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Existence and Uniqueness of Solution to a Terminal Value Problem of First-Order Differential Equation

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Abstract: The terminal value problem of differential equations has an important application background. In this paper, we are concerned with the terminal value problem of a first-order differential equation. Some sufficient conditions are given to obtain the existence and uniqueness results of solutions to the problem. Firstly, some comparison lemmas are established; secondly, an iterative technique and fixed point method are used to set up the main results; Finally, an example is provided to illustrate the application of the main results.

Keywords: terminal value problem; existence; uniqueness; comparison lemma; solution

MSC: 34C99; 34A40; 34A45



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1. Introduction

The terminal value problem (also called the final value problem, initial inverse problem, backward in time problem, abbrev. **TVP**) is an exciting topic within differential equations. It has important applications in many fields, such as aerospace science, mathematical economics, optimal control, and differential games, etc. For example, in aerospace science, the question of how to design the flight path of a spacecraft given its landing site on a planet can be reduced to the terminal value problem of a differential equation.

With the development of nonlinear functional analysis, scholars have made significant progress with the use of the fixed point theory method in the study of the terminal value problem of differential equations. For example, Wang [1] transformed the terminal value problem of fractional differential equations into initial value problems based on the shooting method, and then used the theoretical results of the initial value problem of fractional differential equations in solving the terminal value problem. Finally, the effectiveness of this method to solve the final value problem of fractional differential equations was verified by numerical simulation. Zhang [2] used Monch's fixed point theorem to study the terminal value problem of first-order differential equations in Banach space, obtained a new existence theorem under looser conditions, and improved and generalized some known results. Wang [3] studied the existence and uniqueness of the solution to the terminal value problem of first-order differential equations with discontinuous terms in Banach space by using semi-order theory and the mixed monotone iteration technique, without involving compact conditions, and presented an error estimate of the iterative sequence of approximations to the solution. Zhou [4] used the new comparison results and semi-order theory to study the existence of the minimum and maximum solutions of the terminal value problem of first-order nonlinear differential equations in Banach space, and improved and generalized some known results. In [5], combining the generalized quasi-linearization technique with the upper and lower solutions method, Yakar and Arslan obtained a unique solution to the fractional causal terminal value problem. In [6], Shah and Rehman established a sufficient condition for the existence and uniqueness of the solution of a class of fractional differential equations over infinite intervals. In [7], the authors

discussed the terminal value differential inequality, the existence of extreme value solutions of differential equations, and the corresponding comparison principle. In [8], Benchohra et al. presented the existence results and uniqueness of solutions for a class of boundary value problems of the terminal type for fractional differential equations with the Hilfer–Katugampola fractional derivative. The reasoning was mainly based upon different types of classical fixed point theorems, such as the Banach contraction principle and Krasnoselskii’s fixed point theorem. In [9], Li et al. were concerned with the well-posedness and efficient numerical algorithm for a terminal value problem with a generalized Caputo fractional derivative. They investigated the existence and uniqueness of the solution of the terminal value problem and considered the continuous dependence of the solutions on the given data. In [10], Babak and Wu tempered fractional differential equations with terminal value problems. Discretized collocation methods on piecewise polynomial spaces were proposed for solving these equations. Regularity results were constructed on weighted spaces, and convergence order was studied.

The above results are mainly based on the properties of compact operators or increasing operators.

In this paper, we are concerned with the following **TVP**,

$$u'(t) = f(t, u(t)) \quad t \in [0, T], \quad u(T) = u_T,$$

where $T > 0, u_T \in R$ are two constants and $f : [0, T] \times R \rightarrow R$ is continuous.

By the properties of decreasing operators, we obtain the existence and uniqueness of the solution to this problem. Our contributions are the following:

- (1) we present some comparison lemmas for (**TVP**);
- (2) we establish the existence and uniqueness results of solutions for (**TVP**);
- (3) we set up an iterative scheme of approximation solutions for (**TVP**).

The paper is organized as follows. In Section 2, some comparison lemmas are established; the existence and uniqueness results of (**TVP**) are presented in Section 3 via the iterative technique and fixed point method; an example shown in Section 4 illustrates the application of the results obtained.

2. Comparison Lemmas

The following comparison lemmas are of importance throughout this paper.

Lemma 1. *If $v \in C^1[0, T]$ satisfies*

$$v'(t) + \lambda v(t) \geq 0 \quad v(T) \leq 0 \quad t \in [0, T]$$

where $\lambda \in R$ is a constant, then $v(t) \leq 0$ for $t \in [0, T]$.

Proof. Since $v'(t) + \lambda v(t) \geq 0$, we have

$$e^{\lambda t}(v'(t) + \lambda v(t)) \geq 0$$

that is,

$$(v(t)e^{\lambda t})' \geq 0$$

which implies that $v(t)e^{\lambda t}$ is increasing on $[0, T]$. Hence, for $\forall t \in [0, T]$,

$$v(t)e^{\lambda t} \leq v(T)e^{\lambda T} \leq 0$$

i.e., $v(t) \leq 0, t \in [0, T]$. \square

Lemma 2. *Let $v, w \in C^1[0, T]$, and $\lambda \in R$ be a constant. If*

$$w'(t) + \lambda w(t) \leq v'(t) + \lambda v(t) \quad v(T) \leq u_T \leq w(T) \quad t \in [0, T],$$

then $v(t) \leq w(t)$ for $t \in [0, T]$.

Proof. Let $h(t) = v(t) - w(t)$, then we have

$$h'(t) + \lambda h(t) \geq 0 \quad h(T) \leq 0 \quad t \in [0, T].$$

By Lemma 1, we know $h(t) \leq 0, t \in [0, T]$, i.e., $v(t) \leq w(t)$ for $t \in [0, T]$. \square

Lemma 3. Let $w \in C^1[0, T], h \in C[0, 1]$, and $\lambda \in R$ be a constant. If

$$w'(t) + \lambda w(t) \leq h(t) \quad w(T) \geq u_T \quad t \in [0, T],$$

then

$$w(t) \geq u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} h(s) ds$$

for $t \in [0, T]$.

Proof. If $v \in C^1[0, 1]$ is a solution to the following terminal value problem

$$v'(t) + \lambda v(t) = h(t) \quad v(T) = u_T \quad t \in [0, T],$$

then we have

1.

$$v(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(T-t)} h(s) ds$$

2.

$$w'(t) + \lambda w(t) \leq v'(t) + \lambda v(t), \quad v(T) = u_T \leq w(T) \quad t \in [0, T].$$

By Lemma 2, we obtain

$$w(t) \geq v(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(T-t)} h(s) ds.$$

\square

3. Main Results

In this section, we give some sufficient conditions to ensure the existence and uniqueness of the (TVP).

Firstly, we transform the (TVP) to a fixed point problem; secondly, we construct an iterative sequence by the integral operator; finally, by using comparison lemmas, we verify that the sequence is uniformly convergent to the unique solution of the (TVP).

Let $u, v \in C[0, T]$; if $u(t) \leq v(t)$ for $\forall t \in [0, T]$, we denote $u \leq v$. The order interval $[u, v] = \{x \in C[0, T] | u(t) \leq x(t) \leq v(t), \forall t \in [0, T]\}$.

The main result of this paper is the following.

Theorem 1. Let us say that there exist $v, w \in C^1[0, T], v \leq w$ and a constant λ such that

1. for $\forall t \in [0, T], x, y \in [v, w], x \leq y$,

$$f(t, y(t)) - f(t, x(t)) \geq -\lambda(y(t) - x(t))$$

2. for $\forall t \in [0, T], 0 \leq \ell \leq 1$ and $x, y \in [v, w]$,

$$f(t, \ell x(t) + (1 - \ell)y(t)) \geq \ell f(t, x(t)) + (1 - \ell)f(t, y(t))$$

3. for $\forall t \in [0, T]$,

$$\begin{aligned} (v + w)'(t) + \lambda(v - w)(t) &\geq 2f(t, w(t)) \\ f(t, v(t)) &\geq w'(t) + \lambda(w - v)(t) \end{aligned} \tag{1}$$

4. $v(T) = u_T = w(T)$.

Then, (TVP) has a unique solution \tilde{x} satisfying $v(t) \leq \tilde{x}(t) \leq w(t), t \in [0, T]$ (abbr. $\tilde{x} \in [v, w]$).

Proof. Let $x \in C[0, 1]$. If $h \in C^1[0, T]$ be a solution to the following terminal value problem:

$$h'(t) + \lambda h(t) = f(t, x(t)) + \lambda x(t), h(T) = u_T$$

Then,

$$h(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(T-t)} [f(s, x(s)) + \lambda x(s)] ds$$

Define a mapping T on $C[0, T]$ as follows:

$$(Tx)(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, x(s)) + \lambda x(s)] ds, x \in C[0, 1].$$

It is easy to verify that T maps $C[0, T]$ into $C[0, T]$, and (TVP) has a solution if and only if T has a fixed point in $C[0, T]$.

By Assumptions (1) and (2), we know that T is decreasing and convex on $[v, w]$.

By the first inequality in (1), we have

$$\begin{cases} \left(\frac{v+w}{2}\right)'(t) + \lambda\left(\frac{v+w}{2}\right) \geq f(t, w(t)) + \lambda w(t), \\ \left(\frac{v+w}{2}\right)(T) = u_T. \end{cases}$$

Due to the second inequality in (1), we obtain

$$\begin{cases} w'(t) + \lambda w(t) \leq f(t, v(t)) + \lambda v(t), \\ w(T) = u_T. \end{cases}$$

Let $x_0(t)$ be a solution to the following terminal value problem:

$$\begin{cases} u'(t) + \lambda u(t) = f(t, w(t)) + \lambda w(t), \\ u(T) = u_T, \end{cases}$$

Then,

$$x_0(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, w(s)) + \lambda w(s)] ds.$$

Construct an iterative sequence $\{x_n(t)\}$ as follows:

$$\begin{cases} x'_{n+1}(t) + \lambda x_{n+1}(t) = f(t, x_n(t)) + \lambda x_n(t) \\ x_{n+1}(T) = u_T \end{cases} \quad n = 0, 1, 2, \dots$$

i.e.,

$$x_{n+1}(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, x_n(s)) + \lambda x_n(s)] ds.$$

In what follows, we prove that $\{x_n\}$ is a Cauchy sequence in $C[0, T]$, and converges to the solution of the (TVP) in $C[0, T]$.

Step 1. We assert

$$w(t) \geq x_0(t) \geq \left(\frac{w+v}{2}\right)(t) \geq v(t) \quad t \in [0, T].$$

In virtue of

$$\left(\frac{w+v}{2}\right)'(t) + \lambda\left(\frac{v+w}{2}\right)(t) \geq x_0'(t) + \lambda x_0(t)$$

and

$$\left(\frac{u+v}{2}\right)(T) = u_T = x_0(T)$$

and Lemma 2, we obtain

$$x_0(t) \geq \left(\frac{w+v}{2}\right)(t).$$

Moreover,

$$\begin{aligned} w'(t) + \lambda w(t) &\leq f(t, v(t)) + \lambda v(t) \\ &\leq f(t, w(t)) + \lambda w(t) \\ &= x_0'(t) + \lambda x_0(t), \\ w(T) &= u_T = x_0(T). \end{aligned}$$

Hence, by Lemma 2, we have $x_0(t) \leq w(t)$, and

$$w \geq x_0 \geq \frac{w+v}{2} \geq v.$$

Step 2. For $n = 0, 1, 2, \dots$,

$$w(t) \geq x_{2n+1}(t) \geq x_{2n}(t) \geq \left(\frac{w+v}{2}\right)(t) \geq v(t).$$

On the one hand, since

$$\begin{aligned} x_1(t) &= u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, x_0(s)) + \lambda x_0(s)] ds \\ &= (Tx_0)(t) \end{aligned}$$

and T is decreasing, we obtain

$$Tw \leq Tx_0 = x_1 \leq T\left(\frac{w+v}{2}\right) \leq Tv.$$

Noting that $x_0 = Tw$, we have

$$x_0 \leq x_1.$$

On the other hand, by the second inequality in (1), we have

$$w'(t) + \lambda w(t) \leq f(t, v(t)) + \lambda v(t) \leq x_1'(t) + \lambda x_1(t) = f(t_1, x_0(t)) + \lambda x_0(t)$$

which means

$$x_1'(t) + \lambda x_1(t) \geq w'(t) + \lambda w(t)$$

By the comparison Lemma 2, there holds $x_1(t) \leq w(t)$. Hence,

$$x_0 \leq x_1 \leq w.$$

Noting that $x_0 \geq \frac{w+v}{2}$, we have $w \geq x_1 \geq x_0 \geq \frac{w+v}{2} \geq v$, which implies that the assertion holds for $n = 0$.

Suppose that when $n = k$,

$$w(t) \geq x_{2k+1}(t) \geq x_{2k}(t) \geq \left(\frac{v+w}{2}\right)(t) \geq v(t)$$

then

$$\begin{aligned} f(t, w(t)) + \lambda w(t) &\geq f(t, x_{2k+1}(t)) + \lambda x_{2k+1}(t) \\ &\geq f(t, x_{2k}(t)) + \lambda x_{2k}(t) \\ &\geq f(t, v(t)) + \lambda v(t). \end{aligned}$$

Hence, we have

$$\begin{aligned} \left(\frac{v+w}{2}\right)'(t) + \lambda \left(\frac{v+w}{2}\right)(t) &\geq x'_{2k+2}(t) + \lambda x_{2k+2}(t) \\ &\geq x'_{2k+1}(t) + \lambda x_{2k+1}(t) \\ &\geq w'(t) + \lambda w(t). \end{aligned}$$

By the comparison Lemma 2,

$$\frac{v+w}{2} \leq x_{2k+2}(t) \leq x_{2k+1}(t) \leq w(t).$$

Repeating this process, we can verify

$$\frac{v+w}{2} \leq x_{2k+2}(t) \leq x_{2k+3}(t) \leq w(t),$$

which means the assertion holds for $n = k + 1$.

Hence, for all n , there holds

$$w(t) \geq x_{2n+1}(t) \geq x_{2n}(t) \geq \left(\frac{v+w}{2}\right)(t) \geq v(t)$$

Step 3. $\{x_{2n}(t)\}$ is increasing, while $\{x_{2n+1}(t)\}$ is decreasing.
Since

$$u(t) \leq \left(\frac{v+w}{2}\right)(t) \leq x_1(t) \leq w(t),$$

then

$$\begin{aligned} f(t, v(t)) + \lambda v(t) &\leq f(t, x_1(t)) + \lambda x_1(t) \\ &\leq f(t, w(t)) + \lambda w(t) \end{aligned}$$

and

$$w'(t) + \lambda w(t) \leq x'_2(t) + \lambda x_2(t) \leq x'_0(t) + \lambda x_0(t)$$

By the comparison Lemma 2,

$$x_0(t) \leq x_2(t) \leq w(t)$$

In a similar way to Step 2, we can prove

$$\{x_{2n}(t)\} \text{ is increasing, } \{x_{2n+1}(t)\} \text{ is decreasing.}$$

Step 4. $\{x_n(t)\}$ is uniformly convergent on $[0, 1]$.

By Steps 1–3, we know that $\{x_n(t)\}$ satisfies

$$v(t) \leq \left(\frac{v+w}{2}\right)(t) \leq x_0(t) \leq x_2(t) \leq \dots \leq x_{2n}(t) \leq \dots \leq x_{2n+1}(t) \leq \dots \leq x_1(t) \leq w(t)$$

Let $Z_n(t) = x_n(t) - v(t)$. We have

$$0 \leq \left(\frac{w-v}{2}\right)(t) \leq Z_0(t) \leq Z_2t \leq \dots \leq Z_{2n}(t) \leq \dots \leq Z_{2n+1}(t) \leq \dots \leq Z_1(t) \leq (w-v)(t).$$

Define

$$r_n = \sup\{r \in R \mid Z_{2n}(t) \geq rZ_{2n+1}(t)\}$$

Then, the sequence $\{r_n\}$ is well defined, $\frac{1}{2} \leq r_n \leq 1$, and $\{r_n\}$ is increasing.

Since

1.

$$Z_{2n}(t) \geq \frac{1}{2}(w-v)(t) \geq \frac{1}{2}Z_{2n+1}(t)$$

we have $r_n \geq \frac{1}{2}$

2.

$$Z_{2n}(t) \geq Z_{2n+1}(t)$$

and we obtain $r_n \leq 1$.

3. If r satisfies $Z_{2n}(t) \geq rZ_{2n+1}(t)$, then the monotonicity of $\{Z_{2n}\}$ and $\{Z_{2n+1}\}$ implies

$$Z_{2n+2}(t) \geq Z_{2n}(t) \geq rZ_{2n+1}(t) \geq rZ_{2n+3}(t)$$

i.e., $\{r \in R \mid Z_{2n}(t) \geq rZ_{2n+1}(t)\} \subset \{r \in R \mid Z_{2n+2}(t) \geq rZ_{2n+3}(t)\}$.

By (1-3), we know that $\{r_n\}$ is convergent. Denote $r_0 = \lim_{n \rightarrow \infty} r_n$.

By the comparison Lemma 3,

$$\begin{aligned} Z_{2n+3}(t) &\leq Z_{2n+1}(t) = x_{2n+1}(t) - v(t) \\ &= u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, x_{2n}(s)) + \lambda x_{2n}(s)] ds - v(t) \\ &= u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, Z_{2n}(s) + v(s)) + \lambda(Z_{2n} + v(s))] ds - v(t) \\ &\leq u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, (r_n Z_{2n+1} + v)(s)) + \lambda(r_n Z_{2n+1} + v)(s)] ds - v(t) \\ &= u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} \left[\begin{aligned} &f(s, (r_n x_{2n+1} + (1-r_n)v(s))) + \\ &\lambda(r_n x_{2n+1} + (1-r_n)v(s)) \end{aligned} \right] ds - v(t) \\ &\leq u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} \left[\begin{aligned} &r_n(f(s, x_{2n+1}(s)) + \lambda x_{2n+1}(s)) + \\ &(1-r_n)(f(s, v(s)) + \lambda v(s)) \end{aligned} \right] ds - v(t) \\ &= r_n \left[u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, x_{2n+1}(s)) + \lambda x_{2n+1}(s)] ds - v(t) \right] \\ &\quad + (1-r_n) \left[u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, v(s)) + \lambda v(s)] ds - v(t) \right] \\ &\leq r_n [x_{2n+2}(t) - v(t)] + (1-r_n)[w(t) - v(t)] \\ &= r_n Z_{2n+2} + 2(1-r_n) \left(\frac{w-v}{2}\right)(t) \\ &\leq r_n Z_{2n+2} + 2(1-r_n) Z_{2n+2} \\ &= (2-r_n) Z_{2n+2}, \end{aligned}$$

then

$$r_{n+1} = \sup\{r \in R \mid Z_{2n+3}(t) \geq rZ_{2n+2}(t)\} \geq \frac{1}{2-r_n}.$$

Taking the limit on both sides, we obtain

$$r_0 \geq \frac{1}{2 - r_0}.$$

Noting that $\frac{1}{2} \leq r_0 \leq 1$, we know $r_0 = 1$.
Then, for an even number p ,

$$0 \leq Z_{2n+p} - Z_{2n} \leq Z_{2n+1} - Z_{2n} \leq (1 - r_n)Z_{2n+1} \leq (1 - r_n)(v - u).$$

Since $r_n \rightarrow 1$, $\{Z_{2n}\}$ is convergent. In a similar way, we obtain that $\{Z_{2n+1}\}$ is convergent, and

$$\lim_{n \rightarrow \infty} Z_{2n} = \lim_{n \rightarrow \infty} Z_{2n+1}.$$

Hence, $\{Z_n\}$ is convergent.

Let $Z = \lim_{n \rightarrow \infty} Z_n$ and $\bar{x} = Z + v$, and then

$$\begin{aligned} \lim_{n \rightarrow \infty} x_n(t) &= \bar{x}(t) \\ &= \lim_{n \rightarrow \infty} x_{n+1}(t) \\ &= \lim_{n \rightarrow \infty} \left[u_T e^{\lambda(T-t)} - \int_t^T [f(s, x_n(s)) + \lambda x_n(s)] ds \right] \\ &= u_T e^{\lambda(T-t)} - \int_t^T [f(s, \bar{x}(s)) + \lambda \bar{x}(s)] ds \\ &= (T\bar{x})(t), \end{aligned}$$

which means that $\bar{x}(t)$ is a fixed point of T .

Step 5. \bar{x} is the unique fixed point of T in $[v, w]$.

In fact, if \tilde{x} is a fixed point of T in $[v, w]$, then

$$\begin{aligned} v \leq \tilde{x} \leq w &\Rightarrow Tv \geq T\tilde{x} \geq Tw \\ &\Rightarrow w \geq Tv \geq \tilde{x} \geq x_0. \end{aligned}$$

Continuing this process, we have

$$x_{2n}(t) \leq \tilde{x}(t) \leq x_{2n+1}(t).$$

Taking the limit on both sides, we obtain

$$\tilde{x}(t) = \bar{x}(t).$$

Hence, \bar{x} is the unique fixed point of T in $[v, w]$, i.e., \bar{x} is the unique solution of (TVP) in $[v, w]$. \square

Remark 1. Let

$$y_0(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, v(s)) + \lambda v(s)] ds$$

Define

$$y_{n+1}(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, y_n(s)) + \lambda y_n(s)] ds, n = 0, 1, 2, \dots$$

In the same way as in Theorem 1, we can prove that $\{y_n\}$ is uniformly convergent to \bar{x} on $[0, 1]$.

Corollary 1. Assume that there exist two constants $c > 0$ and $\lambda \in R$ satisfying the following:

1. $f(t, 0) - \lambda c(T - t) \geq -c \geq 2f(t, c(T - t)) + \lambda c(T - t)$;
2. for $\forall t \in [0, T]$, $f(t, \cdot)$ is concave;
3. for $\forall t \in [0, T]$, $x, y \in [0, c(T - t)]$, $x \leq y$,

$$f(t, y) - f(t, x) \geq -\lambda(y - x),$$

and then (TVP)

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(T) = 0 \end{cases}$$

has a unique solution $\bar{x}(t)$ satisfying

$$0 \leq \bar{x}(t) \leq c(T - t), t \in [0, T].$$

Proof. Choose $v(t) = 0, w(t) = c(T - t)$, and we can verify that all conditions of Theorem 1 are fulfilled. \square

Consider the following terminal value problem.

$$\begin{cases} x'(t) = t + g(x(t)) t \in [0, 1] \\ x(1) = 0 \end{cases}$$

Corollary 2. Let $g \in C^2[0, 1]$. If the following conditions are satisfied

1. $g(0) \geq -2$;
2. for $\forall x \in [0, 1]$, $g(x) \leq \frac{3}{2}x - \frac{3}{2}$;
3. for $\forall x \in [0, 1]$, $g'(x) \geq 1$;
4. for $\forall x \in [0, 1]$, $g''(x) \leq 0$.

Then the above (TVP) has a unique solution $\bar{x}(t)$ satisfying

$$0 \leq \bar{x}(t) \leq 1 - t, t \in [0, 1].$$

Proof. Let

$$f(t, x) = t + g(x).$$

and $c = T = 1, \lambda = -1$; we can verify that

$$\begin{aligned} f(t, 0) - \lambda c(T - t) &\geq t + (-2) + 1 - t \\ &= -1 = -c \\ 2f(t, c(T - t)) + \lambda c(T - t) &= 2[t + g(1 - t)] - (1 - t) \\ &\leq 2[t + \frac{3}{2}(1 - t) - \frac{3}{2}] - (1 - t) \leq -1 = -c, \end{aligned}$$

which implies that assumption (1) of Corollary 1 is satisfied.

Moreover,

$$f''_{xx}(t, x) = g''(x) \leq 0$$

means that $f(t, \cdot)$ is concave, i.e., Assumption (2) of Corollary 1 is fulfilled. Noting that

$$f'_x(t, x) = g'(x) \geq 1, x \in [0, 1],$$

hence, f meets condition (3) of Corollary 1.

By Corollary 1, we know that this terminal problem has a unique solution $\bar{x}(t)$ satisfying

$$0 \leq \bar{x}(t) \leq 1 - t.$$

□

4. Application

Example 1. Let $g(x) = x + \sin x - 2$. We can verify that all assumptions of Corollary 2 hold. Hence, the terminal value problem

$$\begin{cases} x'(t) = t + x(t) + \sin x(t) - 2 \\ x(1) = 0 \end{cases}$$

has a unique solution $\bar{x}(t)$ satisfying

$$0 \leq \bar{x}(t) \leq 1 - t.$$

Let $T = 1, \lambda = -1, v_0(t) = 0$. Define

$$v_{n+1}(t) = u_T e^{\lambda(T-t)} - \int_t^T e^{\lambda(s-t)} [f(s, v_n(s)) + \lambda v_n(s)] ds, n = 0, 1, 2, \dots$$

Then, the approximate solutions of the above TVP are

$$\begin{aligned} v_1(t) &= (t - 2) [e^{(t-1)} - 1] \\ v_2(t) &= \left\{ t + \sin \left\{ (t - 2) [e^{(t-1)} - 1] \right\} - 2 \right\} \cdot [e^{(t-1)} - 1] \\ &\dots \end{aligned}$$

The image of the approximate solutions of v_1, v_2 is the Figure 1.

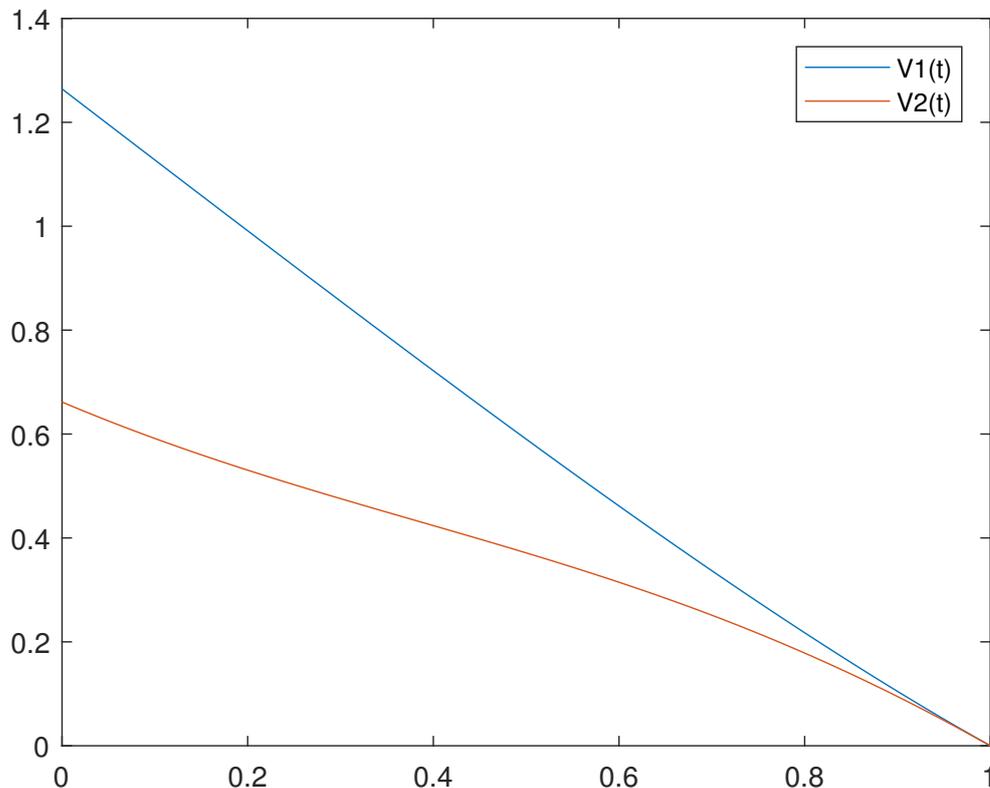


Figure 1. Image of the approximate solutions of v_1, v_2 .

5. Discussion

In this paper, we have constructed an iterative monotone sequence and verified that this sequence is convergent to a solution of problem (TVP). Other assumptions ensure the uniqueness of this solution. Our uniqueness result is a **local result**, which means that the problem may have multiple solutions in the space $C[0, T]$.

6. Conclusions

In this paper, we have used comparison lemmas, an iterative technique and a fixed point method to obtain the existence and uniqueness results of solutions for a terminal value problem of the first-order differential equation. Our discussion lies in a bounded interval. It is an interesting problem to extend the study to an unbounded interval, i.e.,

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(\infty) = \lim_{t \rightarrow \infty} x(t) = u_{\infty} \end{cases}$$

We will try to find appropriate conditions to ensure the existence and uniqueness of the solution to the above problem.

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