

On the Strong Starlikeness of the Bernardi Transform

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Abstract: Many papers concern both the starlikeness and the convexity of Bernardi integral operator. Using the Nunokawa's Lemma, we want to determine conditions for the strong starlikeness of the Bernardi transform of normalized analytic functions g , such that $|\arg\{g'(z)\}| < \frac{\alpha\pi}{2}$ in the open unit disk Δ where $0 < \alpha < 2$. Our results include the results of Mocanu, Nunokawa and others on the Libera transform.

Keywords: Nunokawa's lemma; Libera operator; Bernardi operator; strongly starlike functions

MSC: 30C45; 30C80

1. Introduction

The class of all analytic functions in the open unit disk Δ is shown by \mathcal{H} , and the class of functions $h \in \mathcal{H}$ which is in the form

$$h(z) = z + a_{n+1}z^{n+1} + a_{n+2}z^{n+2} + \dots \quad (z \in \Delta),$$

is denoted by \mathcal{A}_n with $\mathcal{A}_1 = \mathcal{A}$.

Furthermore, the class of strongly starlike functions of order β ($0 < \beta \leq 1$) is denoted by $\mathcal{SS}^*(\beta)$, where

$$\mathcal{SS}^*(\beta) = \left\{ h \in \mathcal{A} : \left| \arg \left\{ \frac{zh'(z)}{h(z)} \right\} \right| < \frac{\beta\pi}{2}, z \in \Delta \right\}$$

as was introduced in [1,2]. We know that $\mathcal{SS}^*(1) \equiv \mathcal{S}^*$ is the class of starlike functions in Δ . Refer to [3–5] for various sufficient conditions for this subject. Let

$$\mathcal{R} = \{h \in \mathcal{A} : \operatorname{Re}\{h'(z)\} > 0, z \in \Delta\},$$

which is the class of functions with bounded turning. For $\xi \geq 1$, we denote using L_ξ ; the Bernardi transform is defined as $L_\xi: \mathcal{A} \rightarrow \mathcal{A}$, where

$$L_\xi[h](z) = \frac{1+\xi}{z^\xi} \int_0^z h(t)t^{\xi-1}dt, \quad (1)$$

is the Bernardi integral operator. Several authors have studied this (for example, see [6,7]). The study presented in [8] concerns aspects regarding both the starlikeness and the convexity of Bernardi integral operator. Investigations on the Bernardi integral operator have continued in recent years. Applications introducing new classes of analytic functions can be seen in [9,10]. Several majorization results for the class of normalized starlike functions are obtained using the Bernardi integral operator in [11], and studies regarding coefficient estimates have been performed for a new class of starlike functions associated with sine



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functions, using the Bernardi integral operator in [12]. Integral transforms have an important role in geometric function theory. The reader can find interesting results in, for instance [13,14]. A thorough review on the importance of integral operators can be seen in [15].

If $\zeta = 1$, we have $L_1[h] = L[h]$ where

$$L[h](z) = \frac{2}{z} \int_0^z h(t) dt, \quad (2)$$

is the well-known Libera integral operator.

The problem of the starlikeness of $L[h]$ was considered by Mocanu [16], and the following result was proved.

Theorem 1 (see [16]). *If h is analytic and $\operatorname{Re}\{h'(z)\} > 0$ in Δ , then $L[h] \in \mathcal{S}^*$.*

Or briefly

$$L[\mathcal{R}] \subset \mathcal{S}^* = \mathcal{SS}^*(1), \quad (3)$$

where $L[\mathcal{R}] = \{L[h] : h \in \mathcal{R}\}$. Relation (3) was improved by Mocanu [17] as follows:

$$L[\mathcal{R}] \subset \mathcal{SS}^*(8/9). \quad (4)$$

Recently, the problem of the strong starlikeness of $L[h]$ for $h \in \mathcal{R}$ was considered also in [18]. Nunokawa et al. in [18] proved the following result, which is an improvement on Mocanu's result (4).

Theorem 2 (see [18]). *If $h \in \mathcal{A}$ and $\operatorname{Re}\{h'(z)\} > 0$ in Δ , then the function (2) satisfies*

$$\left| \arg \left\{ \frac{zL'[h](z)}{L[h](z)} \right\} \right| < \frac{\xi\pi}{2} = 1.368 \dots \quad (z \in \Delta)$$

where

$$\xi = \frac{2}{\pi} \left(\frac{\pi}{2} - \log 2 \right) \left(1 + \frac{\pi}{2} - \log 2 \right) = 0.870907 \dots \quad (5)$$

This result may be written as

$$L[\mathcal{R}] \subset \mathcal{SS}^*(\xi), \quad (6)$$

where ξ is given by (5).

In this paper, motivated by the works mentioned above, we studied the problem of the strong starlikeness of the Bernardi transform, and obtained an improvement on the results of Mocanu and Nunokawa et al. One can continue this work by using other integral operators, for example, the Libera–Pascu operator on alpha-close-to-convex functions (for more details see [19]). Furthermore, these conclusions can be extended by applying q-calculus and constructing positive operators in the future. There are many papers on q-calculus, but one of the more recent papers is [20].

2. Main Results

We need the following Lemmas to prove the main theorem.

Lemma 1 (see [21]). *Suppose that*

$$h(z) = 1 + \sum_{n=m \geq 1}^{\infty} a_n z^n \quad (a_m \neq 0; z \in \Delta) \quad (7)$$

with $h(z) \neq 0$ in Δ . If there exists a point z_0 ($z \in \Delta$) such that

$$|\arg\{h(z)\}| < \frac{\beta\pi}{2} \quad \text{for } |z| < |z_0|$$

and

$$|\arg\{h(z_0)\}| = \frac{\beta\pi}{2}$$

for some $\beta > 0$, then

$$\frac{z_0 h'(z_0)}{h(z_0)} = ik\beta,$$

where

$$k \geq \frac{m(a^2 + 1)}{2a} \geq 1, \quad \text{when } \arg\{h(z_0)\} = \frac{\beta\pi}{2}$$

and

$$k \leq -\frac{m(a^2 + 1)}{2a} \leq -1, \quad \text{when } \arg\{h(z_0)\} = -\frac{\beta\pi}{2},$$

where

$$\{h(z_0)\}^{\frac{1}{\beta}} = \pm ia \quad (a > 0).$$

Theorem 3 (see [18]). Let h be analytic in Δ with $h(0) = 1$ and

$$h(z) + \xi zh'(z) \prec \left(\frac{1+z}{1-z}\right)^\alpha \quad (z \in \Delta),$$

where $0 < \alpha < 2$ and $\xi \leq 1$. Then

$$h(z) \prec \left(\frac{1+z}{1-z}\right)^\beta \quad (z \in \Delta),$$

where

$$\beta = \alpha \left(1 - \frac{2}{\pi} \log 2\right). \quad (8)$$

Lemma 2. If $g \in \mathcal{A}$ and $|\arg\{g'(z)\}| < \frac{\alpha\pi}{2}$ in Δ , then the function (1) satisfies

$$\left|\arg\left\{\frac{L_\xi[g](z)}{z}\right\}\right| < \frac{\alpha\pi}{2} \left(1 - \frac{2}{\pi} \log 2\right)^2 \quad (z \in \Delta),$$

where $0 < \alpha < 2$ and $\xi \geq 1$.

Proof. Let $g \in \mathcal{A}$ and $|\arg\{g'(z)\}| < \frac{\alpha\pi}{2}$ where $z \in \Delta$ and $0 < \alpha < 2$. From (1) we have

$$zL_\xi''[g](z) + (1 + \xi)L_\xi'[g](z) = (1 + \xi)g'(z) \quad (z \in \Delta),$$

and so

$$\left|\arg\left(L_\xi'[g](z) + \frac{1}{1 + \xi}zL_\xi''[g](z)\right)\right| < \frac{\alpha\pi}{2}.$$

Therefore

$$L_{\xi}'[g](z) + \frac{1}{1+\xi} z L_{\xi}''[g](z) \prec \left(\frac{1+z}{1-z} \right)^{\alpha}$$

and by Theorem 3,

$$L_{\xi}'[g](z) \prec \left(\frac{1+z}{1-z} \right)^{\beta},$$

where β is given by (8). Similar to the proof of Lemma 2.3 in [18], let $h(z) = L_{\xi}'[g](z)/z$ ($z \in \Delta$). Then $h(z) + zh'(z) = L_{\xi}'[g](z)$ and by (9) we have

$$|\arg\{h(z) + zh'(z)\}| < \frac{\beta\pi}{2}.$$

Again, using Theorem 3,

$$h(z) \prec \left(\frac{1+z}{1-z} \right)^{\delta} \quad (z \in \Delta),$$

where

$$\delta = \beta \left(1 - \frac{2}{\pi} \log 2 \right) = \alpha \left(1 - \frac{2}{\pi} \log 2 \right)^2$$

and so

$$\left| \arg \left(\frac{L_{\xi}'[g](z)}{z} \right) \right| < \frac{\alpha\pi}{2} \left(1 - \frac{2}{\pi} \log 2 \right)^2 \quad (z \in \Delta).$$

□

Theorem 4. Let $g \in \mathcal{A}$ and $\xi \geq 1$. Furthermore, suppose that for $0 < \alpha < 2$,

$$|\arg\{g'(z)\}| < \frac{\alpha\pi}{2} \quad (z \in \Delta). \quad (9)$$

If Equation (with respect to x)

$$x + \frac{2}{\pi} \tan^{-1} \frac{x}{\xi} = \alpha \left(1 + \left(1 - \frac{2}{\pi} \log 2 \right)^2 \right) \quad (10)$$

has a solution $\beta \in (0, 1]$, then

$$\left| \arg \left\{ \frac{z L_{\xi}'[g](z)}{L_{\xi}[g](z)} \right\} \right| < \frac{\beta\pi}{2}, \quad (11)$$

and $L_{\xi}[g]$ is the strongly starlike of order β .

Proof. Let

$$h(z) = \frac{z L_{\xi}'[g](z)}{L_{\xi}[g](z)} \quad (z \in \Delta).$$

If there exists a point $z_0 \in \Delta$, for which

$$|\arg\{h(z)\}| < \frac{\beta\pi}{2} \quad (|z| < |z_0|)$$

and

$$|\arg\{h(z_0)\}| = \frac{\beta\pi}{2},$$

then from Nunokawa's Lemma 1, we have

$$\frac{z_0 h'(z_0)}{h(z_0)} = ik\beta,$$

where

$$k \geq \frac{a^2 + 1}{2a} \geq 1, \quad \text{when} \quad \arg\{h(z_0)\} = \frac{\beta\pi}{2}$$

and

$$k \leq -\frac{a^2 + 1}{2a} \leq -1, \quad \text{when} \quad \arg\{h(z_0)\} = -\frac{\beta\pi}{2},$$

with $h(z_0) = (\pm ia)^\beta$ ($a > 0$).

If $\arg\{h(z_0)\} = \frac{\beta\pi}{2}$, we have

$$\begin{aligned} \left| \arg\left\{z_0 h'(z_0) + h(z_0)^2 + \xi h(z_0)\right\} \right| &= \left| \arg\left\{h(z_0) \left[\xi + h(z_0) + \frac{z_0 h'(z_0)}{h(z_0)} \right] \right\} \right| \\ &= \left| \arg\{h(z_0)\} + \arg\left\{ \xi + h(z_0) + \frac{z_0 h'(z_0)}{h(z_0)} \right\} \right| \\ &= \left| \frac{\beta\pi}{2} + \tan^{-1} \left\{ \frac{\beta k + a^\beta \sin(\beta\pi/2)}{\xi + a^\beta \cos(\beta\pi/2)} \right\} \right|, \end{aligned} \quad (12)$$

where $h(z_0) = (ia)^\beta$ ($a > 0$) and

$$k \geq \frac{a^2 + 1}{2a} \geq 1.$$

Let us put

$$u(a) = \frac{\beta k + a^\beta \sin(\beta\pi/2)}{\xi + a^\beta \cos(\beta\pi/2)} \quad (a > 0).$$

Then

$$u(a) \geq \frac{\beta + a^\beta \sin(\beta\pi/2)}{\xi + a^\beta \cos(\beta\pi/2)} \quad (a > 0). \quad (13)$$

Putting

$$f(x) = \frac{\beta + x \sin(\beta\pi/2)}{\xi + x \cos(\beta\pi/2)} \quad (x \geq 0),$$

we have

$$f'(x) = \frac{\xi \sin(\beta\pi/2) - \beta \cos(\beta\pi/2)}{(\xi + x \cos(\beta\pi/2))^2} > 0 \quad (x \geq 0),$$

because $\tan(\beta\pi/2) > \beta$ and $\xi \geq 1$. Therefore, for $x > 0$ we obtain $f(x) > f(0) = \beta/\xi$, so from (13) we get

$$u(a) > \frac{\beta}{\xi},$$

which implies that

$$\tan^{-1} \left\{ \frac{\beta k + a^\beta \sin(\beta\pi/2)}{\xi + a^\beta \cos(\beta\pi/2)} \right\} > \tan^{-1} \frac{\beta}{\xi} \quad (a > 0).$$

Therefore, from (12), we have the following inequality

$$\begin{aligned} \left| \arg \left\{ z_0 h'(z_0) + h(z_0)^2 + \xi h(z_0) \right\} \right| &= \frac{\beta\pi}{2} + \tan^{-1} \left\{ \frac{\beta k + a^\beta \sin(\beta\pi/2)}{\xi + a^\beta \cos(\beta\pi/2)} \right\} \\ &> \frac{\beta\pi}{2} + \tan^{-1} \frac{\beta}{\xi}. \end{aligned} \quad (14)$$

Moreover, from Lemma 2, we have

$$|\arg \{H(z_0)\}| = \left| \arg \left\{ \frac{L_\xi[g](z_0)}{z_0} \right\} \right| < \frac{\alpha\pi}{2} \left(1 - \frac{2}{\pi} \log 2 \right)^2, \quad (15)$$

where

$$H(z) = \frac{L_\xi[g](z)}{z} \quad (z \in \Delta).$$

By (14) and (15), we can obtain

$$\begin{aligned} |\arg \{ (1 + \xi)g'(z_0) \}| &= \left| \arg \left\{ (1 + \xi)L'_\xi[g](z_0) + z_0 L''_\xi[g](z_0) \right\} \right| \\ &= \left| \arg \left\{ H(z_0) \left(z_0 h'(z_0) + h(z_0)^2 + \xi h(z_0) \right) \right\} \right| \\ &= \left| \arg \{H(z_0)\} + \arg \left\{ z_0 h'(z_0) + h(z_0)^2 + \xi h(z_0) \right\} \right| \\ &> \frac{\beta\pi}{2} + \tan^{-1} \frac{\beta}{\xi} - \frac{\alpha\pi}{2} \left(1 - \frac{2}{\pi} \log 2 \right)^2 = \frac{\alpha\pi}{2}, \end{aligned}$$

because β is the solution of (10). Therefore, we have

$$|\arg \{g'(z_0)\}| = |\arg \{ (1 + \xi)g'(z_0) \}| > \frac{\alpha\pi}{2}. \quad (16)$$

This contradicts the hypothesis. If $\arg \{h(z_0)\} = -\frac{\beta\pi}{2}$, we have similar calculations, and the proof is completed. \square

By putting $\xi = \alpha = 1$ in Theorem 4, we have:

Corollary 1. *If $g \in \mathcal{A}$ and $\operatorname{Re}\{g'(z)\} > 0$ in Δ , then*

$$\left| \arg \frac{zL'[g](z)}{L[g](z)} \right| < \frac{\beta\pi}{2} \quad (z \in \Delta),$$

where $\beta = 0.860004 \dots$.

Or briefly

$$L[\mathcal{R}] \subset \mathcal{SS}^*(\beta), \quad (17)$$

where $\beta = 0.860004 \dots$, which shows that the result (17) improves the result (6) in Theorem 2 obtained by Nunokawa et al. on the Libera integral operator.

Furthermore, with suitable choices of α and β in Theorem 4, we obtain:

Corollary 2. (i) If $g \in \mathcal{A}$ and

$$|\arg\{g'(z)\}| < \frac{\alpha\pi}{2},$$

with $\alpha \approx 0.6059$, then $L[g] \in \mathcal{SS}^*(1/2)$.

(ii) If $g \in \mathcal{A}$ and

$$|\arg\{g'(z)\}| < \frac{\alpha\pi}{2},$$

with $\alpha \approx 0.7933$, then $L[g] \in \mathcal{SS}^*(2/3)$.

(iii) If $g \in \mathcal{A}$ and

$$|\arg\{g'(z)\}| < \frac{\alpha\pi}{2},$$

with $\alpha \approx 1.1432$, then $L[g] \in \mathcal{SS}^*(1) = \mathcal{S}^*$.

3. Conclusions

In the present investigation, we have found suitable conditions for the Bernardi transform of a special class of analytic functions to be in the class of strongly starlike functions. One can obtain the conditions for other integral transforms to have geometric properties, such as starlikeness, convexity, q -starlikeness and alpha-close-to-convexity.

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