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Centrally Extended Jordan $(*)$-Derivations Centralizing Symmetric or Skew Elements

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Abstract: Let A be a non-commutative prime ring with involution $*$, of characteristic $\neq 2$ (and 3), with Z as the center of A and Π a mapping $\Pi : A \to A$ such that $[\Pi(x), x] \in Z$ for all (skew) symmetric elements $x \in A$. If Π is a non-zero CE-Jordan derivation of A, then A satisfies s_4, the standard polynomial of degree 4. If Π is a non-zero CE-Jordan $*$-derivation of A, then A satisfies s_4 or $\Pi(y) = \lambda(y - y^*)$ for all $y \in A$, and some $\lambda \in C$, the extended centroid of A. Furthermore, we give an example to demonstrate the importance of the restrictions put on the assumptions of our results.

Keywords: prime ring; involution; centrally extended Jordan ($*$-)derivation; (skew) symmetric elements

MSC: 16W10; 16N60; 16W25

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1. Introduction

Throughout this article, A denotes an associative ring with the center Z, and with the maximal symmetric ring of quotients of A, denoted by $Q_{ms} = Q_{ms}(A)$. The center of Q_{ms} is called the extended centroid of A and is denoted by C. Clearly, $A \subseteq Q_{ms}$ and $Z \subseteq C$. Moreover, if A is prime, then C is a field. The ring AC is called the central closure of A. The prime ring A is called centrally closed if $A = AC$. In particular, the prime ring Q_{ms} is centrally closed; more information about these objects can be found in [1]. The symbol $[a, b]$ (resp., $a \circ b$) denotes the commutator (resp., anti-commutator) $ab - ba$ (resp., $ab + ba$) for all $a, b \in A$. A ring A is called prime if, for all $a, b \in A$, $aAb = (0)$ implies either $a = 0$ or $b = 0$, and if $aAa = (0)$ implies $a = 0$, then A is called a semi-prime ring. A ring A is called 2-torsion-free if, for all $a \in A$, $2a = 0$ implies $a = 0$. If $a, ax \in Z$ and A is a prime ring, then $a = 0$ or $x \in Z$ for all $a, x \in A$. Further, if $0 \neq a \in Z$ and A is a prime ring, then a is not a zero divisor for all $a \in A$. An additive map $* : A \to A$ is called an involution if $(a^*)^* = a$ for all $a \in A$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. By a ring with involution, we mean a ring equipped with an involution "$*$", which is also called a $*$-ring. Let $H = \{a \in A : a^* = a\}$ and $S = \{a \in A : a^* = -a\}$; the elements of H are called symmetric, and the elements of S are called skew-symmetric. Thus, for all $a \in A$, we have $a + a^* \in H$ and $a - a^* \in S$. The involution "$*$" can be uniquely extended to the involution of Q_{ms}. The involution "$*$" is said to be of the first kind if $Z \subseteq H$; otherwise, it is of the second kind, i.e., $S \cap Z \neq (0)$. An additive mapping $\Pi : A \to A$ is called a derivation if $\Pi(ab) = \Pi(a)b + a\Pi(b)$ for all $a, b \in A$. For a fixed element $c \in A$, a mapping $a \mapsto [c, a]$ is called an inner derivation induced by 'c'. An additive map Π is called a Jordan derivation if $\Pi(a^2) = \Pi(a)a + a\Pi(a)$ for all $a \in A$. Obviously, every derivation is a Jordan derivation, but the converse is not necessarily true (see [2], Example 3.2.1). Moreover, the question of "When is a Jordan derivation a derivation?" led to a new and significant area of research (see [3–7]). In 1957, Herstein [6] showed that for prime rings of characteristic

$\neq 2$, every Jordan derivation is an ordinary derivation. Later, Brešar and Vukman [5] gave a brief and elegant proof of this result. In the same year, Brešar [4] showed that for a rather wider class of rings—namely, semi-prime rings with 2-torsion-free condition—every Jordan derivation is a derivation. Thenceforth, a considerable number of results have been proved in this direction. Let A be a $*$-ring. An additive mapping $\Pi : A \to A$ is called a $*$-derivation if $\Pi(ab) = \Pi(a)b^* + a\Pi(b)$ for all $a, b \in A$, and is called a Jordan $*$-derivation if $\Pi(a^2) = \Pi(a)a^* + a\Pi(a)$ for all $a \in A$. The notions of $*$-derivation and Jordan $*$-derivation were first mentioned in [8]. Note that the mapping $a \mapsto a^*c - ca$, where c is a fixed element of A, is a Jordan $*$-derivation known as an inner Jordan $*$-derivation. Moreover, Π is called X-inner if there exists $q \in Q_{ms}$ such that $\Pi(a) = aq - qa^*$ for all $a \in A$ (see [9]). The issue of quadratic forms' representability by bilinear forms gave rise to the study of Jordan $*$-derivations (see [10,11]). Since then, there has been a significant interest in studying the algebraic structure of Jordan $*$-derivations in rings and algebras; for a good cross-section, we refer the reader to [12–15]. For further generalizations and recent results, see [9].

Recently, Bell and Daif [16] introduced a centrally extended derivation and defined it as follows: a map $\Pi : A \to A$ is called a centrally extended derivation if $\Pi(a+b) - \Pi(a) - \Pi(b) \in Z$ for all $a, b \in A$ and $\Pi(ab) - \Pi(a)b - a\Pi(b) \in Z$ for all $a, b \in A$. There has been rising literature investigating centrally extended mappings in rings under various settings; e.g., see [16–20].

Let D be a subset of A; a mapping f is called commuting (resp., centralizing) on D, if $[f(a), a] = 0$ (resp., $[f(a), a] \in Z$) for all $a \in D$. In 1955, Divinsky [21] established that a simple Artinian ring is commutative if it admits a commuting non-trivial automorphism, which launched the study of commuting and centralizing mappings. Posner [22] proved another remarkable result: A must be commutative if there is a non-zero centralizing derivation on A. Ali and Dar [23] introduced $*$-commuting and $*$-centralizing mappings and defined them as follows: a mapping f is called $*$-commuting (resp., $*$-centralizing) on a set D if $[f(a), a^*] = 0$ (resp., $[f(a), a^*] \in Z$) for all $a \in D$. For further generalizations and recent results, see [24].

One of the most interesting and revolutionary concepts was the study of derivations in rings. It has been proven in a variety of other derivations over time. Amalgamation endomorphisms, anti-automorphisms, and (anti-) commutators with derivations have opened up a new world of intriguing ideas. Although purely an algebraic concept, derivations have a wide range of applications. Many algebraists are interested in the issue of knowing the structure of rings, and the concept of derivations on rings and modules is convenient for this goal. The relationship between derivations and the structure of rings has been extensively examined in recent years, although more work is needed. The study of derivations in rings was initiated long ago but received impetus only after Posner [22], who in 1957 established two very striking results on derivations in prime rings. The notion of derivation has also been generalized in various directions, such as Jordan derivation, centrally extended Jordan ($*$)-derivation, centrally extended generalized Jordan ($*$)-derivation, etc. Moreover, there has been considerable interest in investigating the commutativity of rings, more often that of prime and semiprime rings, and admitting these mappings, which are centralizing or commuting on some appropriate subsets of R. Kharchenko [25] described identities with derivations, and his results are used effectively as a powerful tool to reduce a differential identity to a generalized polynomial identity.

Recently, Bhushan et al. [17] introduced centrally extended Jordan derivations, which are a generalization of Jordan derivations and derivations, and they discussed the existence of these mappings in rings. Accordingly, a self-mapping Π of A is called a centrally extended Jordan derivation if $\Pi(a+b) - \Pi(a) - \Pi(b) \in Z$ and $\Pi(a \circ b) - \Pi(a) \circ b - a \circ \Pi(b) \in Z$ for all $a, b \in A$. They abbreviated this map as the CE-Jordan derivation. They also established the following result: if A is a non-commutative prime ring with involution "$*$" and Π is a non-zero centrally extended Jordan derivation of A such that $[\Pi(a), a] \in Z$

(resp., $[\Pi(a), a^*] \in Z$) for all $a \in A$, then A satisfies s_4 (in other words, A is an order in a central simple algebra of dimension at most 4 over its center, see Lemma 1).

Motivated by this, we show that if a non-zero centrally extended Jordan derivation Π on a non-commutative prime ring A, $\mathrm{char}(A) \neq 2$ with involution " $*$ " satisfying $[\Pi(a), a] \in Z$ for all $a \in H$ or $a \in S$, then A satisfies s_4. Moreover, we provide analogous studies related to centrally extended Jordan $*$-derivations. Furthermore, we give Example 1 to demonstrate the importance of the primeness A in our results.

2. Preliminary Results

The standard identity in four non-commuting variables, denoted by s_4, is defined by

$$s_4(a_1, a_2, a_3, a_4) = \sum_{\sigma \in S_4} (-1)^{\sigma} a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)} a_{\sigma(4)},$$

where $(-1)^{\sigma}$ is the sign of the permutation $\sigma \in S_4$, S_4 is the symmetric group of degree 4, and $a_{\sigma(i)}$ are the indeterminate variables [26,27]. It is known that if A is a non-commutative prime ring and satisfies s_4, then A is an order in a central simple algebra of dimension at most 4 over its center, see Lemma 1.

Lemma 1 ([28], Lemma 2.1 and [29], Theorem (Posner) 4.4, p.42)**.** *Let A be a non-commutative prime ring. Then, $dim_C AC \leq 4$ if and only if A satisfies s_4.*

Lemma 2 ([30], Lemma 2)**.** *Let A be a semi-prime ring. If $[S^2, S^2] = (0)$, then A satisfies s_4.*

Lemma 3 ([31], Theorem 3)**.** *Let A be a prime ring. If n is a fixed natural number such that $h^n \in Z$ for all $h \in H$, then A satisfies s_4.*

Lemma 4 ([31], Theorem 7)**.** *Let A be a prime ring. If δ is a derivation on A such that $\delta(s) \circ s \in Z$ for all $s \in S$, then $\delta = 0$ or A satisfies s_4.*

Lemma 5 ([31],Theorem 1 and 2)**.** *Let A be a prime ring and $char(A) \neq 2$. If δ is a non-zero derivation on A such that $[\delta(t), t] \in Z$ for all $t \in H$ ($t \in S$), then A satisfies s_4.*

We now introduce the notation of a generalized polynomial identity taken from [32]. With or without involution, let A be a prime ring, $AC_C\langle X\rangle$ a free product over C of AC, and $C\langle X\rangle$ a free algebra on a set X of indeterminates. An additive subgroup A of AC is called a generalized polynomial identity over C (shortly, A is GPI over C) if there exists a non-zero element $\theta(x_1, x_2, ..., x_m)$ of $AC_C\langle X\rangle$ such that $\theta(u_1, u_2, ..., u_m) = 0$ for all $u_i \in A$.

Lemma 6 ([1], Corollary 6.2.5)**.** *Let A be a prime ring, $char(A) \neq 2$ with involution $*$.*

(i) If S is GPI, then A is GPI.
(ii) If H is GPI, then A is GPI.

Lemma 7 ([32], Lemma 3.2)**.** *Let D be any set and A be a prime ring. If functions $F : D \to A$ and $G : D \to A$ satisfy such $F(u)aG(t) = G(u)aF(t)$ for all $u, t \in D$ and $a \in A$, then $F = 0$ or there exists λ in the extended centroid of A, such that $G(u) = \lambda F(u)$ for all $u \in D$.*

Lemma 8 ([33], Lemma 1.3.2)**.** *Let A be a prime ring. Suppose that a_i, b_i are elements in A such that $\sum a_i u b_i = 0$ for all $u \in A$. Then, all $a_i's = 0$ or $b_i's = 0$ unless the $a_i's$ are linearly dependent over C, and the $b_i's$ are linearly dependent over C.*

3. Results on Centrally Extended Jordan Derivations

Let A be a ring with involution " $*$ ". Recently, Bhushan et al. [17] introduced the notion of CE-Jordan derivation. They established the following result: if A is a non-commutative prime ring, $\mathrm{char}(A) \neq 2$ with involution " $*$ ", and Π is a CE-Jordan derivation such that

$[\Pi(a), a] \in Z$ (resp., $[\Pi(a), a^*] \in Z$) for all $a \in A$, then $\Pi = 0$ or $\dim_C AC \leq 4$. They also proved that a CE-Jordan derivation Π of a prime ring is additive. Now, we will show the following results on a CE-Jordan derivation.

Theorem 1. *Let A be a non-commutative prime ring, char$(A) \neq 2$ with involution "$*$", and let Π be a non-zero CE-Jordan derivation of A. Suppose that $[\Pi(h), h] \in Z$ for all $h \in H$. Then, A satisfies s_4.*

Proof. Assume that

$$[\Pi(h), h] \in Z \tag{1}$$

for all $h \in H$. If $Z = (0)$; then, from the definition of Π, we have that Π is a Jordan derivation and by [6], we obtain that Π is a derivation, and so A satisfies s_4, by Lemma 5. Thus, from now on we will assume that $Z \neq (0)$.

Now, by linearizing (1), we see that $[\Pi(h), h_1] + [\Pi(h_1), h] \in Z$ for all $h, h_1 \in H$. Putting $h_1 = h^2$ in the last relation, we find that $[\Pi(h), h^2] + [\Pi(h^2), h] \in Z$, and so

$$[\Pi(h), h]h + h[\Pi(h), h] + [\Pi(h)h + h\Pi(h), h] \in Z.$$

Further,

$$[\Pi(h), h]h + h[\Pi(h), h] + [\Pi(h), h]h + h[\Pi(h), h] \in Z.$$

Using (1) in the above expression, we conclude that $4h[\Pi(h), h] \in Z$, and hence, $h[\Pi(h), h] \in Z$ for all $h \in H$. It follows that $[h[\Pi(h), h], r] = 0$ for all $r \in A$. By applying (1) in the previous relation, we infer that $[h, r][\Pi(h), h] = 0$. Again, by using (1) in the last equation, we find that $[h, r]A[\Pi(h), h] = 0$ for all $r \in A$. Taking r by $\Pi(h)$ in the previous relation, we obtain $[h, \Pi(h)]A[\Pi(h), h] = 0$, and so,

$$[\Pi(h), h] = 0 \tag{2}$$

for all $h \in H$. By linearizing (2), we obtain

$$[\Pi(h), h_1] + [\Pi(h_1), h] = 0 \tag{3}$$

for all $h, h_1 \in H$. Replacing h_1 by $h_1 \circ h_2$ in (3), we obtain

$$[\Pi(h), h_1 \circ h_2] + [\Pi(h_1 \circ h_2), h] = 0$$

for all $h, h_1, h_2 \in H$—that is,

$$[\Pi(h), h_1 \circ h_2] + [\Pi(h_1) \circ h_2 + h_1 \circ \Pi(h_2), h] = 0.$$

It follows that

$$\begin{aligned}
&[\Pi(h), h_1]h_2 + h_1[\Pi(h), h_2] + [\Pi(h), h_2]h_1 + h_2[\Pi(h), h_1] \\
&+ [\Pi(h_1), h]h_2 + \Pi(h_1)[h_2, h] + h_2[\Pi(h_1), h] + [h_2, h]\Pi(h_1) \\
&+ [\Pi(h_2), h]h_1 + \Pi(h_2)[h_1, h] + h_1[\Pi(h_2), h] + [h_1, h]\Pi(h_2) = 0.
\end{aligned}$$

Applying (3) in the above equation, we see that

$$\Pi(h_1)[h_2, h] + [h_2, h]\Pi(h_1) + [\Pi(h_2), h]h_1 + \Pi(h_2)[h_1, h] + [h_1, h]\Pi(h_2) = 0.$$

That is,

$$\Pi(h_1) \circ [h_2, h] + \Pi(h_2) \circ [h_1, h] = 0 \tag{4}$$

for all $h, h_1, h_2 \in H$. By Lemma 6(ii), we have

$$\Pi(h_1) \circ [h_2, x] + \Pi(h_2) \circ [h_1, x] = 0 \tag{5}$$

for all $h_1, h_2 \in H$ and $x \in A$. Replacing x by xy in (5), where $y \in A$, we conclude that

$$\Pi(h_1) \circ [h_2, x]y + \Pi(h_1) \circ x[h_2, y] + \Pi(h_2) \circ [h_1, x]y + \Pi(h_2) \circ x[h_1, y] = 0$$

for all $h_1, h_2 \in H$ and $x, y \in A$—that is,

$$\begin{aligned}(\Pi(h_1) \circ [h_2, x] + \Pi(h_2) \circ [h_1, x])y - ([h_2, x][\Pi(h_1), y] + [h_1, x][\Pi(h_2), y])\\ + x(\Pi(h_1) \circ [h_2, y] + \Pi(h_2) \circ [h_1, y]) + ([\Pi(h_1), x][h_2, y] + [\Pi(h_2), x][h_1, y]) = 0\end{aligned}$$

for all $h_1, h_2 \in H$ and $x, y \in A$. By using (5) in the above expression, we arrive at

$$[\Pi(h_1), x][h_2, y] + [\Pi(h_2), x][h_1, y] - [h_2, x][\Pi(h_1), y] - [h_1, x][\Pi(h_2), y] = 0$$

for all $h_1, h_2 \in H$ and $x, y \in A$. Taking $h_1 = h_2 = h$ in the last relation, we have

$$[\Pi(h), x][h, y] - [h, x][\Pi(h), y] = 0$$

for all $h \in H$ and $x, y \in A$. Putting y by ry in the previous equation and applying it, we obtain

$$[\Pi(h), x]r[h, y] - [h, x]r[\Pi(h), y] = 0 \tag{6}$$

for all $h \in H$ and $x, y, r \in A$. By using Lemma 7, we obtain $[h, x] = \lambda[\Pi(h), x]$ for all $h \in H$, $x \in A$, and some $\lambda \in C$ or $[\Pi(h), x] = 0$ for all $h \in H$ and $x \in A$.

Case (I): Suppose that $[\Pi(h), x] = 0$ for all $h \in H$ and $x \in A$. It follows that

$$\Pi(h) \in Z \tag{7}$$

for all $h \in H$. Taking h by h^2 in (7) and applying it, we have $2h\Pi(h) \in Z$ for all $h \in H$—that is, $h\Pi(h) \in Z$ for all $h \in H$. Using (7) in the last relation, we see that $h \in Z$ or $\Pi(h) = 0$. In the case where $h \in Z$ for all $h \in H$, then by Lemma 3, we obtain that A satisfies s_4. Now, if

$$\Pi(h) = 0 \tag{8}$$

for all $h \in H$. Putting h by s^2 in (8), where $s \in S$, we find that $\Pi(s) \circ s + z(s^2) = 0$, and so,

$$\Pi(s) \circ s \in Z \tag{9}$$

for all $s \in S$. Replacing s by $s + (s \circ h)$ in (9), where $h \in H$, we conclude that

$$\Pi(s \circ h) \circ s + \Pi(s) \circ (s \circ h) \in Z.$$

Applying (8) in the above expression, we have

$$\{\Pi(s) \circ h + z(h, s)\} \circ s + \Pi(s) \circ (s \circ h) \in Z.$$

It follows that

$$(\Pi(s) \circ h) \circ s + 2z(h, s)s + \Pi(s) \circ (s \circ h) \in Z.$$

This implies that

$$(\Pi(s)h + h\Pi(s))s + s(\Pi(s)h + h\Pi(s)) + 2z(h, s)s + \Pi(s)(sh + hs) + (sh + hs)\Pi(s) \in Z.$$

Hence,

$$\Pi(s)hs + h\Pi(s)s + s\Pi(s)h + sh\Pi(s) + 2z(h,s)s + \Pi(s)sh + \Pi(s)hs + sh\Pi(s) + hs\Pi(s) \in Z.$$

That is,

$$h(\Pi(s)s + s\Pi(s)) + h(\Pi(s)s + s\Pi(s)) + 2\Pi(s)hs + 2sh\Pi(s) + 2z(h,s)s \in Z.$$

Using (9) in the previous relation, we obtain

$$2h(\Pi(s)s + s\Pi(s)) + 2\Pi(s)hs + 2sh\Pi(s) + 2z(h,s)s \in Z.$$

This implies that

$$h(\Pi(s)s + s\Pi(s)) + \Pi(s)hs + sh\Pi(s) + z(h,s)s \in Z.$$

Thus,

$$[h(\Pi(s)s + s\Pi(s)) + \Pi(s)hs + sh\Pi(s) + z(h,s)s, s] = 0.$$

Again, by applying (9) in the last equation, we have

$$(\Pi(s) \circ s)[h,s] + [\Pi(s)h, s]s + s[h\Pi(s), s] = 0 \tag{10}$$

for all $s \in S$ and $h \in H$. Using Lemma 6(ii) in (10), we obtain

$$(\Pi(s) \circ s)[x,s] + [\Pi(s)x, s]s + s[x\Pi(s), s] = 0 \tag{11}$$

for all $s \in S$ and $x \in A$. Putting x by sx in (11) and applying (9), left multiplying it by s, and then subtracting them, we arrive that $[[\Pi(s),s]x,s]s = 0$—that is, $[\Pi(s),s]xs^2 - s[\Pi(s),s]xs = 0$. Using Lemma 8 in the previous relation, we obtain $[\Pi(s),s] = 0$ or $s = 0$ unless $\lambda_1(s)s + \lambda_2(s)s^2 = 0$ for some $\lambda_1(s), \lambda_2(s) \in C$.

Subcase (1): If $s = 0$ for all $s \in S$, then $\Pi(s) = 0$ for all $s \in S$, and from (8), we obtain $\Pi(s+h) = 0$—that is $\Pi(x) = 0$ for all $x \in A$—and so $\Pi = 0$, a contradiction.

Subcase (2): If $[\Pi(s),s] = 0$ for all $s \in S$, then $\Pi(s)s = s\Pi(s)$ and by applying the last expression in (9), we obtain $2\Pi(s)s \in Z$; so,

$$\Pi(s)s \in Z \tag{12}$$

for all $s \in S$. Replacing s by $s + (s \circ h)$ in the above relation, where $h \in H$, we conclude that $\Pi(s \circ h)s + \Pi(s)(s \circ h) \in Z$. Using (8) in the previous expression, we have $(\Pi(s) \circ h)s + z(h,s)s + \Pi(s)(s \circ h) \in Z$—that is, $2h\Pi(s)s + 2\Pi(s)hs + z(h,s)s \in Z$. Hence, $[2h\Pi(s)s + 2\Pi(s)hs, s] = 0$; so, $[h\Pi(s)s + \Pi(s)hs, s] = 0$. It follows that $[h\Pi(s)s, s] + [\Pi(s)h, s]s = 0$. Applying (12) in the last equation, we find that $[h,s]\Pi(s)s + [\Pi(s)h,s]s = 0$—that is, $[h,s]\Pi(s)s + [\Pi(s),s]hs + \Pi(s)[h,s]s = 0$. However, from Subcase (2), we have $[\Pi(s),s] = 0$; so, $[h,s]\Pi(s)s + \Pi(s)[h,s]s = 0$. Using Lemma 6(ii) in the previous relation, we obtain $[x,s]\Pi(s)s + \Pi(s)[x,s]s = 0$ for all $x \in A$ and $s \in S$—that is, $xs\Pi(s)s - sx\Pi(s)s + \Pi(s)xs^2 - \Pi(s)sxs = 0$ for all $x \in A$ and $s \in S$. Applying (12) in the last expression, we see that $-sx\Pi(s)s + \Pi(s)xs^2 = 0$ for all $x \in A$ and $s \in S$. Again, using (12) in the previous relation, we have

$$\Pi(s)s^2x - \Pi(s)xs^2 = 0 \tag{13}$$

for all $x \in A$ and $s \in S$. Applying Lemma 8 in (13), we obtain $s^2 = 0$ unless $\lambda_3(s) + \lambda_4(s)s^2 = 0$ for some $\lambda_3(s), \lambda_4(s) \in C$. In case $s^2 = 0$ for all $s \in S$ and by Lemma 2, we obtain that A satisfies s_4. Now, consider the case $\lambda_3(s) + \lambda_4(s)s^2 = 0$ and $s^2 \neq 0$. Since

$\lambda_3(s) \neq 0 \neq \lambda_4(s)$, we obtain $s^2 = -\lambda_3(s)\lambda_4^{-1}(s) \in C$; hence, $s^2 \in C$, and since $s^2 \in A$, we obtain

$$s^2 \in Z \tag{14}$$

for all $s \in S$—that is, $[s^2, x] = 0$ for all $x \in A$. It follows that $[s, x]s + s[s, x] = 0$. We put $\delta(s) = [s, x]$, and so $\delta(s)s + s\delta(s) = 0$; hence, $\delta(s) = 0$ or A satisfies s_4 by Lemma 4. If $\delta(s) = 0$, then $[s, x] = 0$ and, by Lemma 6(i), we obtain $[y, x] = 0$ for all $x, y \in A$; so, A is commutative, a contradiction.

Subcase (3): If

$$\lambda_1(s)s + \lambda_2(s)s^2 = 0 \tag{15}$$

for all $s \in S$, then $(\lambda_1(s)s + \lambda_2(s)s^2)^* = 0$; so,

$$-\lambda_1(s)^*s + \lambda_2(s)^*s^2 = 0 \tag{16}$$

for all $s \in S$.

First: Suppose that " $*$ " is the first kind. From (16), we see that $-\lambda_1(s)s + \lambda_2(s)s^2 = 0$ and by using the last expression in (15) we obtain $2\lambda_1(s)s = 0$—that is, $\lambda_1(s)s = 0$, and since $\lambda_1(s) \neq 0$, we find that $s = 0$. Now, the same as in the above, we obtain that A satisfies s_4.

Second: Suppose that " $*$ " is the second kind. Let $0 \neq s' \in S \cap Z$. Assume that $\Pi(s') = 0$. Replacing h by $s \circ s'$ in (8), where $s \in S$, we have $\Pi(s \circ s') = 0$; so, $\Pi(s) \circ s' + z(s, s') = 0$, which implies that $\Pi(s) \circ s' \in Z$ and, hence, $2\Pi(s)s' \in Z$—that is, $\Pi(s) \in Z$. Taking s by $s \circ h$ in the previous relation, and applying it and (8), where $h \in H$, we see that $\Pi(s)h \in Z$. By Lemma 6(i), we obtain $\Pi(s)x \in Z$ for all $x \in A$ and $s \in S$. Since $\Pi(s) \in Z$, we find that $\Pi(s) = 0$ for all $s \in S$ or $x \in Z$ for all $x \in A$. If $x \in Z$ for all $h \in A$, then A is commutative, a contradiction. If $\Pi(s) = 0$ for all $s \in S$, then by using (8), we infer that $\Pi(x) = 0$ for all $x \in A$, a contradiction. Now, assume that $\Pi(s') \neq 0$. Putting h by $s \circ s'$ in (8), where $s \in S$, we have $\Pi(s \circ s') = 0$—that is, $\Pi(s) \circ s' + \Pi(s') \circ s + z(s, s') = 0$. It follows that $\Pi(s) \circ s' + \Pi(s') \circ s \in Z$; so,

$$2s'\Pi(s) + \Pi(s') \circ s \in Z \tag{17}$$

for all $s \in S$. Taking $s = s'$ in (17), we obtain $4s'\Pi(s') \in Z$; so, $\Pi(s') \in Z$. Applying the last relation in (17), we see that $2s'\Pi(s) + 2\Pi(s')s \in Z$. This implies that $[\Pi(s), s] = 0$ for all $s \in S$. Now, the same as in Subcase (2), we obtain that A satisfies s_4.

Case (II): Suppose that $[h, x] = \lambda[\Pi(h), x]$ for all $h \in H$ and $x \in A$. It follows that $[h - \lambda\Pi(h), x] = 0$ for all $h \in H$ and $x \in A$—that is,

$$h - \lambda\Pi(h) \in Z \tag{18}$$

for all $h \in H$. Replacing h by $h \circ h_1$ in (18), where $h_1 \in H$, we have

$$hh_1 + h_1h - \lambda\Pi(h)h_1 - \lambda h_1\Pi(h) - \lambda\Pi(h_1)h - \lambda h\Pi(h_1) \in Z.$$

This implies that

$$h(h_1 - \lambda\Pi(h_1)) + (h_1 - \lambda\Pi(h_1))h - \lambda\Pi(h)h_1 - \lambda h_1\Pi(h) \in Z.$$

Using (18) in the above expression, we obtain

$$2h(h_1 - \lambda\Pi(h_1)) - \lambda\Pi(h)h_1 - \lambda h_1\Pi(h) \in Z.$$

That is, $[\lambda\Pi(h)h_1 + \lambda h_1\Pi(h), h] = 0$; so, $\lambda[\Pi(h)h_1 + h_1\Pi(h), h] = 0$. Hence, $\lambda = 0$ or $[\Pi(h)h_1 + h_1\Pi(h), h] = 0$. If $\lambda = 0$, then, from (18), we obtain $h \in Z$ for all $h \in H$ and, by Lemma 3, A satisfies s_4. From now on, we will assume that $\lambda \neq 0$, and so, $[\Pi(h)h_1 + h_1\Pi(h), h] = 0$. Applying (2) in the last equation, we see that $\Pi(h)[h_1, h] + [h_1, h]\Pi(h) = 0$. Using Lemma 6(ii) in the previous relation, we find that $\Pi(h)[x, h] + [x, h]\Pi(h) = 0$ for all $x \in A$ and $h \in H$. Taking h by $h + h_1$ in the last expression, where $h_1 \in H$, we have $\Pi(h)[x, h_1] + \Pi(h_1)[x, h] + [x, h]\Pi(h_1) + [x, h_1]\Pi(h) = 0$. Again, taking x by s, and h_1 by s^2 in the previous equation, where $s \in S$, we obtain $\Pi(s^2)[s, h] + [s, h]\Pi(s^2) = 0$. Applying Lemma 6(ii) in the last relation, we see that $\Pi(s^2)[s, x] + [s, x]\Pi(s^2) = 0$ for all $x \in A$ and $s \in S$—that is,

$$\Pi(s^2)sx - \Pi(s^2)xs + sx\Pi(s^2) - xs\Pi(s^2) = 0 \tag{19}$$

for all $s \in S$ and $x \in A$. Taking h by s^2 in (18), we see that $s^2 - \lambda\Pi(s^2) \in Z$; so, $[s^2 - \lambda\Pi(s^2), s] = 0$—that is, $\lambda[\Pi(s^2), s] = 0$. Since $\lambda \neq 0$, we obtain $[\Pi(s^2), s] = 0$, and hence, $\Pi(s^2)s = s\Pi(s^2)$; then, by using the previous expression in (19), we have $\Pi(s^2)sx - \Pi(s^2)xs + sx\Pi(s^2) - x\Pi(s^2)s = 0$. Replacing x by xy in the last equation, right multiplying it by y, and then subtracting them, where $y \in A$, we see that $-\Pi(s^2)x[y, s] + sx[y, \Pi(s^2)] - x[y, \Pi(s^2)s] = 0$. Again, replacing x by rx in the last relation, left multiplying it by r, and then subtracting them, where $r \in A$, we find that $[-\Pi(s^2), r]x[y, s] + [s, r]x[y, \Pi(s^2)] = 0$. This implies that $[\Pi(s^2), r]x[s, y] - [s, r]x[\Pi(s^2), y] = 0$. Applying Lemma 8 in the previous expression, we infer that $[\Pi(s^2), r] = \lambda_0[s, r]$ or $[s, r] = 0$. If $[s, r] = 0$, then $S \subseteq Z$; by Lemma 2, we obtain that A satisfies s_4. Now, if $[s, r] \neq 0$, then $[\Pi(s^2), r] = \lambda_0[s, r]$ for all $s \in S$ and $r \in A$. Putting s by $-s$ in the last relation and using it, we obtain $2\lambda_0[s, r] = 0$, and so, $\lambda_0[s, r] = 0$; since $[s, r] \neq 0$, we obtain $\lambda_0 = 0$, and so, $[\Pi(s^2), r] = 0$. Hence, $\Pi(s^2) \in Z$ for all $s \in S$. Taking h by s^2 in (18) and applying the previous expression, we have $s^2 \in Z$ for all $s \in S$. Now, the same as in Subcase (2) in (14), we obtain that A satisfies s_4. □

Corollary 1 ([17], Theorem 3.6)**.** *Let A be a non-commutative prime ring, char$(A) \neq 2$ with involution "$*$", and let Π be a CE-Jordan derivation of A. Suppose that $[\Pi(x), x] \in Z$ for all $x \in A$. Then, $\Pi = 0$ or $\dim_C AC \leq 4$.*

Corollary 2 ([17], Theorem 3.7)**.** *Let A be a non-commutative prime ring, char$(A) \neq 2$ with involution "$*$", and let Π be a CE-Jordan derivation of A. Suppose that $[\Pi(x), x^*] \in Z$ for all $x \in A$. Then, $\Pi = 0$ or $\dim_C AC \leq 4$.*

Theorem 2. *Let A be a non-commutative prime ring, char$(A) \neq 2, 3$ with involution "$*$", and let Π be a non-zero CE-Jordan derivation of A. Suppose that $[\Pi(s), s] \in Z$ for all $s \in S$. Then, A satisfies s_4.*

Proof. Let $Z = (0)$, the same as in Theorem 1. Now, suppose that $Z \neq (0)$. Assume that

$$[\Pi(s), s] \in Z \tag{20}$$

for all $s \in S$. By linearizing (20), we have $[\Pi(s), s_1] + [\Pi(s_1), s] \in Z$ for all $s, s_1 \in S$. Putting s_1 by s^3 in the last relation, we obtain $[\Pi(s), s^3] + [\Pi(s^3), s] \in Z$. Using (20) in the previous expression, we obtain $3[\Pi(s), s]s^2 + [\Pi(s^3), s] \in Z$—that is, $6[\Pi(s), s]s^2 + 2[\Pi(s^3), s] \in Z$. Hence,

$$6[\Pi(s), s]s^2 + [\Pi(s^2 \circ s), s] \in Z \tag{21}$$

for all $s \in S$. Again, applying (20) in (21), we see that $6[\Pi(s), s]s^2 + 6[\Pi(s), s]s^2 \in Z$. Thus, $12[\Pi(s), s]s^2 \in Z$, and so, $[\Pi(s), s]s^2 \in Z$. Using (20) in the last relation, we find that $[\Pi(s), s] = 0$ or $s^2 \in Z$. Suppose that $s^2 \in Z$; the same as in the proof of Theorem 2, we obtain that A satisfies s_4. Now, suppose that

$$[\Pi(s), s] = 0 \tag{22}$$

for all $s \in S$. By linearizing (20), we see that

$$[\Pi(s), s_1] + [\Pi(s_1), s] = 0 \tag{23}$$

for all $s, s_1 \in S$. Taking s_1 by $s \circ h$ in (23), where $h \in H$, we find that

$$[\Pi(s), sh + hs] + [\Pi(s) \circ h + s \circ \Pi(h), s] = 0.$$

Applying (22) in the above equation, we infer that

$$s[\Pi(s), h] + [\Pi(s), h]s + \Pi(s)[h, s] + [h, s]\Pi(s) + s[\Pi(h), s] + [\Pi(h), s]s = 0.$$

Putting h by $s_1 \circ s$ in the above relation, where $s_1 \in S$, we conclude that

$$\begin{aligned} &s[\Pi(s), s_1 s + s s_1] + [\Pi(s), s_1 s + s s_1]s + \Pi(s)[s_1 s + s s_1, s] + [s_1 s + s s_1, s]\Pi(s) \\ &+ s[\Pi(s_1)s + s\Pi(s_1) + s_1\Pi(s) + \Pi(s)s_1, s] \\ &+ [\Pi(s_1)s + s\Pi(s_1) + s_1\Pi(s) + \Pi(s)s_1, s]s = 0 \end{aligned}$$

for all $s, s_1 \in S$. Using (22) in the above expression, we arrive at

$$\begin{aligned} &s[\Pi(s), s_1]s + s^2[\Pi(s), s_1] + [\Pi(s), s_1]s^2 + s[\Pi(s), s_1]s + \Pi(s)[s_1, s]s + \Pi(s)s[s_1, s] \\ &+ [s_1, s]s\Pi(s) + s[s_1, s]\Pi(s) + s[\Pi(s_1), s]s + s^2[\Pi(s_1), s] + s[s_1, s]\Pi(s) + s\Pi(s)[s_1, s] \\ &+ [\Pi(s_1), s]s^2 + s[\Pi(s_1), s]s + [s_1, s]\Pi(s)s + \Pi(s)[s_1, s]s = 0 \end{aligned}$$

for all $s, s_1 \in S$. Applying (23) in the above relation, we have

$$\begin{aligned} &\Pi(s)[s_1, s]s + \Pi(s)s[s_1, s] + [s_1, s]s\Pi(s) + s[s_1, s]\Pi(s) \\ &+ s[s_1, s]\Pi(s) + s\Pi(s)[s_1, s] + [s_1, s]\Pi(s)s + \Pi(s)[s_1, s]s = 0 \end{aligned}$$

for all $s, s_1 \in S$. Using (22) in the above equation, we obtain

$$2\Pi(s)[s_1, s]s + 2\Pi(s)s[s_1, s] + 2[s_1, s]s\Pi(s) + 2s[s_1, s]\Pi(s) = 0.$$

That is, $\Pi(s)[s_1, s]s + \Pi(s)s[s_1, s] + [s_1, s]s\Pi(s) + s[s_1, s]\Pi(s) = 0$. Applying Lemma 6(i) in the previous expression, we obtain $\Pi(s)[x, s]s + \Pi(s)s[x, s] + [x, s]s\Pi(s) + s[x, s]\Pi(s) = 0$ for all $s \in S$ and $x \in A$. It follows that

$$\Pi(s)xs^2 - \Pi(s)sxs + \Pi(s)sxs - \Pi(s)s^2x + xs^2\Pi(s) - sxs\Pi(s) + sxs\Pi(s) - s^2x\Pi(s) = 0$$

for all $s \in S$ and $x \in A$. This implies that $\Pi(s)xs^2 - \Pi(s)s^2x + xs^2\Pi(s) - s^2x\Pi(s) = 0$ for all $s \in S$ and $x \in A$. Replacing x by xy in the last relation, right multiplying it by y, and then subtracting them, where $y \in A$, we find that $\Pi(s)x[y, s^2] + x[y, s^2\Pi(s)] - s^2x[y, \Pi(s)] = 0$ for all $s \in S$ and $x, y \in A$. Again, replacing x by rx in the previous equation, left multiplying it by r, and then subtracting them, where $r \in A$, we conclude that $[\Pi(s), r]x[y, s^2] + [r, s^2]x[y, \Pi(s)] = 0$ for all $s \in S$ and $x, y, r \in A$—that is, $[\Pi(s), r]x[s^2, y] - [s^2, r]x[\Pi(s), y] = 0$ for all $s \in S$ and $x, y, r \in A$. Using Lemma 7 in the last relation, we arrive at $[\Pi(s), r] = \lambda[s^2, r]$ for all $s \in S$, $r \in A$, and some $\lambda \in C$ or $[s^2, r] = 0$ for all $s \in S$ and $r \in A$. If $[s^2, r] = 0$, then $s^2 \in Z$; so, the same as in the proof of Theorem 2, we obtain A satisfies s_4. Now, suppose that $[\Pi(s), r] = \lambda[s^2, r]$ for all $s \in S$, $r \in A$, and some $\lambda \in C$. Taking s by $-s$ in the last expression and applying it, and since Π is additive, we obtain $[\Pi(s), r] = 0$ for all $s \in S$, $r \in A$; so,

$$\Pi(s) \in Z \tag{24}$$

for all $s \in S$. Putting s by $s \circ h$ in (24) and using it, where $h \in H$, we obtain $2\Pi(s)h + s\Pi(h) + \Pi(h)s \in Z$. Applying (24) in the previous relation, we see that $[s\Pi(h) + \Pi(h)s, h] = 0$ for all $s \in S$ and $h \in H$. By using Lemma 6(i) in the last expression, we find that $[x\Pi(h) + \Pi(h)x, h] = 0$ for all $x \in A$ and $h \in H$. Taking x by $0 \neq z \in Z$ in the previous equation, we infer that $2z[\Pi(h), h] = 0$ for all $h \in H$; so, $[\Pi(h), h] = 0$ for all $h \in H$ and, by Theorem 2, we obtain that A satisfies s_4. $\square$

In 1998, T. Lee ([34], Theorem 1) proved the following result: Let A be a prime ring with involution "$*$" and an additive map $f : S \to A$ such that $[f(s), s] \in Z$ for all $s \in S$. Then, there exist $\lambda \in C$ and an additive map $\mu : S \to C$, such that $f(s) = \lambda s + \mu(s)$ for all $s \in S$, $\dim_C AC = 4$ or 16. Now, from Theorem 2 and Theorem 1 of [34], we have the following result.

Corollary 3. *Let A be a non-commutative prime ring, $char(A) \neq 2, 3$ with involution "$*$", and let Π be a non-zero CE-Jordan derivation of A. Suppose that $[\Pi(s), s] \in Z$ for all $s \in S$. Then, $dim_C AC = 4$.*

4. Results on Centrally Extended Jordan $*$-Derivations

Let A be a ring with involution "$*$". Recently, Bhushan et al. [17] introduced the notion of CE-Jordan $*$-derivation: a self-mapping Π of A is called a CE-Jordan $*$-derivation if $\Pi(a + b) - \Pi(a) - \Pi(b) \in Z$ and $\Pi(a \circ b) - \Pi(a)b^* - \Pi(b)a^* - a\Pi(b) - b\Pi(a) \in Z$ for all $a, b \in A$. They established the following result: if A is a non-commutative prime ring, char$(A) \neq 2$ with involution "$*$", and Π is a CE-Jordan $*$-derivation such that $[\Pi(a), a] \in Z$ (resp., $[\Pi(a), a^*] \in Z$) for all $a \in A$, then $\Pi = 0$ or $\dim_C AC \leq 4$. They also proved that a CE-Jordan $*$-derivation Π of a prime ring is additive. Now, we will prove the following result on CE-Jordan $*$-derivation.

Theorem 3. *Let A be a non-commutative prime ring, $char(A) \neq 2$ with involution "$*$", and let Π be a non-zero CE-Jordan $*$-derivation of A. Suppose that $[\Pi(h), h] \in Z$ for all $h \in H$. Then, $\Pi(x) = \lambda(x - x^*)$ for all $x \in A$ and some $\lambda \in C$, or A satisfies s_4.*

Proof. Assume that $[\Pi(h), h] \in Z$ for all $h \in H$. If $Z = (0)$, then from the definition of Π, we have that Π is a Jordan $*$-derivation and, by ([14], Theorem 1.2), we obtain that Π is X-inner—that is, $\Pi(x) = xq - qx^*$ for all $x \in A$ and some Q_{ms} in the case where A satisfies s_4, as desired. Now, suppose that A does not satisfy s_4. We will prove that $q \in C$. Applying our hypothesis in the last relation, we obtain $[hq - qh^*, h] = 0$ for all $h \in H$. Hence, $[hq - qh, h] = 0$ for all $h \in H$. Using Lemma 6(ii) in the previous equation, we see that $[xq - qx, x] = 0$ for all $x \in A$. This implies that $x[q, x] - [q, x]x = 0$ for all $x \in A$. Note that $\delta(x) = [q, x]$ is a derivation; so, $x\delta(x) - \delta(x)x = 0$ for all $x \in A$—that is, $[\delta(x), x] = 0$ for all $x \in A$. In particular, $[\delta(x), x] = 0$ for all $x \in H$. By applying Lemma 5 in the last expression, we find that $\delta = 0$, for all $x \in A$, and so $[q, x] = 0$ for all $x \in A$; hence, $q \in C$, as desired. Thus, from now on, we will assume that $Z \neq (0)$.

Since $h^* = h$ for all $h \in H$, we obtain two cases as in the proof of Theorem 1:

Case (I): Suppose that $[\Pi(h), x] = 0$ for all $h \in H$ and $x \in A$. From (8), we obtain

$$\Pi(h) = 0 \tag{25}$$

for all $h \in H$. Putting h by s^2 in (25), where $s \in S$, we find that $[\Pi(s), s] + z(s^2) = 0$, and so,

$$[\Pi(s), s] \in Z \tag{26}$$

for all $s \in S$. Replacing s by $s + s^3$ in (26) and using it, we obtain $[\Pi(s), s^3] + [\Pi(s^3), s] \in Z$—that is, $[\Pi(s), s]s^2 + s[\Pi(s), s^2] + [\Pi(s^3), s] \in Z$. Applying (26) in the last relation, we obtain $3[\Pi(s), s]s^2 + [\Pi(s^3), s] \in Z$. This implies that $6[\Pi(s), s]s^2 + [\Pi(s^2 \circ s), s] \in Z$. Hence, $6[\Pi(s), s]s^2 + [-\Pi(s^2)s + \Pi(s)s^2 + s\Pi(s^2) + s^2\Pi(s), s] \in Z$. Putting $h = s^2$ in (25) and using it in the last expression, we see that $6[\Pi(s), s]s^2 + [\Pi(s)s^2 + s^2\Pi(s), s] \in Z$. It follows that $6[\Pi(s), s]s^2 + [\Pi(s), s]s^2 + s^2[\Pi(s), s] \in Z$. Applying (26) in the previous relation, we find that $8[\Pi(s), s]s^2 \in Z$—that is, $[\Pi(s), s]s^2 \in Z$. Thus, $[\Pi(s), s] = 0$ or $s^2 \in Z$. Suppose that $s^2 \in Z$; the same as in the proof of Theorem 2, we obtain that A satisfies s_4. Now, suppose that

$$[\Pi(s), s] = 0 \tag{27}$$

for all $s \in S$. Taking s by $s + s \circ h$ in (27) and using it, where $h \in H$, we have $[\Pi(s), s \circ h] + [\Pi(s \circ h), s] = 0$. Applying (25) in the previous equation, we obtain $[\Pi(s), s \circ h] + [\Pi(s) \circ h, s] = 0$. Using (27) in the last relation, we obtain $s \circ [\Pi(s), h] + \Pi(s) \circ [h, s] = 0$. Putting h by s_1^2 in the previous expression, where $s_1 \in S$, we see that $s \circ [\Pi(s), s_1^2] + \Pi(s) \circ [s_1^2, s] = 0$. Again, putting s_1 by $s_1 + s_2$ in the last relation and applying it, where $s_2 \in S$, we infer that $s \circ [\Pi(s), s_1 \circ s_2] + \Pi(s) \circ [s_1 \circ s_2, s] = 0$. Using Lemma 6(i) in the previous equation, we find that $s \circ [\Pi(s), x \circ y] + \Pi(s) \circ [x \circ y, s] = 0$ for all $s \in S$ and $x, y \in A$. Taking y by $0 \neq z \in Z$ in the last relation, we conclude that $s \circ [\Pi(s), x] + \Pi(s) \circ [x, s] = 0$ for all $s \in S$ and $x \in A$—that is,

$$s\Pi(s)x - 2sx\Pi(s) + 2\Pi(s)xs - x\Pi(s)s - \Pi(s)sx + xs\Pi(s) = 0$$

for all $s \in S$ and $x \in A$. Applying (27) in the above expression, we arrive at $\Pi(s)xs - sx\Pi(s) = 0$ for all $s \in S$ and $x \in A$. Using Lemma 7, we have $\Pi(s) = \lambda_0 s$ for all $s \in S$ and some $\lambda_0 \in C$ or $s = 0$ for all $s \in S$. If $s = 0$ for all $s \in S$, then $S = (0)$ and, by Lemma 2, we obtain that A satisfies s_4. Now, suppose that

$$\Pi(s) = \lambda_0 s \tag{28}$$

for all $s \in S$. Since $x + x^* \in H$ and $x - x^* \in S$ for all $x \in A$, we obtain $2\Pi(x) = \Pi((x + x^*) + (x - x^*)) = \Pi(x + x^*) + \Pi(x - x^*)$ and, by applying (25) and (28), we see that $2\Pi(x) = \Pi((x + x^*) + (x - x^*)) = 0 + \lambda_0(x - x^*)$—that is, $2\Pi(x) = \lambda_0(x - x^*)$. Taking $\lambda = \frac{\lambda_0}{2} \in C$ in the last relation, we find that $\Pi(x) = \lambda(x - x^*)$ for all $x \in A$ and some $\lambda \in C$, as desired.

Case (II): The same as in Case (II) of Theorem 1. $\square$

Corollary 4 ([17], Theorem 4.6)**.** *Let A be a non-commutative prime ring, char$(A) \neq 2$ with involution "$*$", and let Π be a CE-Jordan $*$-derivation of A. Suppose that $[\Pi(x), x] \in Z$ for all $x \in A$. Then, $\Pi = 0$ or $\dim_C AC \leq 4$.*

Proof. Assume that

$$[\Pi(x), x] \in Z \tag{29}$$

for all $x \in A$. Thus, $[\Pi(h), h] \in Z$ for all $h \in H$ and, by Theorem 3, we obtain that A satisfies s_4, $\Pi(x) = \lambda(x - x^*)$ for all $x \in A$, or $\Pi = 0$. If A satisfies s_4 and by Lemma 1, we obtain $\dim_C AC \leq 4$ in case $\Pi = 0$, as desired. Now, consider the case where

$$\Pi(x) = \lambda(x - x^*) \tag{30}$$

for all $x \in A$. In this case, we will prove that it is equivalent to $\dim_C AC \leq 4$, under the assumption of (29). Using (30) in (29), we see that $\lambda[x^*, x] \in Z$ for all $x \in A$, and so, $\lambda = 0$ or $[x^*, x] \in Z$. If $\lambda = 0$, then $\Pi = 0$, as desired. Suppose that $\lambda \neq 0$; hence, $[x^*, x] \in Z$ for all $x \in A$. Applying [35] (Proposition 3.1) in the previous relation, we see that $[x^*, x] = 0$ for all $x \in A$. Using [35], (Theorem 3.2) in the last equation, there exists $\lambda_0 \in C$ and an additive map $\mu : A \to C$, such that $x^* = \lambda_0 x + \mu(x)$ for all $x \in A$—that is,

$$x^* - \lambda_0 x \in C \tag{31}$$

for all $x \in A$. Putting x by $x \circ h$ in (31) and applying it, where $h \in H$, we obtain $2h(x^* - \lambda_0 x) \in C$; so, $h(x^* - \lambda_0 x) \in C$ and, by using (31) in the last expression, we obtain $h \in C$ or $x^* - \lambda_0 x = 0$. Suppose that $h \in C$ and, by Lemma 3, we find that A satisfies s_4, as desired. If $x^* - \lambda_0 x = 0$, then $x^* = \lambda_0 x$ for all $x \in A$. Applying the previous equation in (30), we have $\Pi(x) = \lambda(1 - \lambda_0)x$ for all $x \in A$. Note that if $\lambda_0 = 1$, then $\Pi = 0$, as desired. Suppose that $1 - \lambda_0 \neq 0$. We put $\lambda_1 = \lambda(1 - \lambda_0) \neq 0$; so,

$$\Pi(x) = \lambda_1 x \tag{32}$$

for all $x \in A$. From the definition of Π, we have $\Pi(x^2) - \Pi(x)x^* - x\Pi(x) \in Z$ for all $x \in A$, and by using (32) in the previous relation, we obtain $\lambda_1 x^2 - \lambda_1 x x^* - \lambda_1 x^2 \in Z$ for all $x \in A$; so, $\lambda_1 x x^* \in Z$ for all $x \in A$ and, since $\lambda_1 \neq 0$, we obtain $xx^* \in Z$ for all $x \in A$. In particular, $h^2 \in Z$ for all $h \in H$, by Lemma 3 we infer that A satisfies s_4, and by Lemma 1 we obtain $\dim_C AC \leq 4$. $\square$

Corollary 5 ([17], Theorem 4.7)**.** *Let A be a non-commutative prime ring, char$(A) \neq 2$ with involution "$*$", and let Π be a CE-Jordan $*$-derivation of A. Suppose that $[\Pi(x), x^*] \in Z$ for all $x \in A$. Then, $\Pi = 0$ or $\dim_C AC \leq 4$.*

Theorem 4. *Let A be a non-commutative prime ring, char$(A) \neq 2, 3$ with involution "$*$", and let Π be a non-zero CE-Jordan $*$-derivation of A. Suppose that $[\Pi(s), s] \in Z$ for all $s \in S$. Then, $\Pi(x) = \lambda(x - x^*)$ for all $x \in A$ and some $\lambda \in C$, or A satisfies s_4.*

Proof. Let $Z = (0)$, the same as in Theorem 3. Now, suppose that $Z \neq (0)$. Assume that

$$[\Pi(s), s] \in Z \tag{33}$$

for all $s \in S$. Now, the same as in Theorem 2 in (21)—that is,

$$6[\Pi(s), s]s^2 + [\Pi(s^2 \circ s), s] \in Z \tag{34}$$

for all $s \in S$. By applying (33) and definition of Π in (34), we have $8[\Pi(s), s]s^2 \in Z$ for all $s \in S$. This implies that $[\Pi(s), s]s^2 \in Z$ for all $s \in S$. Hence, $[\Pi(s), s] = 0$ for all $s \in S$ or $s^2 \in Z$ for all $s \in S$. Suppose that $s^2 \in Z$ for all $s \in S$; the same as in Theorem 1, we obtain that A satisfies s_4. Now, suppose that

$$[\Pi(s), s] = 0 \tag{35}$$

for all $s \in S$. By linearizing (35), we see that

$$[\Pi(s), s_1] + [\Pi(s_1), s] = 0 \tag{36}$$

for all $s, s_1 \in S$. Taking s_1 by $s \circ h$ in (36), where $h \in H$, we find that

$$[\Pi(s), s \circ h] + [\Pi(s)h - \Pi(h)s + s\Pi(h) + h\Pi(s), s] = 0.$$

Using (35) in the above expression, we infer that

$$s[\Pi(s), h] + [\Pi(s), h]s + \Pi(s)[h, s] + [h, s]\Pi(s) + s[\Pi(h), s] - [\Pi(h), s]s = 0.$$

Putting h by $s_1 \circ s$ in the last equation, where $s_1 \in S$, we conclude that

$$\begin{aligned} &s[\Pi(s), s_1 s + s s_1] + [\Pi(s), s_1 s + s s_1]s + \Pi(s)[s_1 s + s s_1, s] \\ &+ [s_1 s + s s_1, s]\Pi(s) + s[\Pi(s_1 s + s s_1), s] - [\Pi(s_1 s + s s_1), s]s = 0 \end{aligned}$$

for all $s, s_1 \in S$—that is,

$$\begin{aligned} &s[\Pi(s), s_1]s + s^2[\Pi(s), s_1] + [\Pi(s), s_1]s^2 + s[\Pi(s), s_1]s + \Pi(s)[s_1, s]s + \Pi(s)s[s_1, s] \\ &+[s_1, s]s\Pi(s) + s[s_1, s]\Pi(s) + s[-\Pi(s_1)s - \Pi(s)s_1 + s\Pi(s_1) + s_1\Pi(s), s] \\ &-[-\Pi(s_1)s - \Pi(s)s_1 + s\Pi(s_1) + s_1\Pi(s), s]s = 0 \end{aligned}$$

for all $s, s_1 \in S$. Hence,

$$\begin{aligned} &s[\Pi(s), s_1]s + s^2[\Pi(s), s_1] + [\Pi(s), s_1]s^2 + s[\Pi(s), s_1]s + \Pi(s)[s_1, s]s + \Pi(s)s[s_1, s] \\ &+[s_1, s]s\Pi(s) + s[s_1, s]\Pi(s) - s[\Pi(s_1), s]s - s\Pi(s)[s_1, s] + s^2[\Pi(s_1), s] + s[s_1, s]\Pi(s) \\ &-[\Pi(s_1), s]s^2 - \Pi(s)[s_1, s]s + s[\Pi(s_1), s]s + [s_1, s]\Pi(s)s = 0 \end{aligned}$$

for all $s, s_1 \in S$. Applying (36) and (35) in the above relation, we obtain

$$s[\Pi(s), s_1]s + [\Pi(s), s_1]s^2 + 2[s_1, s]\Pi(s)s + 2s[s_1, s]\Pi(s) - s[\Pi(s_1), s]s - [\Pi(s_1), s]s^2 = 0.$$

Again, using (36) in the last equation, we obtain

$$2s[\Pi(s), s_1]s + 2[\Pi(s), s_1]s^2 + 2[s_1, s]s\Pi(s) + 2s[s_1, s]\Pi(s) = 0.$$

That is,

$$s[\Pi(s), s_1]s + [\Pi(s), s_1]s^2 + [s_1, s]s\Pi(s) + s[s_1, s]\Pi(s) = 0.$$

Applying Lemma 6(i) in the last expression, we see that

$$s[\Pi(s), x]s + [\Pi(s), x]s^2 + [x, s]s\Pi(s) + s[x, s]\Pi(s) = 0$$

for all $s, s_1 \in S$ and $x \in A$. By using (35) in the above relation, we find that

$$s\Pi(s)xs - sx\Pi(s)s + \Pi(s)xs^2 - s^2x\Pi(s) = 0 \tag{37}$$

for all $s \in S$ and $x \in A$. Replacing s by $s + s_1$ in (37) and applying it, replacing s_1 by $-s_1$, and then subtracting them, where $s_1 \in S$, we obtain

$$\begin{aligned} &s\Pi(s)xs_1 + s_1\Pi(s)xs + s\Pi(s_1)xs - s_1x\Pi(s)s + \Pi(s_1)xs^2 - sx\Pi(s_1)s \\ &-sx\Pi(s)s_1 + \Pi(s)xs_1s + \Pi(s)xss_1 - s^2x\Pi(s_1) - s_1sx\Pi(s) - ss_1x\Pi(s) = 0 \end{aligned}$$

for all $s, s_1 \in S$ and $x \in A$. Taking x by xs in the last equation, right multiplying it by s, and then subtracting them, we arrive at

$$\begin{aligned} &s\Pi(s)x[s, s_1] - sx[s, \Pi(s_1)]s - sx\Pi(s)[s, s_1] \\ &+\Pi(s)x[s, s_1]s + \Pi(s)xs[s, s_1] - s^2x[s, \Pi(s_1)] = 0 \end{aligned}$$

for all $s, s_1 \in S$ and $x \in A$—that is,

$$\begin{aligned} &\{s\Pi(s)xs - sx\Pi(s)s + \Pi(s)xs^2\}s_1 - s\Pi(s)xs_1s + sx\Pi(s)s_1s - \Pi(s)xs_1s^2 \\ &-sx[s, \Pi(s_1)]s - s^2x[s, \Pi(s_1)] = 0 \end{aligned}$$

for all $s, s_1 \in S$ and $x \in A$. By using (37) in the previous expression, we see that

$$s^2x\Pi(s)s_1 - s\Pi(s)xs_1s + sx\Pi(s)s_1s - \Pi(s)xs_1s^2 - sx[s, \Pi(s_1)]s - s^2x[s, \Pi(s_1)] = 0 \tag{38}$$

for all $s, s_1 \in S$ and $x \in A$. Putting x by xs_1 in (37), we obtain $s\Pi(s)xs_1s - sxs_1\Pi(s)s + \Pi(s)xs_1s^2 - s^2xs_1\Pi(s) = 0$; so, $-sxs_1\Pi(s)s - s^2xs_1\Pi(s) = -s\Pi(s)xs_1s - \Pi(s)xs_1s^2$. Applying the last relation in (38), we infer that

$$s^2x\Pi(s)s_1 - sxs_1\Pi(s)s - s^2xs_1\Pi(s) + sx\Pi(s)s_1s - sx[s,\Pi(s_1)]s - s^2x[s,\Pi(s_1)] = 0.$$

That is,

$$s^2x[\Pi(s),s_1] + sx[\Pi(s),s_1]s - sx[s,\Pi(s_1)]s - s^2x[s,\Pi(s_1)] = 0.$$

Using (36) in the previous equation, we obtain

$$s^2x[\Pi(s),s_1] + sx[\Pi(s),s_1]s + sx[\Pi(s),s_1]s + s^2x[\Pi(s),s_1] = 0.$$

Hence, $2s^2x[\Pi(s),s_1] + 2sx[\Pi(s),s_1]s = 0$; so, $s^2x[\Pi(s),s_1] + sx[\Pi(s),s_1]s = 0$ for all $s, s_1 \in S$ and $x \in A$. Applying Lemma 6(i) in the last expression, we obtain

$$s^2x[\Pi(s),y] + sx[\Pi(s),y]s = 0 \tag{39}$$

for all $s \in S$ and $x, y \in A$. By using Lemma 8 in (39), we conclude that $s = 0$ for all $s \in S$ or $[\Pi(s),y] = 0$ for all $s \in S$ and $y \in A$, unless $\lambda_1(s)s + \lambda_2(s)s^2 = 0$ for all $s \in S$ and some $\lambda_1(s), \lambda_2(s) \in C$. If $s = 0$ for all $s \in S$, then A satisfies s_4. Now, we have the following:

Case (I): If $[\Pi(s),y] = 0$ for all $s \in S$ and $y \in A$, then

$$\Pi(s) \in Z \tag{40}$$

for all $s \in S$. Taking s by $s \circ h$ in (40) and applying it, where $h \in H$, we have $2\Pi(s)h + [s,\Pi(h)] \in Z$. Again, by using (40) in the previous relation, we obtain $[[s,\Pi(h)],h] = 0$ for all $s \in S$ and $h \in H$. Applying Lemma 6(i) in the last equation, we see that $[[x,\Pi(h)],h] = 0$ for all $x \in A$ and $h \in H$. Putting x by $\Pi(h)x$ in the last relation and using it, we find that $[\Pi(h),h][x,\Pi(h)] = 0$. Again, putting x by xh in the previous expression and applying it, we infer that $[\Pi(h),h]x[h,\Pi(h)] = 0$. Hence, $[\Pi(h),h] = 0$ for all $h \in H$ and, by Theorem 3, we obtain $\Pi(x) = \lambda(x - x^*)$ for all $x \in A$ and some $\lambda \in C$, or A satisfies s_4.

Case (II): Assume that $\lambda_1(s)s + \lambda_2(s)s^2 = 0$ for all $s \in S$ and some $\lambda_1(s), \lambda_2(s) \in C$.

First: Suppose that " $*$ " is the first kind. Now, the same as in (15) and the "First" of Theorem 1, we obtain that A satisfies s_4.

Second: Suppose that " $*$ " is the second kind. Let $0 \neq s' \in S \cap Z$. Replacing s_1 by s' in (36), we find that $[\Pi(s'),s] = 0$ for all $s \in S$. Using Lemma 6(i) in the previous relation, we see that $[\Pi(s'),x] = 0$ for all $x \in A$—that is, $\Pi(s') \in Z$. Taking s by $s + s'$ in (39) and applying the last equation, we have $(s+s')^2x[\Pi(s),y] + (s+s')x[\Pi(s),y](s+s') = 0$, and by using (39) in the last expression, we obtain

$$2ss'x[\Pi(s),y] + s'^2x[\Pi(s),y] + sx[\Pi(s),y]s' + s'x[\Pi(s),y]s + s'x[\Pi(s),y]s' = 0.$$

That is, $3sx[\Pi(s),y] + 2x[\Pi(s),y]s' + x[\Pi(s),y]s = 0$. Putting x by rx in the previous relation, left multiplying it by r, and then subtracting them, where $r \in A$, we have $3[s,r]x[\Pi(s),y] = 0$; so, $[s,r]x[\Pi(s),y] = 0$, and hence, $[s,r] = 0$ for all $s \in S$ and $r \in A$, or $[\Pi(s),y] = 0$ for all $s \in S$ and $y \in A$. Suppose that $[\Pi(s),y] = 0$ for all $s \in S$ and $y \in A$, the same as in Case (I). Now, if $[s,r] = 0$ for all $s \in S$ and $r \in A$, then $s \in Z$ for all $s \in S$ and, by Lemma 2, we obtain that A satisfies s_4. □

The same as in Corollary 3, we have the following result.

Corollary 6. *Let A be a non-commutative prime ring, char$(A) \neq 2,3$ with involution " $*$ ", and let Π be a non-zero CE-Jordan $*$-derivation of A. Suppose that $[\Pi(s),s] \in Z$ for all $s \in S$. Then, $\Pi(x) = \lambda(x - x^*)$ for all $x \in A$ and some $\lambda \in C$, or $\dim_C AC = 4$.*

We will now give an example to verify the necessity of the various conditions stipulated in the hypothesis of Theorems 1 and 3.

Example 1. *Let $A_1 = M_n(\mathbb{F})$ be a ring over a field $\mathbb{F}$ with involution $*_1$, $n \in \mathbb{Z}^+$ such that $n > 4$, let*

$$A_2 = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} : a, b, w \in \mathbb{Z} \right\} \quad \textit{with center} \quad Z(A_2) = \left\{ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : b \in \mathbb{Z} \right\},$$

and let $A = A_1 \times A_2$ be a ring with center $Z = Z(A_1) \times Z(A_2)$. Define $\Pi : A \to A$ by $\Pi(X,Y) = (0, Y')$ for all $(X,Y) \in A$, where

$$Y = \begin{pmatrix} 0 & a & b \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad Y' = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, Π is a CE-Jordan derivation (moreover, it is a CE-Jordan $*$-derivation) of A and an involution is given $* : A \to A$ by $(X,Y)^* = (X^{*_1}, Y^{*_2})$ for all $(X,Y) \in A$, where $Y^{*_2} = \begin{pmatrix} 0 & w & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{pmatrix}$, but A is non-commutative, it is not prime, and $[\Pi(N), N] \in Z$ for all $N \in H$. Moreover, A does not satisfy s_4 because $n > 4$ (see Lemma 1).

5. Future Research

Future studies could examine our results by using generalized CE-Jordan ($*$)-derivations in place of the CE-Jordan ($*$)-derivations that we used; further, they could substitute semiprime rings for prime rings in our results. What can be said about the structures of $A, S, H, \Pi, *$, and $\mathrm{char}(A)$ then?

6. Conclusions

Unlike the results in [17], the assumptions in this article do not need to be fulfilled for every $x \in A$ in the identities $[\Pi(x), x] \in Z$ or $[\Pi(x), x^*] \in Z$; it is sufficient for every x to be in a subset of A as $x \in H$ or $x \in S$. Therefore, our results are more general than [17]. Recall that every Jordan derivation (resp., $*$-derivation) is a CE-Jordan derivation (resp., $*$-derivation), and every derivation is a Jordan derivation; so, our results are more general than those of [31].

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