\top\widetilde{z}s^{\alpha-1}ds \\ &= (e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}})^2\int_0^{\zeta_1}\|\widetilde{z}^\top B^{-1}\mathcal{P}(\zeta_1-\kappa-s)J\|^2 s^{\alpha-1}ds, \end{aligned}$$

which implies that

$$\widetilde{z}^\top B^{-1}\mathcal{P}(\zeta_1-\kappa-s)J = (\overbrace{0,\cdots,0}^{n}) := \mathbf{0}^\top, \quad \forall s \in I. \tag{22}$$

Given that (19) exhibits relative controllability, according to Definition 2, there is one control $v_1(\zeta)$ that can steer the initial state to $\mathbf{0}$ at ζ_1, i.e.,

$$
\begin{aligned}
z(\zeta_1) &= \mathcal{P}(\zeta_1)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds \\
&+B^{-1}\int_0^{\zeta_1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}Jv_1(s)s^{\alpha-1}ds \\
&= \mathbf{0}.
\end{aligned} \tag{23}
$$

Likewise, there is one control $v_2(\zeta)$ capable of directing the initial state to the state $\widetilde{z}$ at ζ_1, which can be represented as

$$
\begin{aligned}
z(\zeta_1) &= \mathcal{P}(\zeta_1)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds \\
&+B^{-1}\int_0^{\zeta_1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}Jv_2(s)s^{\alpha-1}ds \\
&= \widetilde{z}.
\end{aligned} \tag{24}
$$

From (23) and (24), one has

$$
\widetilde{z} = B^{-1}\int_0^{\zeta_1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}J[v_2(s)-v_1(s)]s^{\alpha-1}ds. \tag{25}
$$

Multiply both sides of (25) by $\widetilde{z}^\top$ and via (22) we have

$$
\begin{aligned}
\widetilde{z}^\top\widetilde{z} &= \int_0^{\zeta_1}\widetilde{z}^\top B^{-1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}J[v_2(s)-v_1(s)]s^{\alpha-1}ds \\
&= e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\int_0^{\zeta_1}\widetilde{z}^\top B^{-1}\mathcal{P}(\zeta_1-\kappa-s)J[v_2(s)-v_1(s)]s^{\alpha-1}ds \\
&= 0.
\end{aligned}
$$

This leads to the conclusion that $\widetilde{z} = \mathbf{0}$, which contradicts the non-zero nature of $\widetilde{z}$. Hence, $M_c[0,\zeta_1]$ is nonsingular. Thus, the proof is concluded. □

4.2. Nonlinear Problem

Let us examine the following requirements:
$[AW]$: The operator $M : L^2(I,\mathbb{R}^n) \to \mathbb{R}^n$ defined as

$$
Mv = B^{-1}\int_0^{\zeta_1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}Jv(s)s^{\alpha-1}ds.
$$

The operator M has an inverse, denoted as M^{-1}, which operates on the space $L^2(I,\mathbb{R}^n)/kerM$.
Set $U_1 = \|M^{-1}\|_{L_b(\mathbb{R}^n, L^2(I,\mathbb{R}^n)/kerM)}$. According to the Remark in Wang et al. (2017) [15], it can be inferred that

$$
U_1 = \sqrt{\|M_c^{-1}[0,\zeta_1]\|}, \tag{26}
$$

here, $M_c^{-1}[0,\zeta_1]$ is determined to (20). According to Theorem 4, $M_c^{-1}[0,\zeta_1]$ will be well-defined if (19) exhibits relative controllability.

$[AF]$: The function $y : I\times\mathbb{R}^n \to \mathbb{R}^n$ is continuous and there is one constant $h > 1$ and $L_y(\cdot) \in L^h(I,\mathbb{R}^+)$ meeting

$$
\|y(\zeta,z) - y(\zeta,v)\| \le L_y(\zeta)\|z - v\|,\ z, v \in \mathbb{R}^n.
$$

Theorem 5. *Assuming that* (H_1), (H_2), $[AW]$ *and* $[AF]$ *hold true, it can be concluded that system* (4) *exhibits relative controllability under the condition that*

$$b\Big[1 + W^2 U_1 \|B^{-1}\| \|J\| e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \frac{\zeta_1^\alpha}{\alpha}\Big] < 1, \tag{27}$$

where $b = \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}} \big[\frac{1}{g\alpha-g+1}\zeta_1^{g\alpha-g+1}\big]^{\frac{1}{g}} \|L_y\|_{L^h(I,\mathbb{R}^+)}$, *and* $\frac{1}{g}+\frac{1}{h}=1$, g, $h>1$.

Proof. By exploiting hypothesis $[AW]$ for any $z(\cdot) \in C(I,\mathbb{R}^n)$, defining one control function $v_z(t)$ as follows:

$$\begin{aligned} v_z(\zeta) \quad = \quad & M^{-1}\Big[z_1 - \mathcal{P}(\zeta_1) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} \psi(-\kappa) - B^{-1}\int_{-\kappa}^{0} \mathcal{P}(\zeta_1-\kappa-s) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} [B\mathfrak{D}_\alpha^0 \psi(s) - F\psi(s)] s^{\alpha-1} ds \\ & -B^{-1}\int_0^{\zeta_1} \mathcal{P}(\zeta_1-\kappa-s) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} y(s,z(s)) s^{\alpha-1} ds\Big](\zeta),\ \zeta \in I. \end{aligned} \tag{28}$$

We demonstrate that by employing this control, the operator $\mathcal{T} : C(I,\mathbb{R}^n) \to C(I,\mathbb{R}^n)$ defined as follows:

$$\begin{aligned} (\mathcal{T}z)(\zeta) \quad = \quad & \mathcal{P}(\zeta) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} \psi(-\kappa) + B^{-1}\int_{-\kappa}^{0} \mathcal{P}(\zeta-\kappa-s) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} [B\mathfrak{D}_\alpha^0 \psi(s) - F\psi(s)] s^{\alpha-1} ds \\ & +B^{-1}\int_0^{\zeta} \mathcal{P}(\zeta-\kappa-s) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} y(s,z(s)) s^{\alpha-1} ds \\ & +B^{-1}\int_0^{\zeta} \mathcal{P}(\zeta-\kappa-s) e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} J v_z(s) s^{\alpha-1} ds, \end{aligned}$$

The operator $\mathcal{T}$ has one fixed point z, which corresponds to a solution of (4).

We must verify $(\mathcal{T}z)(\zeta_1) = z_1$ and $(\mathcal{T}z)(0) = z_0$, indicating that v_z guides the system (4) from z_0 to z_1 within a finite time ζ_1. This confirmation implies that system (4) exhibits relative controllability over I.

For every positive number r, consider the set $\mathcal{S}_r = \{z \in C(I,\mathbb{R}^n) : \|z\|_C \leq r\}$. Let $N = \sup_{\zeta\in I} \|y(\zeta,0)\|$. The proof will be divided into a couple of procedures.

Step 1. It is asserted that there is one positive constant r meeting $\mathcal{T}(\mathcal{S}_r) \subseteq \mathcal{S}_r$.

Note that

$$\int_0^{\zeta} s^{\alpha-1} L_y(s) ds \leq \Big[\frac{1}{g\alpha-g+1}\zeta_1^{g\alpha-g+1}\Big]^{\frac{1}{g}} \|L_y\|_{L^h(I,\mathbb{R}^+)},$$

and

$$\int_0^{\zeta} \|y(s,0)\| s^{\alpha-1} ds \leq \frac{\zeta_1^\alpha}{\alpha} N.$$

Taking into account (28), and using (H_1), (H_2), $[AW]$ and $[AF]$, we can get

$$\begin{aligned}
\|v_z\| &\leq \|M^{-1}\|_{L_{\zeta_1}(\mathbb{R}^n, L^2(I,\mathbb{R}^n)/kerM)}\Big(\|z_1\| + \|\mathcal{P}(\zeta_1)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|\psi(-\kappa)\| \\
&\quad + \|B^{-1}\|\int_{-\kappa}^{0}\|\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
&\quad + \|B^{-1}\|\int_{0}^{\zeta_1}\|\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|y(s,z(s))\|s^{\alpha-1}ds\Big) \\
&\leq U_1\Big(\|z_1\| + W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}}\int_{-\kappa}^{0}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}}\int_{0}^{\zeta_1}L_y(s)\|z(s)\|s^{\alpha-1}ds \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\int_{0}^{\zeta_1}\|y(s,0)\|s^{\alpha-1}ds\Big) \\
&\leq U_1\Big(\|z_1\| + W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}}\int_{-\kappa}^{0}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}}\left[\frac{1}{g\alpha-g+1}\zeta_1^{g\alpha-g+1}\right]^{\frac{1}{g}}\|L_y\|_{L^h(I,\mathbb{R}^+)}\|z\|_C \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} N\frac{\zeta_1^\alpha}{\alpha}\Big) \\
&\leq U_1\|z_1\| + U_1 a + U_1 b\|z\|_C,
\end{aligned}$$

where

$$\begin{aligned}
a &= W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\|\psi(-\kappa)\| + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}}\int_{-\kappa}^{0}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} N\frac{\zeta_1^\alpha}{\alpha}.
\end{aligned}$$

Based on the assumptions $[AW]$ and $[AF]$, we can derive

$$\begin{aligned}
\|(\mathcal{T}z)(\zeta)\| &\leq W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\|\psi(-\kappa)\| \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}}\int_{-\kappa}^{0}\|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\left[\frac{1}{g\alpha-g+1}\zeta_1^{g\alpha-g+1}\right]^{\frac{1}{g}}\|L_y\|_{L^h(I,\mathbb{R}^+)}\|z\|_C \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} N\frac{\zeta_1^\alpha}{\alpha} \\
&\quad + \|B^{-1}\|W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\|J\|[U_1\|z_1\| + U_1 a + U_1 b\|z\|_C]\frac{\zeta_1^\alpha}{\alpha}
\end{aligned}$$

$$
\begin{aligned}
&\leq W^2 e^{\beta\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|\psi(-\kappa)\| \\
&+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}} \int_{-\kappa}^{0} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\| s^{\alpha-1}ds \\
&+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \Big[\frac{1}{g\alpha-g+1}\zeta_1^{g\alpha-g+1}\Big]^{\frac{1}{g}} \|L_y\|_{L^h(I,\mathbb{R}^+)} \|z\|_C \\
&+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} N\frac{\zeta_1^\alpha}{\alpha} \\
&+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|J\| U_1 \|z_1\| \frac{\zeta_1^\alpha}{\alpha} \\
&+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|J\| U_1 a\frac{\zeta_1^\alpha}{\alpha} \\
&+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|J\| U_1 b\|z\|_C\frac{\zeta_1^\alpha}{\alpha} \\
\leq\ & a[1+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|J\| U_1 \frac{\zeta_1^\alpha}{\alpha}] \\
&+\|E^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|J\| U_1 \frac{\zeta_1^\alpha}{\alpha}\|z_1\| \\
&+b[1+\|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \|J\| U_1 \frac{\zeta_1^\alpha}{\alpha}]r \\
\leq\ & a\big(1+\frac{U_1\|J\|}{N}a\big) + \frac{U_1\|J\|}{N}a\|z_1\| + b\big(1+\frac{U_1\|J\|}{N}a\big)r = r,
\end{aligned}
$$

where

$$r = \frac{a\big(1+\frac{U_1\|J\|}{N}a + \frac{U_1\|J\|}{N}\|z_1\|\big)}{1-b\big(1+\frac{U_1\|J\|}{N}a\big)}.$$

Thus, we deduce that $\mathcal{T}(\mathcal{S}_r)$ is a subset of $\mathcal{S}_r$ for this particular value of r.

Throughout the remaining proof, the aforementioned value of r will be the one under consideration. Operators $\mathcal{T}_1$ and $\mathcal{T}_2$ are defined as follows:

$$
\begin{aligned}
(\mathcal{T}_1 z)(\zeta) &= \mathcal{P}(\zeta)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\psi(-\kappa) + B^{-1}\int_{-\kappa}^{0}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}[B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)]s^{\alpha-1}ds \\
&\quad + B^{-1}\int_0^{\zeta}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} Jv_z(s)s^{\alpha-1}ds, \\
(\mathcal{T}_2 z)(\zeta) &= B^{-1}\int_0^{\zeta}\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} y(s,z(s))s^{\alpha-1}ds,
\end{aligned}
$$

for $\zeta \in I$.

Step 2. The operator $\mathcal{T}_1$ exhibits the properties of a contraction mapping.

Let $z,\omega \in \mathcal{S}_r$. In view of $[AW]$ and $[AF]$, for every $\zeta \in I$, we obtain

$$
\begin{aligned}
\|u_z(\zeta) - u_\omega(\zeta)\| &\leq U_1\|B^{-1}\| \int_0^{\zeta_1} \|\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\| \|y(s,z(s)) - y(s,w(s))\| s^{\alpha-1}ds \\
&\leq U_1\|B^{-1}\| \int_0^{\zeta_1} W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta-s)^\alpha}{\alpha}} L_y(s)\|z(s)-\omega(s)\| s^{\alpha-1}ds \\
&\leq U_1 b\|z-\omega\|_C.
\end{aligned}
$$

Thus,

$$\begin{aligned}\|(\mathcal{T}_1 z)(\zeta)-(\mathcal{T}_1\omega)(\zeta)\| &\le \|B^{-1}\|\int_0^{\zeta}\|\mathcal{P}(\zeta-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\|\|J\|\|u_z(s)-u_\omega(s)\|s^{\alpha-1}ds\\ &\le \|B^{-1}\|\int_0^{\zeta}W^2e^{w\frac{\kappa^\alpha}{\alpha}}e^{\frac{(\beta-w)(\zeta-s)^\alpha}{\alpha}}s^{\alpha-1}ds\|J\|U_1 b\|z-\omega\|_C\\ &\le \|B^{-1}\|W^2e^{w\frac{\kappa^\alpha}{\alpha}}e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\|J\|U_1 b\frac{\zeta_1^\alpha}{\alpha}\|z-\omega\|_C\\ &= bW^2U_1\|B^{-1}\|\|J\|e^{w\frac{\kappa^\alpha}{\alpha}}e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\frac{\zeta_1^\alpha}{\alpha}\|z-\omega\|_C,\end{aligned}$$

so we obtain

$$\|\mathcal{T}_1 z-\mathcal{T}_1\omega\|_C\le\Lambda\|z-\omega\|_C,$$

where $\Lambda=bW^2U_1\|B^{-1}\|\|J\|e^{w\frac{\kappa^\alpha}{\alpha}}e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\frac{\zeta_1^\alpha}{\alpha}$.

Since (27) implies $\Lambda<1$, it follows that the operator $\mathcal{T}$ is a contraction mapping.

Step 3. The operator $\mathcal{T}_2:\mathcal{S}_r\to C(I,\mathbb{R}^n)$ is both compact and continuous.

Consider $z_n\in\mathcal{S}_r$ with $z_n\to z$ in $\mathcal{S}_r$ as $n\to\infty$. By utilizing $[AF]$, we obtain $y(\cdot,z_n)\to y(\cdot,z)$ in $C(I,\mathbb{R}^n)$ as $n\to\infty$ and thus

$$\begin{aligned}\|(\mathcal{T}_2 z_n)(\zeta)-(\mathcal{T}_2 z)(\zeta)\| &\le \|B^{-1}\|\int_0^{\zeta}W^2e^{w\frac{\kappa^\alpha}{\alpha}}e^{\frac{(\beta-w)(\zeta-s)^\alpha}{\alpha}}\|y(s,z_n(s))-y(s,z(s))\|s^{\alpha-1}ds\\ &\le \|B^{-1}\|W^2e^{w\frac{\kappa^\alpha}{\alpha}}e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}}\frac{\zeta_1^\alpha}{\alpha}\|y(s,z_n(s))-y(s,z(s))\|_C\\ &\quad\to 0\quad \textit{while}\quad n\to\infty.\end{aligned}$$

This indicates that $\mathcal{T}_2$ is continuous on $\mathcal{S}_r$.

To demonstrate the compactness of $\mathcal{T}_2$ on $\mathcal{S}_r$, we need to establish that $\mathcal{T}_2(\mathcal{S}_r)$ is both equicontinuous and bounded. For any $z\in\mathcal{S}_r$, $\zeta_1\ge\zeta+\nu\ge\zeta>0$, we proceed with the following proof:

$$\begin{aligned}&(\mathcal{T}_2 z)(\zeta+\nu)-(\mathcal{T}_2 z)(\zeta)\\ =\;& B^{-1}\int_{\zeta}^{\zeta+\nu}\mathcal{P}(\zeta+\nu-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}y(s,z(s))s^{\alpha-1}ds\\ &+B^{-1}\int_0^{\zeta}e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}}\left(e_{\kappa,\alpha}^{P_1B^{-1}(\zeta+\nu-\kappa-s)}-e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}\right)y(s,z(s))s^{\alpha-1}ds\\ &+B^{-1}\int_0^{\zeta}\left(e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}}-e^{FB^{-1}\frac{(\zeta-\kappa-s)^\alpha}{\alpha}}\right)e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}y(s,z(s))s^{\alpha-1}ds.\end{aligned}$$

Let

$$\begin{aligned}Q_1 &= B^{-1}\int_{\zeta}^{\zeta+\nu}\mathcal{P}(\zeta+\nu-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}y(s,z(s))s^{\alpha-1}ds,\\ Q_2 &= B^{-1}\int_0^{\zeta}e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}}\left(e_{\kappa,\alpha}^{P_1B^{-1}(\zeta+\nu-\kappa-s)}-e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}\right)y(s,z(s))s^{\alpha-1}ds,\\ Q_3 &= B^{-1}\int_0^{\zeta}\left(e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}}-e^{FB^{-1}\frac{(\zeta-\kappa-s)^\alpha}{\alpha}}\right)e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}y(s,z(s))s^{\alpha-1}ds.\end{aligned}$$

Based on the aforementioned analysis, we can conclude that

$$\|(\mathcal{T}_2 z)(\zeta+\nu)-(\mathcal{T}_2 z)(\zeta)\|\le\|Q_1\|+\|Q_2\|+\|Q_3\|.$$

Now, it remains to verify that $\|Q_i\| \to 0$ as $\nu \to 0$ for $i = 1,2,3$. Note

$$\begin{aligned}
\|Q_1\| &\leq \|B^{-1}\| \int_{\zeta}^{\zeta+\nu} \|\mathcal{P}(\zeta+\nu-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\| \|y(s,z(s))\| s^{\alpha-1} ds \\
&\leq \|B^{-1}\| \int_{\zeta}^{\zeta+\nu} W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta+\nu-s)^\alpha}{\alpha}} L_y(s)\|z(s)\| s^{\alpha-1} ds \\
&\quad + \|B^{-1}\| \int_{\zeta}^{\zeta+\nu} W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta+\nu-s)^\alpha}{\alpha}} \|y(s,0)\|) s^{\alpha-1} ds \\
&\leq \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\nu^\alpha}{\alpha}} \frac{1}{g\alpha-g+1}\left[(\zeta+\nu)^{g\alpha-g+1} - \zeta^{g\alpha-g+1}\right]^{\frac{1}{g}} \|L_y\|_{L^h(I,\mathbb{R}^+)} r \\
&\quad + \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\nu^\alpha}{\alpha}} N\frac{1}{\alpha}[(\zeta+\nu)^\alpha - \zeta^\alpha] \to 0 \quad as \quad \nu \to 0,
\end{aligned}$$

and

$$\begin{aligned}
\|Q_2\| &\leq \|B^{-1}\| e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} e^{FB^{-1}\frac{(\zeta+\nu-\kappa)^\alpha}{\alpha}} \int_0^{\zeta} \|e_{\kappa,\alpha}^{P_1B^{-1}(\zeta+\nu-\kappa-s)} - e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}\| \\
&\quad \cdot (L_y(s)\|z(s)\| + \|y(s,0)\|) s^{\alpha-1} ds \\
&\leq \|B^{-1}\| e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} e^{FB^{-1}\frac{(\zeta+\nu-\kappa)^\alpha}{\alpha}} \|L_y\|_{L^h(I,\mathbb{R}^+)} \frac{\zeta_1^\alpha}{\alpha} r \\
&\quad \times \Big(\int_0^{\zeta} \|e_{\kappa,\alpha}^{P_1B^{-1}(\zeta+\nu-\kappa-s)} - e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}\|^g ds \Big)^{\frac{1}{g}} \\
&\quad + \|B^{-1}\| e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}} e^{FB^{-1}\frac{(\zeta+\nu-\kappa)^\alpha}{\alpha}} N\frac{\zeta_1^\alpha}{\alpha} \int_0^{\zeta} \|e_{\kappa,\alpha}^{P_1B^{-1}(\zeta+\nu-\kappa-s)} - e_{\kappa,\alpha}^{P_1B^{-1}(\zeta-\kappa-s)}\| ds \\
&\quad \to 0 \quad as \quad \nu \to 0,
\end{aligned}$$

where the uniformly continuity of $e_{\kappa,\alpha}^{P_1B^{-1}\cdot}$ is used. Next

$$\|Q_3\| \leq \|B^{-1}\| We^{\beta\frac{\zeta_1^\alpha}{\alpha}} \int_0^{\zeta} \|e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}} - e^{FB^{-1}\frac{(\zeta-\kappa-s)^\alpha}{\alpha}}\| (L_y(s)\|z\|_C + N) s^{\alpha-1} ds.$$

Note that $\lim\limits_{\nu\to 0}\left\|e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}} - e^{FB^{-1}\frac{(\zeta-\kappa-s)^\alpha}{\alpha}}\right\| = 0$. So

$$\begin{aligned}
\|Q_3\| &\leq \|B^{-1}\| We^{\beta\frac{\zeta_1^\alpha}{\alpha}} \int_0^{\zeta} \|e^{FB^{-1}\frac{(\zeta+\nu-\kappa-s)^\alpha}{\alpha}} - e^{FB^{-1}\frac{(\zeta-\kappa-s)^\alpha}{\alpha}}\| (L_y(s)\|z\|_C + N) s^{\alpha-1} ds \\
&\quad \to 0 \quad while \quad \nu \to 0,
\end{aligned}$$

from utilizing the dominated convergence theorem of Lebesgue.

Consequently, we receive

$$\|(\mathcal{T}_2 z)(\zeta+\nu) - (\mathcal{T}_2 z)(\zeta)\| \to 0 \quad while \quad \nu \to 0,$$

for all $z \in \mathcal{S}_r$, thus establishing the equicontinuity of $\mathcal{T}_2(\mathcal{S}_r)$.

Continuing with the aforementioned calculations, we can further deduce that

$$\begin{aligned}
\|(\mathcal{T}_2 z)(\zeta)\| &\leq \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \int_0^{\zeta} s^{\alpha-1} L_y(s)\|z(s)\| ds \\
&\quad + \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \int_0^{\zeta} \|y(s,0)\| s^{\alpha-1} ds \\
&\leq \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \left[\frac{1}{g\alpha-g+1}\zeta_1^{g\alpha-g+1}\right]^{\frac{1}{g}} \|L_y\|_{L^h(I,\mathbb{R}^+)} r \\
&\quad + \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \frac{\zeta_1^\alpha}{\alpha} N.
\end{aligned}$$

Hence $\mathcal{T}_2(\mathcal{S}_r)$ is bounded. From the Arzela-Ascoli theorem, $\mathcal{T}_2(\mathcal{S}_r)$ is relatively compact in $C(I, \mathbb{R}^n)$. Hence, $\mathcal{T}_2 : \mathcal{S}_r \to C(I, \mathbb{R}^n)$ is an operator that is both compact and continuous.

By invoking Krasnoselskii's fixed point theorem, we can ensure the existence of one fixed point z for the operator $\mathcal{T}$ on $\mathcal{S}_r$. Obviously, z is a solution to the system (4) that satisfies $z(\zeta_1) = z_1$. Additionally, the boundary condition $z(\zeta) = \psi(\zeta)$, for $-\kappa \leq \zeta \leq 0$ holds true based on Equation (18). Thus, the proof is concluded. □

5. Examples

This section will take a few numerical examples to verify our theoretical results. For simulation purposes, we utilize the infinite-norm.

Example 1. *Let us examine the subsequent linear conformable delay system, which is nonsingular:*

$$\begin{cases} B\mathfrak{D}^0_{0.5}z(\zeta) = Fz(\zeta) + Pz(\zeta - 0.2), \zeta \geq 0, \\ z(\zeta) = \psi(\zeta) = (0.2, 0.1)^\top, \ -0.2 \leq \zeta \leq 0. \end{cases} \tag{29}$$

where $\kappa = 0.2$ *and we set*

$$F = \begin{bmatrix} -1.2 & -0.4 \\ 0 & -2 \end{bmatrix}, \quad P = \begin{bmatrix} 0.6 & 0.1 \\ 0 & 0.8 \end{bmatrix}, \quad B = \begin{bmatrix} 1.4 & 0 \\ 0 & 1.4 \end{bmatrix}.$$

Note

$$FP = \begin{bmatrix} -0.72 & -0.44 \\ 0 & -1.6 \end{bmatrix} = PF, \quad FB = \begin{bmatrix} 1.68 & -0.56 \\ 0 & -2.8 \end{bmatrix} = BF,$$

$$PB = \begin{bmatrix} 0.84 & 0.14 \\ 0 & 1.12 \end{bmatrix} = BP, \quad B^{-1} = \begin{bmatrix} 0.7143 & 0 \\ 0 & 0.7143 \end{bmatrix},$$

$$\|P_1 B^{-1}\| = \left\| \begin{bmatrix} -9.6 & -7.44 \\ 0 & -2.16 \end{bmatrix} \right\| \leq \beta e^{0.2\beta}, \quad \textit{choosing} \quad \beta = 0.9354.$$

Clearly, for $\alpha = \frac{1}{2}$, $\|e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}\| = e^{-2.8\sqrt{\zeta}} \leq We^{-2w\sqrt{\zeta}}$ *with* $W = 1$ *and* $w = 1$*. Next* $\|\psi\|_1 = 0.2 < \delta := 0.25$ *and*

$$\begin{aligned} U(\psi, \mathfrak{D}^0_\alpha \psi) &= W^2 \Big(e^{\beta \frac{\kappa^\alpha}{\alpha}} \|\psi(-\kappa)\| \\ &\quad + \|B^{-1}\| \int_{-\kappa}^0 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta - w)(-s)^\alpha}{\alpha}} \|B\mathfrak{D}^0_\alpha \psi(s) - F\psi(s)\| s^{\alpha-1} ds \Big) \\ &= 0.1616 > 0. \end{aligned}$$

and $\beta - w = -0.0646 < 0$.

At present, those requirements stipulated in Theorem 1 are fulfilled. Hence,

$$\|z(\zeta)\| \leq U(\psi, \mathfrak{D}^0_\alpha \psi) e^{(\beta - w)2\sqrt{\zeta}} = 0.1616 e^{-0.1292\sqrt{\zeta}} \to 0 \quad \textit{while} \quad \zeta \to \infty,$$

Therefore, the solution to (29) exhibits α*-exponential stability.*

Example 2. *Let us contemplate the given linear conformable system:*

$$\begin{cases} B\mathfrak{D}^0_{0.5}z(\zeta) = Fz(\zeta) + Pz(\zeta - 0.2) + l(\zeta, z(\zeta - 0.2)), \zeta \geq 0, \\ z(\zeta) = \psi(\zeta) = (0.1, 0.2)^\top, \ -0.2 \leq \zeta \leq 0. \end{cases} \tag{30}$$

where $\kappa = 0.2$ *and suppose*

$$F = \begin{bmatrix} -2.3 & 0.5 \\ 0 & -2.4 \end{bmatrix}, \quad P = \begin{bmatrix} 1.6 & 0.5 \\ 0 & 1.5 \end{bmatrix}, \quad B = \begin{bmatrix} 1.6 & 0 \\ 0 & 1.6 \end{bmatrix}.$$

Note

$$l(\zeta, z(\zeta - 0.2)) = \begin{bmatrix} 0.2z_1(\zeta - 0.2)sin\zeta \\ 0.15z_2(\zeta - 0.2)sin\zeta \end{bmatrix}.$$

Note

$$FP = \begin{bmatrix} -3.68 & -0.4 \\ 0 & -3.6 \end{bmatrix} = PF, \quad FB = \begin{bmatrix} -3.68 & 0.8 \\ 0 & -3.84 \end{bmatrix} = BF,$$

$$PB = \begin{bmatrix} 2.56 & 0.8 \\ 0 & 2.4 \end{bmatrix} = BP, \quad B^{-1} = \begin{bmatrix} 0.625 & 0 \\ 0 & 0.625 \end{bmatrix},$$

$$\|P_1B^{-1}\| = \left\| \begin{bmatrix} 1.5527 & 1.0946 \\ 0 & 1.8264 \end{bmatrix} \right\| \leq \beta e^{0.4\beta}, \quad \text{choosing} \quad \beta = 0.8211.$$

Clearly, for $\alpha = \frac{1}{2}$, $\|e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}\| = e^{-3\sqrt{\zeta}} \leq We^{-2w\sqrt{\zeta}}$ *with* $W = 1$ *and* $w = 1.3$. $\|l(\zeta, z(\zeta - 0.2))\| \leq Q\|z(\zeta - 0.2)\|$ *with* $Q = 0.2$.

Subsequently $\|\psi\|_1 = 0.2 < \delta := 0.25$ *and*

$$\begin{aligned} U(\psi, \mathfrak{D}_\alpha^0\psi) &= W^2\Big(e^{\beta\frac{\kappa^\alpha}{\alpha}}\|\psi(-\kappa)\| \\ &\quad + \|B^{-1}\| \int_{-\kappa}^{0} e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta - w)(-s)^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0\psi(s) - F\psi(s)\|s^{\alpha-1}ds\Big) \\ &= 0.1587 > 0. \end{aligned}$$

$U_*(\psi, \mathfrak{D}_\alpha^0\psi) = [U(\psi, \mathfrak{D}_\alpha^0\psi) + \|B^{-1}\| \int_0^\kappa W^2Qe^{(w-\beta)s}\|\psi(s-\kappa)\|s^{\alpha-1}ds]e^{-\|B^{-1}\|W^2Q\frac{\zeta_1^\alpha}{\alpha}e^{(w-\beta)\kappa}\kappa}$ $= 0.1690 > 0$ *and* $\|B^{-1}\|W^2Qe^{w\frac{\kappa^\alpha}{\alpha}} + \beta - w = -0.1445 < 0$.

It is obvious those requirements stipulated in Theorem 3 are fulfilled. Hence,

$$\|z(\zeta)\| \leq U_*(\psi, \mathfrak{D}_\alpha^0\psi)e^{(\|B^{-1}\|W^2Qe^{w\frac{\kappa^\alpha}{\alpha}} + \beta - w)\frac{\zeta^\alpha}{\alpha}} = 0.1690e^{-0.2890\sqrt{\zeta}} \to 0 \quad \text{while} \quad \zeta \to \infty,$$

Hence, the solution of (30) exhibits α*-exponential stability.*

Example 3. *Let us contemplate the given conformable system:*

$$\begin{cases} B\mathfrak{D}_{0.5}^0 z(\zeta) = Fz(\zeta) + Pz(\zeta - 0.2) + m(\zeta, z(\zeta)), \zeta \geq 0, \\ z(\zeta) = \psi(\zeta) = (0.2, 0.15, 0.1)^\top, \ -0.2 \leq \zeta \leq 0. \end{cases} \tag{31}$$

where $\kappa = 0.2$ *and setting*

$$F = \begin{bmatrix} -3.6 & 0 & 4 \\ 0 & -6.8 & 0.8 \\ 0 & 0 & -2.2 \end{bmatrix}, \quad P = \begin{bmatrix} 2.8 & 0 & -1 \\ 0 & 2.45 & 0 \\ 0 & 0 & 2.45 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.8 & 0 & 0 \\ 0 & 1.8 & 0 \\ 0 & 0 & 1.8 \end{bmatrix}, \quad m(\zeta, z(\zeta)) = \begin{bmatrix} 0.2z_1(\zeta)\sin\zeta \\ 0.1z_2(\zeta)\sin\zeta \\ 0.2z_3(\zeta)\sin\zeta \end{bmatrix}.$$

Note

$$FP = \begin{bmatrix} -10.08 & 0 & 13.4 \\ 0 & -16.66 & 1.96 \\ 0 & 0 & -5.39 \end{bmatrix} = PF,$$

$$FB = \begin{bmatrix} 6.48 & 0 & 7.2 \\ 0 & -12.24 & 1.44 \\ 0 & 0 & -3.96 \end{bmatrix} = BF,$$

$$PB = \begin{bmatrix} 5.04 & 0 & -1.8 \\ 0 & 4.41 & 0 \\ 0 & 0 & 4.41 \end{bmatrix} = BP, \quad B^{-1} = \begin{bmatrix} 0.5556 & 0 & 0 \\ 0 & 0.5556 & 0 \\ 0 & 0 & 0.5556 \end{bmatrix},$$

$$\|P_1 B^{-1}\| = \left\| \begin{bmatrix} 3.6248 & 0 & -3.4163 \\ 0 & 3.7828 & -0.2463 \\ 0 & 0 & 2.5999 \end{bmatrix} \right\| \le \beta e^{0.8\beta}, \quad \textit{choosing} \quad \beta = 0.3996.$$

Clearly, for $\alpha = \frac{1}{2}$, $\|e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}\| = e^{-7.556\zeta} \le We^{-2w\sqrt{\zeta}}$ *where* $W = 1$ *and* $w = 3$. $\|m(\zeta, z(\zeta))\| \le L\|z(\zeta)\|$, *here* $L = 0.2$. *Together with* $\|\psi\|_1 = 0.2 < \delta := 0.25$,

$$\begin{aligned} U(\psi, \mathfrak{D}_\alpha^0 \psi) &= W^2 \Big(e^{\beta \frac{\kappa^\alpha}{\alpha}} \|\psi(-\kappa)\| \\ &\quad + \|B^{-1}\| \int_{-\kappa}^{0} e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta - w)(-s)^\alpha}{\alpha}} \|B\mathfrak{D}_\alpha^0 \psi(s) - F\psi(s)\| s^{\alpha-1} ds \Big) \\ &= 0.1641 > 0, \end{aligned}$$

with $\|B^{-1}\| W^2 L e^{w\frac{\kappa^\alpha}{\alpha}} + \beta - w = -0.9744 < 0$.

It is evident those requirements stipulated in Theorem 2 are fulfilled. Hence,

$$\|z(\zeta)\| \le U(\psi, \mathfrak{D}_\alpha^0 \psi) e^{(\|B^{-1}\| W^2 L e^{w\frac{\kappa^\alpha}{\alpha}} + \beta - w)\frac{\zeta^\alpha}{\alpha}} = 0.1641 e^{-1.9488\sqrt{\zeta}} \to 0 \quad \textit{while} \quad \zeta \to \infty,$$

Thus, the solution of (31) exhibits α-exponential stability.

Example 4. *Set* $t_1 = 0.9$. *Let us contemplate the given conformable differential controlled system:*

$$\begin{cases} B\mathfrak{D}_{0.6}^0 z(\zeta) = Fz(\zeta) + Pz(\zeta - 0.2) + y(\zeta, z(\zeta)) + Jv(\zeta), \ \zeta \in I := [0, 0.9], \\ z(\zeta) = \psi(\zeta) = (0.3, 0.2)^\top, \ -0.2 \le \zeta \le 0. \end{cases} \tag{32}$$

where $\kappa = 0.2$ *and let*

$$F = \begin{bmatrix} -2 & -0.2 \\ 0 & -1.8 \end{bmatrix}, \quad P = \begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -0.2 \\ 0 & 1.2 \end{bmatrix},$$

$$J = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad y(\zeta, z(\zeta)) = \begin{bmatrix} 0.2\zeta z_1(\zeta) \\ 0.1\zeta z_2(\zeta) \end{bmatrix}.$$

Note

$$FP = \begin{bmatrix} -1.6 & -0.52 \\ 0 & -1.08 \end{bmatrix} = PF, \quad FB = \begin{bmatrix} -2 & 0.16 \\ 0 & -2.16 \end{bmatrix} = BF,$$

$$PB = \begin{bmatrix} 0.8 & 0.08 \\ 0 & 0.72 \end{bmatrix} = BP, \quad B^{-1} = \begin{bmatrix} 1 & 0.1667 \\ 0 & 0.8333 \end{bmatrix},$$

$$\|P_1 B^{-1}\| = \left\| \begin{bmatrix} 1.8221 & 0.3644 \\ 0 & 1.4577 \end{bmatrix} \right\| \leq \beta e^{0.4\beta}, \quad choosing \quad \beta = 0.7993.$$

Clearly, for $\alpha = 0.6$, $\|e^{FB^{-1}\frac{\zeta^\alpha}{\alpha}}\| = e^{-3.3\zeta^{0.6}} \leq We^{-2w\zeta^{0.6}}$ *where* $W = 1$ *and* $w = 1.2$. *Together with* $\|J\| = 1$ *and* $\|B^{-1}\| = 1$.

Subsequently, we will utilize Equation (26) to estimate U_1. *To achieve this, we first need to obtain* $M_c[0, \zeta_1]$, *followed by the calculation of its inverse,* $M_c[0, \zeta_1]^{-1}$. *The delay Grammian matrix takes the form of*

$$\begin{aligned} M_c[0,\zeta_1] &= \int_0^{\zeta_1} B^{-1}\mathcal{P}(\zeta_1-\kappa-s)e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}JJ^\top e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}\mathcal{P}^\top(\zeta_1-\kappa-s){B^{-1}}^\top s^{\alpha-1}ds \\ &= \int_0^{\zeta_1} B^{-1}e^{FB^{-1}\frac{(\zeta_1-\kappa-s)^\alpha}{\alpha}}e_{\kappa,\alpha}^{P_1B^{-1}(\zeta_1-\kappa-s)}e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}J \\ &\times J^\top e^{FB^{-1}\frac{\kappa^\alpha}{\alpha}}e_{\kappa,\alpha}^{(P_1B^{-1})^\top(\zeta_1-\kappa-s)}e^{(FB^{-1})^\top\frac{(\zeta_1-\kappa-s)^\alpha}{\alpha}}{B^{-1}}^\top s^{\alpha-1}ds \\ &= \int_0^{\zeta_1} B^{-1}e^{F'\frac{(\zeta_1-\kappa-s)^\alpha}{\alpha}}e_{\kappa,\alpha}^{P'(\zeta_1-\kappa-s)}e^{F'\frac{\kappa^\alpha}{\alpha}}JJ^\top e^{F'^\top\frac{\kappa^\alpha}{\alpha}}e_{\kappa,\alpha}^{P'^\top(\zeta_1-\kappa-s)}e^{F'^\top\frac{(\zeta_1-\kappa-s)^\alpha}{\alpha}}{B^{-1}}^\top s^{\alpha-1}ds \\ &= \int_0^{0.9} B^{-1}e^{F'\frac{(0.7-s)^\alpha}{\alpha}}e_{\zeta,\alpha}^{P'(0.7-s)}e^{F'\frac{\kappa^\alpha}{\alpha}}e^{F'^\top\frac{\kappa^\alpha}{\alpha}}e_{\kappa,\alpha}^{P'^\top(0.7-s)}e^{F'^\top\frac{(0.7-s)^\alpha}{\alpha}}s^{\alpha-1}ds \\ &= M_1+M_2+M_3. \end{aligned}$$

where $F' = FB^{-1}$ *and* $P' = P_1B^{-1}$,

$$\begin{aligned} M_1 &= \int_0^{0.3} B^{-1}e^{F'\frac{(0.7-s)^\alpha}{\alpha}}\left[\mathcal{I}+P'\frac{(0.7-s)^\alpha}{\alpha}+P'^2\frac{1}{2!}\left(\frac{(0.5-s)^\alpha}{\alpha}\right)^2\right]e^{F'\frac{\kappa^\alpha}{\alpha}}e^{F'^\top\frac{\kappa^\alpha}{\alpha}} \\ &\times\left[\mathcal{I}+P'^\top\frac{(0.7-s)^\alpha}{\alpha}+(P'^\top)^2\frac{1}{2!}\left(\frac{(0.5-s)^\alpha}{\alpha}\right)^2\right]e^{F'^\top\frac{(0.7-s)^\alpha}{\alpha}}{B^{-1}}^\top s^{\alpha-1}ds, \end{aligned}$$

and

$$\begin{aligned} M_2 &= \int_{0.3}^{0.6} B^{-1}e^{F'\frac{(0.7-s)^\alpha}{\alpha}}\left[\mathcal{I}+P'\frac{(0.7-s)^\alpha}{\alpha}\right]e^{F'\frac{\kappa^\alpha}{\alpha}}e^{F'^\top\frac{\kappa^\alpha}{\alpha}} \\ &\times\left[\mathcal{I}+P'^\top\frac{(0.7-s)^\alpha}{\alpha}\right]e^{F'^\top\frac{(0.7-s)^\alpha}{\alpha}}{B^{-1}}^\top s^{\alpha-1}ds, \end{aligned}$$

and

$$M_3 = \int_{0.6}^{0.9} B^{-1}e^{F'\frac{(0.7-s)^\alpha}{\alpha}}\mathcal{I}^2e^{F'\frac{\kappa^\alpha}{\alpha}}e^{F'^\top\frac{\kappa^\alpha}{\alpha}}e^{F'^\top\frac{(0.7-s)^\alpha}{\alpha}}{B^{-1}}^\top s^{\alpha-1}ds.$$

Note

$$M_1 = \begin{bmatrix} 19.4212 & 11.0558 \\ 12.2618 & 7.6964 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 6.6098 & 3.9537 \\ 4.2875 & 2.9241 \end{bmatrix},$$

$$M_3 = \begin{bmatrix} 1.0284 & 0.7608 \\ 0.7767 & 0.4579 \end{bmatrix}.$$

Therefore, we obtain

$$M_c[0,\zeta_1] = \begin{bmatrix} 27.0594 & 15.7703 \\ 17.3260 & 11.0784 \end{bmatrix}, \quad M_c^{-1}[0,\zeta_1] = \begin{bmatrix} 0.4174 & -0.5942 \\ -0.6529 & 1.0196 \end{bmatrix}.$$

and

$$U_1 = \sqrt{\|M_c^{-1}[0,\zeta_1]\|} = \sqrt{0.4254} = 0.6522.$$

Moreover, $\forall u, v \in \mathbb{R}^n$,

$$\begin{aligned} \|y(\zeta,u) - y(\zeta,v)\| &= \max\{0.2\zeta\|u_1 - v_1\|, 0.1\zeta\|u_2 - v_2\|\} \\ &\le 0.2\zeta \max\{\|u_1 - v_1\|, \|u_2 - v_2\|\} \\ &= 0.2\zeta\|u - v\|. \end{aligned}$$

Now, setting $L_y(\zeta) = 0.2\zeta \in L^g(I, \mathbb{R}^+)$ *with* $g = h = 2$.

Observing that $\|L_y\|_{L^2(I,\mathbb{R}^+)} = (\int_0^{0.9} (0.2s)^2 ds)^{\frac{1}{2}} = 0.0441$,

$$b = \|B^{-1}\| W^2 e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)(\zeta_1+\kappa)^\alpha}{\alpha}} [\frac{1}{g\alpha - g + 1} \zeta_1^{g\alpha - g + 1}]^{\frac{1}{g}} \|L_y\|_{L^h(I,\mathbb{R}^+)} = 0.1030,$$

and

$$b\Big[1 + W^2 U_1 \|B^{-1}\| \|J\| e^{w\frac{\kappa^\alpha}{\alpha}} e^{\frac{(\beta-w)\zeta_1^\alpha}{\alpha}} \frac{\zeta_1^\alpha}{\alpha}\Big] = 0.2814 < 1.$$

As a result, all the requirements stipulated in Theorem 5 are satisfied, indicating that (32) exhibits relative controllability on the interval $[0, 0.9]$.

6. Conclusions

This article is a generalization of literature [20]. Based on literature [20], this paper introduces a class of nonsingular conformable delay systems with non-singular term B. By giving an estimate of the delayed matrix exponential with non-singular term B, which makes our analysis of systems (1)–(3) more refined. In the first part, we give the sufficient conditions for α-exponential stability of systems (1)–(3) through introducing the definition of α-exponential stability. In the second part, by constructing a Grammian matrix with non-singular terms B, the relative controllability of the linear and nonlinear problems discussed is provided. Finally, we validate our theoretical results with several examples.

Further work will be able to discuss some issues such as periodic solutions and their stability of related systems.

Funding: This research was funded by Guizhou University of Finance and Economics grant number 2019XYB11 and National Natural Science Foundation of China (12261018).

Data Availability Statement: The authors confirm that the data supporting the findings of this study are available within the article.

Acknowledgments: The authors are grateful to the referees for their careful reading of the manuscript and valuable comments. The authors thank the help from the editor too.

Conflicts of Interest: The authors declare no conflict of interest.

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