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The Independence Number Conditions for 2-Factors of a Claw-Free Graph

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Abstract: In 2014, some scholars showed that every 2-connected claw-free graph G with independence number $\alpha(G) \leq 3$ is Hamiltonian with one exception of family of graphs. If a nontrivial path contains only internal vertices of degree two and end vertices of degree not two, then we call it a *branch*. A set S of branches of a graph G is called a *branch cut* if we delete all edges and internal vertices of branches of S leading to more components than G. We use a *branch bond* to denote a minimal branch cut. If a branch-bond has an odd number of branches, then it is called *odd*. In this paper, we shall characterize all 2-connected claw-free graphs G such that every odd branch-bond of G has an edge branch and such that $\alpha(G) \leq 5$ but has no 2-factor. We also consider the same problem for those 2-edge-connected claw-free graphs with $\alpha(G) \leq 4$.

Keywords: line graph; essentially k-edge-connected; super-block; closure

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1. Introduction

For graph theory terms not covered in this article, readers can refer to [1]. We consider only simple graphs in this paper. Let $G = (V(G), E(G))$ be a graph having vertex set $V(G)$ and edge set $E(G)$. The girth (the circumference, respectively) of G, denoted by $g(G)$ ($c(G)$, respectively), is the length of a shortest (longest, respectively) cycle of G. A cycle of even length, which is of even order, is defined as an even cycle. For a vertex x of G, we denote the neighborhood (the degree, respectively) of x in G by $N_G(x)$ ($d_G(x)$, respectively). The neighbors of S in G is denoted by $N_G(S) = \{y : y \in N_G(x) \text{ and } x \in S\}$. For a positive integer l, we denote $V_l(G) = \{v \in V(G) | d_G(v) = l\}$ and let $V_{\geq l}(G) = \bigcup_{m \geq l} V_m(G)$. For a vertex $x \in V(G)$, we define the local completion of G at x as the graph G_x^* having $V(G_x^*) = V(G)$ and $E(G_x^*) = E(G) \cup \{uv | u, v \in N_G(x)\}$. We denote the *distance* in G of two vertices $x, y \in V(G)$ by $d_G(x, y)$. Denoted by $\alpha(G)$, $\alpha'(G)$ and $\kappa(G)$ are the independence number, the maximum matching number and the connectivity of a graph G, respectively. We denote the line graph of a graph H by $L(H)$. The vertex set of $L(H)$ is $E(H)$. Two vertices in $L(H)$ are adjacent if and only if the corresponding edges in H have at least one vertex in common.

A clique is a (not necessarily maximal) subgraph of a graph G in which any two vertices in it are adjacent. For an edge $e \in E(G)$, the largest order of a clique having e is denoted by $\omega_G(e)$. Let C_k be a cycle with even length $k \geq 4$. For two edges e_1, $e_2 \in E(G)$, if $d_{C_k}(e_1, e_2) = \frac{k}{2} - 1$, then we define them as antipodal in C_k. For any two antipodal edges e_1, $e_2 \in E(G)$, if $\min\{\omega_G(e_1), \omega_G(e_2)\} = 2$, then we define an even cycle C_k in a graph G as edge-antipodal, abbreviated EA. Analogously, for two vertices $x_1, x_2 \in V(C_k)$, if $d_{C_k}(x_1, x_2) = \frac{k}{2}$), then we define them as antipodal in C_k. For any two antipodal vertices $x_1, x_2 \in V(C_k)$, if $\min\{d_G(x_1), d_G(x_2)\} = 2$, then we define C_k as vertex-antipodal, abbreviated VA.

In 1972, Chvátal and Erdős gave the following well-known sufficient condition for a graph to be Hamiltonian.

Theorem 1 (Chvátal and Erdős, [2])**.** *If G is a graph on at least 3 vertices such that $\alpha(G) \leq \kappa(G)$, then G is Hamiltonian.*

If a graph is $K_{1,3}$-free, then we define it as claw-free. If a graph has a Hamilton cycle, the we define it as *Hamiltonian*. A *2-factor* of a graph G is a spanning subgraph of G where each vertex has the identical degree 2. Therefore, a Hamiltonian cycle equals a connected 2-factor.

Flandrin and Li considered the largest possible independence number of a claw-free graph G with 3-connected.

Theorem 2 (Flandrin and Li, [3])**.** *Every claw-free graph G with connectivity $\kappa(G) \geq 3$ and independence number $\alpha(G) \leq 2\kappa(G)$ is Hamiltonian.*

Xu et al. considered the independence number conditions for Hamiltonicity of 2-connected claw-free graphs.

Theorem 3 (Xu et al. [4])**.** *Let G be a claw-free graph with $\kappa(G) \geq 2$ and $\alpha(G) \leq 3$. Then, G is Hamiltonian with one exceptional family of graphs.*

For results related to Hamiltonicity of claw-free graphs, the reader may refer to the literature; see [5].

Ryjáček [6] proposed the line graph *closure* of a claw-free graph G. For a vertex $x \in V(G)$, if $G[N_G(x)]$ is a connected graph, then we define it as *locally connected* , if $G[N_G(x)]$ is a clique, then we define it as *simplicial*, and if x is locally connected and nonsimplicial, then we define it as *eligible*. We use $EL(G)$ ($SI(G)$, respectively) to denote the set of eligible (simplicial, respectively) vertices of a graph G. If there exists a sequence of graphs $G_1, \cdots, G_k$ satisfying

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in EL(G_i)$, $i = 1, \cdots, k-1$,
- $G_k = cl(G)$ and $EL(G_k) = \emptyset$,

then, we define graph $cl(G)$ as Ryjáček closure of a claw-free graph G. Ryjáček et al. [7] also came up with a new closure $cl^{2f}(G)$ which reinforce the closure $cl(G)$ of G keeping the (non)-existence of a 2-factor of a claw-free graphs. If the set of vertices satisfies

- $x \in EL(G)$ or,
- $x \notin EL(G)$ and x is in an induced cycle of length 4 or 5 or in an induced EA-cycle of length 6,

then it can be denoted by $EL^{2f}(G)$. If there exists a sequence of graphs $G_1, \cdots, G_k$ satisfying

- $G_1 = G$,
- $G_{i+1} = (G_i)^*_{x_i}$ for some $x_i \in EL^{2f}(G)$, $i = 1, \ldots, k-1$,
- $G_k = cl^{2f}(G)$ and $EL^{2f}(G_k) = \emptyset$,

then we call $cl^{2f}(G)$ as a 2-factor-closure of a claw-free graph G.

Theorem 4 (Ryjáček et al. [7])**.** *Let G be a claw-free graph. Then*

(i) *the closure $cl^{2f}(G)$ is uniquely determined,*
(ii) *there is a graph H satisfying*
  (a) $L(H) = cl^{2f}(G)$,
  (b) $g(H) \geq 6$,
  (c) *H does not have any vertex-antipodal cycle of length 6,*
(iii) *G has a 2-factor if and only if $cl^{2f}(G)$ has a 2-factor.*

For results related to the concept of closure of claw-free graph, the reader may refer to the literature; see [8].

If the degree of internal vertices in a nontrivial path is 2 and the degree of end vertices is not 2, then we define this nontrivial path as a *branch*. The *length* of a *branch* is the number of its edges. It is obvious that an edge branch has no internal vertex. A set $\mathcal{B}$ of branches of G is defined as a *branch cut* if the subgraph of G acquired from $G[E(G) \setminus \bigcup_{B\in\mathcal{B}} E(B)]$ by erasing all internal vertices in any branch of $\mathcal{B}$ contains more components than G. We define minimal branch cut as *branch-bond*. If branch-bond has an odd number of branches, then we define it as *odd*. For results related to the concept of branch-bonds, the reader may refer to the literature; see [9,10]. For results related to 2-factor of claw-free graph, see [11].

2. Results and Discussion

It is routine to verify that for a graph G to have a 2-factor, it is necessary that every odd branch-bond of G contains an edge branch. In this paper, we consider the problem of determining the largest possible independence number of a claw-free graph G with the above-mentioned necessary condition to have a 2-factor, as well as other related problems.

We can state our principal theorem after we define two auxiliary graphs. For $i \in \{6,7\}$, $C_{i,3}$ is obtained from a cycle $C = v_0v_1\cdots v_{i-1}v_0$ by adding a path $x_0x_1x_2x_3$ with two vertices v_0 and v_3. In the following, $C_{i,3}$ is depicted in Figure 1 for $i \in \{6,7\}$. Now we use the above two auxiliary graphs to define a family of graphs.

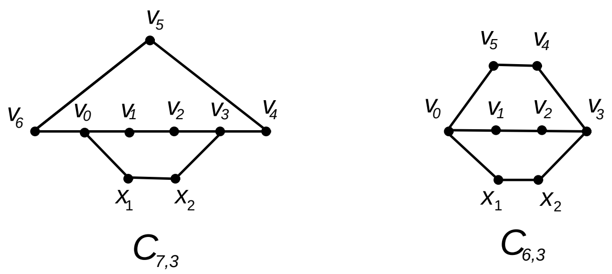

Figure 1. Two 2-connected graphs whose line graphs have no 2-factor.

Let $\mathcal{G}_0$ be the family of graphs obtained from the graphs $C_{7,3}$ and $C_{6,3}$ in the following way: either add some pendent edges (possibly zero) to exactly one vertex w(say) of degree three in $C_{7,3}$ and add exactly one pair of pendent edges to those two vertices in the branch of length four that have distances two and three from w in $C_{7,3}$, respectively, or add exactly one pair of pendent edges to exactly one pair of inner vertices in the same branch of length three in $C_{7,3}$ and $C_{6,3}$, respectively, and in $C_{6,3}$, add some pendent edges (possibly zero) to exactly one vertex of degree three.

Theorem 5. *Let G be a 2-connected claw-free graph with $\alpha(G) \leq 5$ such that every odd branch-bond of G has an edge branch. Then G has a 2-factor if and only if the closure $cl^{2f}(G)$ of G is not isomorphic to the line graph of a member of $\mathcal{G}_0$.*

As the matching number of any graph in $\mathcal{G}_0$ is at least 5, the following corollary follows immediately from Theorem 5.

Corollary 1. *Let G be a 2-connected claw-free graph with $\alpha(G) \leq 4$ such that every odd branch-bond of G has an edge branch. Then G has a 2-factor.*

In this paper, we also investigate the similar problem for 2-edge-connected graphs. We can state our principal theorem after we define some graphs. Let F_1 be the tree obtained from a claw $K_{1,3}$ by adding exactly two leaves to each vertex of $K_{1,3}$, respectively. For $i \in \{2,3,4\}$, let F_i be the tree acquired from a path P_i by adding exactly two leaves on each vertices of P_i, respectively. Let F_5 be the tree acquired from a path P_3 by adding exactly two leaves on each end vertices of P_3 and adding exactly one leaf on the other vertex of P_3, respectively. Let F_6 be the tree acquired from a P_3 by adding exactly two leaves on one end vertex of P_3 and by adding exactly one leaf on the other two vertices of P_3, respectively. For those F_i, see Figure 2.

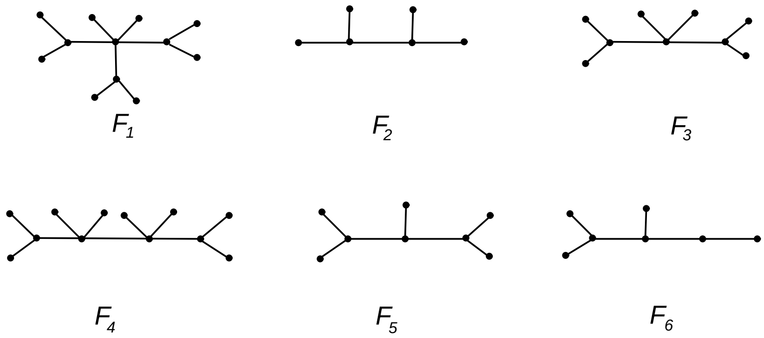

Figure 2. Six trees whose line graphs have no 2-factor.

We first define a family of $\mathcal{F}_0$. Let $\mathcal{F}_0$ be the family of graphs obtained from the graph F_i $(i \in \{2,3,5,6\})$ such that exactly one of the following holds:

(1) Add at least two pendent edges to either exactly one leaf in F_2 or exactly one leaf with a neighbor of degree 4 in F_3;
(2) Add at least two pendent edges to either any pair of leaves whose distance is maximum in F_2 or any pair of leaves whose distance is two in F_5 or a pair of vertices in which one of them has degree 2 and its neighbor is a leaf in F_6;
(3) Add at least three pendent edges to exactly one leaf with a neighbor of degree 3 in F_3.

Now, we may state our result.

Theorem 6. *Let G be a 2-edge-connected claw-free graph with $\alpha(G) \leq 4$ such that each odd branch-bond of G contains an edge branch. Then, G has a 2-factor if and only if the closure $cl^{2f}(G)$ of G is not isomorphic to the line graph of a member of $\mathcal{F}_0 \cup \{F_1, F_2, F_3, F_4\}$.*

Note that the size of any graph in $\{F_1, F_2, F_3, F_4\}$ is at most 11, therefore, we can immediately obtain the following result.

Corollary 2. *Let G be a 2-edge-connected claw-free graph of order $n \geq 12$ with $\alpha(G) \leq 4$ such that each odd branch-bond of G contains an edge branch. Then G has a 2-factor if and only if the closure $cl^{2f}(G)$ of G is not isomorphic to the line graph of a member of $\mathcal{F}_0$.*

3. Preliminaries and Basic Results

Let G be a graph and let X be a proper subset of $V(G)$. We say a subgraph obtained by deleting a set of vertices is an *induced subgraph*. If X is the set of vertices deleted, We use $G - X$ to denote the resulting graph. If S is the set of deleted edges, this subgraph of G is denoted $G \backslash S$. For $x \in V(G)$, we denote all the edges incident with x in G by $E(x)$. If we write $C = x_1 x_2 \cdots x_m x_1$, we assume that an orientation of C is given such that x_2 is the successor of x_1 and operations in the subscripts of x_i's will be taken modulo m in $\{1, 2, \ldots, m\}$.

If $G \backslash X$ contains at least two non-trivial components, then we call an edge cut X of G as essential. For an integer $k > 0$, if G does not contain an essential edge-cut X such that $|X| < k$, then we call G as *essentially k-edge-connected*. Note that a graph G is essentially k-edge-connected if and only if $L(G)$ is k-connected or complete.

We use G_0 to denote the *core* of a graph G which is acquired by deleting all the vertices of degree 1 in G. We define $\Lambda(G)$ to be the set of the vertices in G which are also vertices in G_0 and adjacent to a vertex of degree 1 in G.

The following notations are introduced in [12].

Let G be a 2-connected graph and let C be a cycle of G. Then any component D of $G - C$ contains at least two different neighbors on C. For any path P of D, if the end vertices (which may be identical) of P has two different neighbors on C, then P is called a *two-attaching path* of D. Furthermore, if D has a longest two-attaching path P of length k, then D is called a $(k+1)$-*component* of G. Let C be a cycle of G and let D be a component of $G-C$, we denote $P_C(D) = \{P : P \text{ is a two-attaching path of D}\}$. Moreover, let P be a two-attaching path of D, by $End(P)$ we denote the two endvertices of P and we define the following set

$$A_C(P,D) = \{\{u,v\} : \{u,v\} \subseteq N_G(End(P)) \cap V(C) \ \ and \ \ u \neq v, |(N_G(u) \cup N_G(v)) \cap End(P)| = 2\}.$$

Let G be a 2-connected graph and let C be a cycle of G with an orientation $\overrightarrow{C}$. Let D_1 and D_2 be two components of $G-C$, and let P, P' be two two-attaching paths of D_1 and D_2, respectively. Let $\{v_i, v_j\} \in A_C(P, D_1)$ and $\{v_k, v_l\} \in A_C(P', D_2)$, if v_i, v_k, v_j, v_l are four different vertices that lie along the direction of $\overrightarrow{C}$, then we say that D_1 overlaps D_2 on C.

Let G be essentially 2-edge-connected and let $B_1, B_2, \cdots, B_t$ be all the blocks of G_0. Let $H_1 = B_1 \cup \{e : e$ be a pendent edge of G and e has one end in $V(B_1) \cap \Lambda(G)\}$, $H_i = B_i \cup \{e : e$ be a pendent edge of G and e have one end in $(V(B_i) - \bigcup_{j=1}^{i-1} V(B_j)) \cap \Lambda(G)\}$ for $i \in \{2, \cdots, t\}$. H_i is called a *super-block* of G. Then, by the definition of super-block, for any pendant edges e of G, it holds that e is in exactly one super-block H_i.

In order to prove Theorem 5, we should introduce the following lemmas.

Lemma 1. *If each odd branch-bond of G contains an edge-branch, then each odd branch-bond of $cl^{2f}(G)$ contains an edge-branch.*

Proof of Lemma 1. Otherwise, there exists an odd branch-bond $\mathcal{B}$ of $cl^{2f}(G)$ in which each branch has length at least two. By the definition of $cl^{2f}(G)$, there exists a new edge $e = uv$ in some branch P of $\mathcal{B}$: $e \in E(cl^{2f}(G)) \setminus E(G)$. Note that $|V(P)| \geq 3$. Then, one of u, v is an inner vertex of P, say u. Thus, $d_{cl^{2f}(G)}(u) = 2$, contradicts the fact that e is in a clique of size at least 4 of $cl^{2f}(G)$. □

Lemma 2 (Xiong et al. [13])**.** *Let $P = u_1u_2\cdots u_s (s \geq 3)$ be a path of G and $e_i = u_iu_{i+1}$. Then $P \in \mathcal{B}(G)$ if and only if $P' = e_1e_2\cdots e_{s-1} \in \mathcal{B}(L(G))$.*

From Lemma 2, we deduce the following fact.

Lemma 3. *Each odd branch-bond of $L(G)$ contains an edge branch if and only if each odd branch-bond of G has a shortest branch of length at most 2.*

We call a connected nontrivial even graph a circuit, and the complete bipartite graph $K_{1,m}$ a star. In particular, we call $K_{1,3}$ claw. If F is a subgraph of graph H and each edge of H has at least one vertex in $V(F)$, then we call this phenomenon F dominates H. Let $\mathcal{D}$ be a set of edge-disjoint circuits and stars satisfying at least three edges in H. We say that $\mathcal{D}$ is a dominating system (abbreviated d-system) in H if each edge of H that is not in a star of $\mathcal{D}$ is dominated by a circuit in $\mathcal{D}$.

Lemma 4 (Gould et al. [14])**.** *Let H be a graph. Then, $L(H)$ contains a 2-factor with c components if and only if H contains a d-system with c elements.*

Lemma 5 (Wang et al. [12])**.** *Let G be a 2-connected graph with circumference $c(G)$ and let C be a longest cycle of G. For each k-component D of $G-C$, then $k \leq \lfloor \frac{c(G)}{2} \rfloor - 1$.*

Lemma 6 (Wang et al. [12])**.** *Let G be a 2-connected graph and let C be a longest cycle of G, and let D be a 2-component of $G-C$. Then D is a star.*

Lemma 7 ((Wang et al. [12]). *Let G be a 2-connected graph and let C be a longest cycle of G. If $|V(C)| \leq 7$, then two components of $G - C$ do not overlap on C.*

4. The Proof of Theorem 5

For proving Theorem 5, it suffices to show the following two theorems.

Theorem 7. *Let G be an essentially 2-edge-connected graph with $g(G) \geq 6$, $\alpha'(G) \leq 5$ such that each odd branch-bond of G has a shortest branch of length at most 2. If the core G_0 of G is 2-connected, then G has a d-system if and only if G is not a member of $\mathcal{G}_0$.*

Proof of Theorem 7. Note that every member of $\mathcal{G}_0$ has no d-system, the necessity of Theorem 7 clearly holds.

Suppose that G has no d-system, it suffices to show that $G \in \mathcal{G}_0$. Let $C = v_0 v_1 \cdots v_{c(G)-1} v_0$ be a longest cycle of G, where the subscripts are taken modulo $c(G)$ in the following. Then $c(G) \leq 11$, since otherwise $\alpha'(G) \geq 6$. Moreover, $E(G - C) \neq \emptyset$: Otherwise $G[E(C)]$ is a d-system that dominates all the edges of G. If $10 \leq c(G) \leq 11$, then $\alpha'(G) \geq \alpha'(G[E(C)]) + \alpha'(G[E(G - C)]) \geq 5 + 1 = 6$, a contradiction. Therefore, $6 \leq c(G) \leq 9$. Since $g(G) \geq 6$ and $c(G) \leq 9$, C is also an induced cycle of G.

Claim 1. *$G_0 - C$ has at least one s-component with $s \geq 2$.*

Proof. Suppose, by contradiction, that each component of $G_0 - C$ is a 1-component. Let $x_1, x_2, \ldots, x_t$ be all the components of $G_0 - C$ such that $|E(x_i)| \geq 3$. Then $G[E(C) \cup (\bigcup_{i=1}^{t} E(x_i))]$ is a d-system of G, a contradiction. □

By Claim 1, $G_0 - C$ has at least one s-component D(say) with $s \geq 2$. Let $P = x_1 x_2 \cdots x_s$ be a longest two-attaching path of D joining two different vertices v_{i^1} and v_{i^2} on C.

Claim 2. *For any 2-component D' of $G_0 - C$, it holds that D' is isomorphic to P_2. Moreover, $V(D') \subseteq V_2(G_0)$.*

Proof. By Lemma 6, D' is a star, denoted by $G[x; y_1, y_2, \cdots, y_t]$. Suppose that $t > 1$. Since G_0 is 2-connected and D is a star, $N_{G_0}(y_i) \cap V(C) \neq \emptyset$ for $i \in \{1, 2, \cdots, t\}$. By the definition of 2-component, $N_{G_0}(y_{i_0}) \cap V(C)$ and $N_{G_0}(y_{j_0}) \cap V(C)$ have the same vertex v_0(say) for any pair of $\{i_0, j_0\} \subseteq \{1, 2, \cdots, t\}$. Then, there will produce a cycle $x y_{i_0} v_0 y_{j_0} x$ of length 4, contradicting $g(G) \geq 6$. Therefore, $t = 1$. Moreover, $d_{G_0}(x) = d_{G_0}(y_1) = 2$: Otherwise, at least one of $\{x, y_1\}$ has two neighbors on C, then by $6 \leq |V(C)| \leq 9$, it will produce a cycle of length either at most 5 or at least 10, a contradiction. □

In the following, we need distinguish the following two cases.

Case 1. $8 \leq c(G) \leq 9$.

Note that $\alpha'(G) \leq 5$ and $\alpha'(C) \geq 4$, the following statement clearly holds.

Claim 3. *$\alpha'(G - C) \leq 1$.*

By Claim 3, D is the unique nontrivial component of G.

Claim 4. *If $|V(C)| = 9$, then $G - C$ has no P_3 that one of whose end-vertex is adjacent to C.*

Proof. Suppose, by contradiction, that $G - C$ has a path $y_1 y_2 y_3$ such that $y_3 v_j \in E(G)$ for some $j \in \{0, 1, 2, \ldots, 8\}$. Then $\{y_1 y_2, v_j y_3, v_{j+1} v_{j+2}, v_{j+3} v_{j+4}, v_{j+5} v_{j+6}, v_{j+7} v_{j+8}\}$ is a matching of G with size 6, contradicting $\alpha'(G) \leq 5$. □

Suppose that $c(G) = 9$. Recall that D is a s-component of $G_0 - C$ and by Lemma 5, $2 \leq s \leq 3$. Then by the definition of s-component and Claim 4, $s = 2$. Moreover,

$N_{G-C-D}(V(C)) = \emptyset$: Otherwise we assume that $z \in N_{G-C-D}(V(C))$, say $v_j z \in E(G)$ ($v_j \in V(C)$) for some $j \in \{0, 1, \cdots, 8\}$. Then, $\{v_j z, v_{j+1}v_{j+2}, v_{j+3}v_{j+4}, v_{j+5}v_{j+6}, v_{j+7}v_{j+8}, x_1x_2\}$ is a matching of G with size 6, contradicting $\alpha'(G) \leq 5$. Therefore, D is the only component of $G_0 - C$ and $V(C) \cap \Lambda(G) = \emptyset$. By Claim 2, D is the two-attaching path x_1x_2 joining two different vertices v_{i^1}, v_{i^2}, and $\{x_1, x_2\} \subseteq V_2(G_0)$. Again by Claim 4, $\{x_1, x_2\} \subseteq V_2(G)$. Since C is the longest cycle, $3 \leq d_C(v_{i^1}, v_{i^2}) \leq 4$. Then, without loss of generality, assume that $i^1 = i$, $i^2 \in \{i+3, i+4\}$ for some $i \in \{0, 1, 2, \ldots, 8\}$. Therefore, G has an odd branch-bond $\{v_i x_1 x_2 v_{i+3}, v_i v_{i+1} v_{i+2} v_{i+3}, v_i v_{i+8} v_{i+7} v_{i+6} v_{i+5} v_{i+4} v_{i+3}\}$ ($i^2 = i+3$) or $\{v_i x_1 x_2 v_{i+4}, v_i v_{i+1} v_{i+2} v_{i+3} v_{i+4}, v_i v_{i+8} v_{i+7} v_{i+6} v_{i+5} v_{i+4}\}$ ($i^2 = i+4$) with a shortest branch of length three, a contradiction.

In the following, we assume that $c(G) = 8$. Note that D is a s-component of $G_0 - C$ and by Lemma 5, $2 \leq s \leq 3$.

Claim 5. *If $s = 3$, then D is isomorphic to P_3. Consequently, $V(D) \subseteq V_2(G_0)$.*

Proof. Since D is 3-component of $G - C$, we let $x_1x_2x_3$ be a longest two-attaching path of D joining two different vertices v_{i^1} and v_{i^2} on C. Since C is the longest cycle, we have $d_C(v_{i^1}, v_{i^2}) = 4$. By $g(G) \geq 6$ and $c(G) = 8$, $N(x_1) \cap V(C) = \{v_{i^1}\}$ and $N(x_3) \cap V(C) = \{v_{i^2}\}$. Then, by Claim 3, $N_{D-x_2}(x_i) = \emptyset$ for $i \in \{1, 3\}$. Moreover, $N_{D-x_1-x_3}(x_2) = \emptyset$: Otherwise, we may assume that $N_{D-x_1-x_3}(x_2) = \{z'\}$. By the definition of 3-component, D has no cycle containing the vertices x_1, x_2, z' or x_2, x_3, z'. Then, by Fan Lemma, there exists a path Q of G_0 joining z' and C such that $\{x_1, x_2, x_3\} \cap V(Q) = \emptyset$. This will produce a cycle of length either at most 5 or at least 9, a contradiction. Therefore, D is isomorphic to P_3. Moreover, note that $D \cong P_3$ and $|V(C)| = 8$. Since C is the longest cycle and $g(G) \geq 6$, then $V(D) \subseteq V_2(G_0)$. □

Note that $\alpha'(D) \geq 1$. By Claim 3, $E(G - D - C) = \emptyset$. By Claims 2 and 5, D is the two-attaching path $x_1 \cdots x_s$ ($2 \leq s \leq 3$) joining two different vertices v_{i^1}, v_{i^2}. Since C is the longest cycle, $3 \leq d_C(v_{i^1}, v_{i^2}) \leq 4$.

Claim 6. *$G_0 - C$ has no component other than D.*

Proof. Suppose, by contradiction, that $G_0 - C$ has other component D'(say). Note that $E(G - D - C) = \emptyset$. Then, D' is a 1-component of $G_0 - C$, say y. Note that $|V(C)| = 8$. By $g(G) \geq 6$, we have $|N(y) \cap V(C)| = 2$, say $N(y) \cap V(C) = \{v_{j^1}, v_{j^2}\}$. Again, by $g(G) \geq 6$ and $|V(C)| = 8$, $d_C(v_{j^1}, v_{j^2}) = 4$.

Suppose, first, that $\{v_{i^1}, v_{i^2}\} \cap \{v_{j^1}, v_{j^2}\} = \emptyset$. Recall that $3 \leq d_C(v_{i^1}, v_{i^2}) \leq 4$ and $d_C(v_{j^1}, v_{j^2}) = 4$, so D overlaps D'. Without loss of generality, we assume that $v_{i^1}, v_{j^1}, v_{i^2}, v_{j^2}$ are four different vertices that lie along the direction of $\overrightarrow{C}$. This will produce a cycle of G of length of at least $c(G) + 1$, a contradiction.

Suppose, now, that $\{v_{i^1}, v_{i^2}\} \cap \{v_{j^1}, v_{j^2}\} \neq \emptyset$. Then, $|\{v_{i^1}, v_{i^2}\} \cap \{v_{j^1}, v_{j^2}\}| = 1$: Otherwise, $\{v_{i^1}, v_{i^2}\} = \{v_{j^1}, v_{j^2}\}$. Then, by $E(G - C - D) = \emptyset$, Claims 2 and 5, $G[E(C) \cup E(v_{i^1} P v_{i^2} y v_{i^1})]$ is a d-system of G, a contradiction. Therefore, without loss of generality, we assume $v_{i^2} = v_{j^1}$. By $|V(C)| = 8$ and $d_C(v_{j^1}, v_{j^2}) = 4$, we have $d_C(v_{i^1}, v_{i^2}) = 3$, and thus $d_C(v_{i^1}, v_{j^2}) = 1$. Then, by Claim 2, $E(G - C - D) = \emptyset$ and $d_C(v_{i^1}, v_{j^2}) = 1$, $G[E(v_{i^1} P v_{i^2}) \cup E(v_{i^2} y v_{j^2}) \cup (E(C) \setminus \{v_{i^1} v_{j^2}\})]$ is a d-system of G, a contradiction. □

By Claim 6, $V(G_0) = V(C) \cup V(D)$, $\{v_{i^1}, v_{i^2}\} \subseteq V_3(G_0)$ and $(V(C) \setminus \{v_{i^1}, v_{i^2}\}) \subseteq V_2(G_0)$. If $s = 3$, then, C is the longest cycle, $d_C(v_{i^1}, v_{i^2}) = 4$. Assume, without loss of generality, that $i^1 = i$, $i^2 = i + 4$ for some $i \in \{0, 1, 2, \ldots, 7\}$. By Claim 3, $\{x_1, x_3\} \cap \Lambda(G) = \emptyset$. Then $x_2 \notin \Lambda(G)$: Otherwise, by $\{x_1, x_3\} \cap \Lambda(G) = \emptyset$, then $G[E(C) \cup E(x_2)]$ is a d-system of G, a contradiction. Therefore, $\{x_1, x_2, x_3\} \cap \Lambda(G) = \emptyset$. By symmetry, $\{v_{i+1}, v_{i+2}, v_{i+3}\} \cap \Lambda(G) = \emptyset$ and $\{v_{i+5}, v_{i+6}, v_{i+7}\} \cap \Lambda(G) = \emptyset$. Then, $(\{x_1, x_2, x_3\} \cup (V(C) \setminus \{v_i, v_{i+4}\})) \subseteq V_2(G)$.

Therefore, G has an odd branch-bond $\{v_ix_1x_2x_3v_{i+4}, v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}, v_iv_{i+7}v_{i+6}v_{i+5}v_{i+4}\}$ with a shortest branch of length four, a contradiction.

In the following, we assume that $s = 2$. Since C is the longest cycle, $3 \leq d_C(v_{i^1}, v_{i^2}) \leq 4$. Then, without loss of generality, assume that $i^1 = i$, $i^2 \in \{i+3, i+4\}$ for some $i \in \{0, 1, 2, \ldots, 7\}$.

Claim 7. *For any edge $y_1y_2 \in E(G_0)$, it holds that $|\{y_1, y_2\} \cap \Lambda(G)| \leq 1$.*

Proof. By contradiction, suppose that $y_1, y_2 \in \Lambda(G)$. Let y_1z_1 and y_2z_2 be two pendant edges of G. By Claim 3, $\{y_1, y_2\} \nsubseteq V(D)$. If $\{y_1, y_2\} \subseteq V(C)$, then $\alpha'(G) \geq \alpha'(G[V(C) \cup \{z_1, z_2\}]) + \alpha'(G[V(D)]) \geq 5 + 1 = 6$, a contradiction. Hence, we have that $|\{y_1, y_2\} \cap V(D)| = 1$ and $|\{y_1, y_2\} \cap V(C)| = 1$. Without loss of generality, we assume $y_1 = v_i$ and $y_2 = x_1$. By Claim 3, $x_2 \notin \Lambda(G)$, then $G[E(C) \cup E(x_1)]$ is a d-system of G, a contradiction. □

Note that $N(D) \cap V(C) = \{v_{i^1}, v_{i^2}\}$, and $i^1 = i$, $i^2 \in \{i+3, i+4\}$ for some $i \in \{0, 1, 2, \ldots, 7\}$. In the following, we need distinguish the following two cases.

Case 1.1. $i^1 = i, i^2 = i+3$.

Then $\{v_{i+1}, v_{i+2}\} \cap \Lambda(G) = \emptyset$ and $\{x_1, x_2\} \cap \Lambda(G) = \emptyset$: Otherwise, $\{v_{i+1}, v_{i+2}\} \cap \Lambda(G) \neq \emptyset$ or $\{x_1, x_2\} \cap \Lambda(G) \neq \emptyset$. By Claim 7, $|\{v_{i+1}, v_{i+2}\} \cap \Lambda(G)| \leq 1$ or $|\{x_1, x_2\} \cap \Lambda(G)| \leq 1$. By Claim 2, we can find a d-system

$$\mathcal{D}_1 = \begin{cases} G[E(v_iPv_{i+3}\overrightarrow{C}v_i) \cup E(v_{i+1})], & \text{if } v_{i+1} \in \Lambda(G) \\ G[E(v_iPv_{i+3}\overrightarrow{C}v_i) \cup E(v_{i+2})], & \text{if } v_{i+2} \in \Lambda(G) \\ G[E(C) \cup E(x_1)], & \text{if } x_1 \in \Lambda(G) \\ G[E(C) \cup E(x_2)], & \text{if } x_2 \in \Lambda(G) \end{cases}$$

of G, a contradiction.

Again by Claim 7, $|\{v_{i+5}, v_{i+6}\} \cap \Lambda(G)| \leq 1$. Suppose, first, that $|\{v_{i+5}, v_{i+6}\} \cap \Lambda(G)| = 1$. Without loss of generality, we may suppose that $v_{i+5} \in \Lambda(G)$, then, by Claim 7, $\{v_{i+4}, v_{i+6}\} \cap \Lambda(G) = \emptyset$. Therefore, $v_{i+7} \notin \Lambda(G)$: Otherwise, by $\{v_{i+4}, v_{i+6}\} \cap \Lambda(G) = \emptyset$ and Claim 2, $G[E(v_iPv_{i+3}\overleftarrow{C}v_i) \cup E(v_{i+5}) \cup E(v_{i+7})]$ is a d-system of G, a contradiction. Then, $\{x_1, x_2, v_{i+1}, v_{i+2}, v_{i+6}, v_{i+7}\} \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+3}, v_iv_{i+1}v_{i+2}v_{i+3}, v_iv_{i+7}v_{i+6}v_{i+5}\}$ of G with a shortest branch of length three, a contradiction. Suppose, now, that $|\{v_{i+5}, v_{i+6}\} \cap \Lambda(G)| = 0$. If $\{v_{i+4}, v_{i+7}\} \cap \Lambda(G) = \emptyset$, then $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+3}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+3}, v_iv_{i+1}v_{i+2}v_{i+3}, v_iv_{i+7}v_{i+6}v_{i+5}v_{i+4}v_{i+3}\}$ of G with a shortest branch of length three, a contradiction. Then, we may assume that $\{v_{i+4}, v_{i+7}\} \cap \Lambda(G) \neq \emptyset$. Therefore, $|\{v_{i+4}, v_{i+7}\} \cap \Lambda(G)| = 1$: Otherwise we assume that $v_{i+4}z_1$ and $v_{i+7}z_2$ are two pendant edges of G_0, then $\{v_ix_1, v_{i+1}v_{i+2}, v_{i+3}x_2, v_{i+4}z_1, v_{i+5}v_{i+6}, v_{i+7}z_2\}$ is a matching of G with size 6, contradicting $\alpha'(G) \leq 5$. Hence, without loss of generality, we assume $v_{i+4} \in \Lambda(G)$, then $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+3}, v_{i+4}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+3}, v_iv_{i+1}v_{i+2}v_{i+3}, v_iv_{i+7}v_{i+6}v_{i+5}v_{i+4}\}$ of G with a shortest branch of length three, a contradiction.

Case 1.2. $i^1 = i, i^2 = i+4$.

Then, $\{x_1, x_2\} \cap \Lambda(G) = \emptyset$: Otherwise, $\{x_1, x_2\} \cap \Lambda(G) \neq \emptyset$. By Claim 7, $|\{x_1, x_2\} \cap \Lambda(G)| \leq 1$. Then, we can find a d-system

$$\mathcal{D}_2 = \begin{cases} G[E(C) \cup E(x_1)], & \text{if } x_1 \in \Lambda(G) \\ G[E(C) \cup E(x_2)], & \text{if } x_2 \in \Lambda(G) \end{cases}$$

of G, a contradiction.

Suppose, first, that $\{v_{i+1}, v_{i+3}, v_{i+5}, v_{i+7}\} \cap \Lambda(G) \neq \emptyset$. If there exists a vertex $v_j \in \{v_{i+1}, v_{i+3}, v_{i+5}, v_{i+7}\}$ such that $v_j \in \Lambda(G)$. Then, by symmetry, we may assume that $j = i+1$. Let $v_{i+1}z_1$ be a pendant edge of G_0. By Claim 7, $\{v_i, v_{i+2}\} \cap \Lambda(G) = \emptyset$. Moreover,

$\{v_{i+5}, v_{i+7}\} \cap \Lambda(G) = \emptyset$: Otherwise, we assume either $v_{i+5}z_2$ or $v_{i+7}z_2$ is a pendant edge of G_0, then either $\{v_ix_1, v_{i+1}z_1, v_{i+2}v_{i+3}, v_{i+4}x_2, v_{i+5}z_2, v_{i+6}v_{i+7}\}$ or $\{v_ix_1, v_{i+1}z_1, v_{i+2}v_{i+3}, v_{i+4}x_2, v_{i+5}v_{i+6}, v_{i+7}z_2\}$ is a matching of size 6, contradicting $\alpha'(G) \leq 5$. Hence, we have $\{v_{i+3}, v_{i+6}\} \cap \Lambda(G) = \emptyset$: Otherwise, by $\{v_{i+5}, v_{i+7}\} \cap \Lambda(G) = \emptyset$ and Claim 2, we can find a d-system

$$\mathcal{D}_3 = \begin{cases} G[E(v_iPv_{i+4}\overrightarrow{C}v_i) \cup E(v_{i+1}) \cup E(v_{i+3})], & \text{if } v_3 \in \Lambda(G) \\ G[E(v_iPv_{i+4}\overleftarrow{C}v_i) \cup E(v_{i+6})], & \text{if } v_6 \in \Lambda(G) \end{cases}$$

of G, a contradiction. Then, $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+1}, v_{i+4}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+4}, v_{i+1}v_{i+2}v_{i+3}v_{i+4}, v_iv_{i+7}v_{i+6}v_{i+5}v_{i+4}\}$ of G with a shortest branch of length three, a contradiction.

Suppose, now, that $\{v_{i+1}, v_{i+3}, v_{i+5}, v_{i+7}\} \cap \Lambda(G) = \emptyset$. Then, $\{v_{i+2}, v_{i+6}\} \cap \Lambda(G) = \emptyset$: Otherwise, by Claim 2, we have either $G[E(v_iPv_{i+4}\overrightarrow{C}v_i) \cup E(v_{i+2})]$ or $G[E(v_iPv_{i+4}\overleftarrow{C}v_i) \cup E(v_{i+6})]$ is a d-system of G, a contradiction. Then, $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+4}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+4}, v_iv_{i+1}v_{i+2}v_{i+3}v_{i+4}, v_iv_{i+7}v_{i+6}v_{i+5}v_{i+4}\}$ of G with a shortest branch of length three, a contradiction.

Case 2. $6 \leq c(G) \leq 7$.

Recall that D is an s-component of $G_0 - C$ and $6 \leq |V(C)| \leq 7$, by Lemma 5, $s = 2$. By Claim 2, then D is the two-attaching path x_1x_2 joining two different vertices v_{i^1}, v_{i^2}. Since C is the longest cycle, $d_C(v_{i^1}, v_{i^2}) = 3$.

Claim 8. *$G_0 - C$ has no 1-component.*

Proof. Suppose, by contradiction, that $G_0 - C$ has a 1-component, say v. Since G_0 is 2-connected, $|N_G(v) \cap V(C)| \geq 2$, this will produce a cycle of length at most 5, contradicting $g(G) \geq 6$. □

Claim 9. *$G_0 - C$ has no component other than D.*

Proof. Suppose, by contradiction, that $G_0 - C$ has another component D'(say). By Claim 8 and Lemma 5, D' is a 2-component of $G_0 - C$. By Claim 2, $D \cong P_2$ and $D' \cong P_2$. Let P' be a longest two-attaching path of D' joining two different vertices v_{i^3} and v_{i^4} on C. By Claim 2, $N(D') \cap V(C) = \{v_{i^3}, v_{i^4}\}$. Since C is the longest cycle, $d_C(v_{i^3}, v_{i^4}) = 3$. By Lemma 7, D and D' do not overlap on C. Then, $\{v_{i^1}, v_{i^2}\} \cap \{v_{i^3}, v_{i^4}\} \neq \emptyset$, without loss of generality, we assume $v_{i^2} = v_{i^3}$. Suppose, first, that $v_{i^1} = v_{i^4}$. By Claim 2, we can find a d-system $G[E(C) \cup E(v_{i^1}Pv_{i^2}P'v_{i^1})]$ of G, a contradiction. Suppose, now, that $v_{i^1} \neq v_{i^4}$. By $d_C(v_{i^1}, v_{i^2}) = 3$ and $d_C(v_{i^3}, v_{i^4}) = 3$, we have $c(G) = 7$, and thus, $d_C(v_{i^1}, v_{i^4}) = 1$. Then, by Claim 2, we can find a d-system $G[E(v_{i^1}Pv_{i^2}) \cup E(v_{i^2}P'v_{i^4}) \cup (E(C) \setminus \{v_{i^1}v_{i^4}\})]$ of G, a contradiction. □

By Claim 9, $V(G_0) = V(C) \cup V(D)$, $\{v_{i^1}, v_{i^2}\} \subseteq V_3(G_0)$ and $(V(C) \setminus \{v_{i^1}, v_{i^2}\}) \subseteq V_2(G_0)$. By Claim 2, $V(D) \subseteq V_2(G_0)$. Note that $\alpha'(C) \geq 3$ and $\alpha'(D) = 1$, then

Claim 10. *The following two statements hold.*

(1) If $c(G) = 7$, then no triple of vertices in $\Lambda(G)$ is consecutive on C;
(2) If $c(G) = 6$, then no quadruple of vertices in $\Lambda(G)$ is consecutive on C.

Note that $N(D) \cap V(C) = \{v_{i^1}, v_{i^2}\}$ and $d_C(v_{i^1}, v_{i^2}) = 3$. Then, without loss of generality, assume that $i^1 = i$, $i^2 = i + 3$ for some $i \in \{0, 1, 2, \ldots, c(G) - 1\}$.

Claim 11. *If $v_{i+1} \in \Lambda(G)$, then $v_{i+2} \in \Lambda(G)$. Furthermore, $d_G(v_{i+1}) = 3$ and $d_G(v_{i+2}) = 3$.*

Proof. Suppose, by contradiction, that $v_{i+2} \notin \Lambda(G)$. Then, by Claim 2, $G[E(v_iPv_{i+3}\overrightarrow{C}v_i) \cup E(v_{i+1})]$ is a d-system of G, a contradiction.

Suppose, by contradiction, that $d_G(v_{i+1}) \neq 3$ or $d_G(v_{i+2}) \neq 3$. Note that $v_{i+1} \in \Lambda(G)$ and $v_{i+2} \in \Lambda(G)$. Without loss of generality, we may suppose that $d_G(v_{i+1}) > 3$, Then, by Claim 2, $G[E(v_iPv_{i+3}\overrightarrow{C}v_i) \cup (E(v_{i+1}) \setminus \{v_{i+1}v_{i+2}\}) \cup E(v_{i+2})]$ is a d-system of G, a contradiction. $\square$

Suppose, first, that $\{v_{i+1}, v_{i+2}, x_1, x_2\} \cap \Lambda(G) \neq \emptyset$. Without loss of generality, we assume that $x_1 \in \Lambda(G)$, then $x_2 \in \Lambda(G)$: Otherwise, by Claim 2, $G[E(C) \cup E(x_1)]$ is a d-system of G, a contradiction. Moreover, $d_G(x_1) = 3$ and $d_G(x_2) = 3$: Otherwise, $d_G(x_1) > 3$ or $d_G(x_2) > 3$. Note that $x_1 \in \Lambda(G)$ and $x_2 \in \Lambda(G)$. Without loss of generality, we may suppose that $d_G(x_1) > 3$. Then, by Claim 2, $G[E(C) \cup (E(x_1) \setminus \{x_1x_2\}) \cup E(x_2)]$ is a d-system of G, a contradiction. Hence, $\{v_{i+1}, v_{i+2}\} \cap \Lambda(G) = \emptyset$: Otherwise, without loss of generality, we may suppose that $v_{i+1} \in \Lambda(G)$, by Claim 11, $\{v_{i+1}, v_{i+2}, x_1, x_2\} \subseteq \Lambda(G)$. We assume that $v_{i+1}z_1$, $v_{i+2}z_2$, x_1z_3 and x_2z_4 are four pendant edges of G_0. Then, either $\{v_{i+1}z_1, v_{i+2}z_2, x_1z_3, x_2z_4, v_{i+3}v_{i+4}, v_{i+5}v_i\}$ ($c(G) = 6$) or $\{v_{i+1}z_1, v_{i+2}z_2, x_1z_3,\ x_2z_4, v_{i+3}v_{i+4}, v_{i+5}v_{i+6}\}$ ($c(G) = 7$) is a matching of G with size 6, contradicting $\alpha'(G) \leq 5$. If $c(G) = 6$, then, by symmetry, $\{v_{i+4}, v_{i+5}\} \cap \Lambda(G) = \emptyset$. By Claim 10(2), $|\{v_i, v_{i+3}\} \cap \Lambda(G)| \leq 1$. Since $D \cong P_2$, $|N_G(D) \cap V(C)| = 2$, $\{x_1, x_2\} \subseteq V_3(G)$ and $(V(C) \setminus \{v_i, v_{i+3}\}) \subseteq V_2(G)$, then $G \in \mathcal{G}_0$. Hence, we assume that $c(G) = 7$. By Claim 10(1), $\{v_i, v_{i+3}\} \cap \Lambda(G) = \emptyset$. Moreover, $\{v_{i+4}, v_{i+6}\} \cap \Lambda(G) = \emptyset$: Otherwise, without loss of generality, we assume that $v_{i+4} \in \Lambda(G)$, and x_1z_1 and x_2z_2, $v_{i+4}z_3$ are three pendant edges of G_0, then $\{x_1z_1, x_2z_2, v_iv_{i+1}, v_{i+2}v_{i+3}, v_{i+4}z_3, v_{i+5}v_{i+6}\}$ is a matching of G with size 6, contradicting $\alpha'(G) \leq 5$. Then, $v_{i+5} \notin \Lambda(G)$: Otherwise, by $\{v_{i+4}, v_{i+6}\} \cap \Lambda(G) = \emptyset$ and Claim 2, $G[E(v_iPv_{i+3}\overleftarrow{C}v_i) \cup E(v_{i+5})]$ is a d-system of G, a contradiction. Since $D \cong P_2$, $|N_G(D) \cap V(C)| = 2$, $\{x_1, x_2, v_i, v_{i+3}\} \subseteq V_3(G)$ and $(V(C) \setminus \{v_i, v_{i+3}\}) \subseteq V_2(G)$, then $G \in \mathcal{G}_0$.

Suppose, now, that $\{v_{i+1}, v_{i+2}, x_1, x_2\} \cap \Lambda(G) = \emptyset$. If $c(G) = 6$, then, by symmetry, $\{v_{i+4}, v_{i+5}\} \cap \Lambda(G) = \emptyset$. Then $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+3}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+3}, v_iv_{i+1}v_{i+2}v_{i+3}, v_iv_{i+5}v_{i+4}v_{i+3}\}$ of G with a shortest branch of length three, a contradiction. Hence, we assume that $c(G) = 7$. If $\{v_{i+4}, v_{i+6}\} \cap \Lambda(G) = \emptyset$, then $v_{i+5} \notin \Lambda(G)$: Otherwise, by Claim 2, $G[E(v_iPv_{i+3}\overleftarrow{C}v_i) \cup E(v_{i+5})]$ is a d-system of G, a contradiction. Then $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+3}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+3}, v_iv_{i+1}v_{i+2}v_{i+3}, v_iv_{i+6}v_{i+5}v_{i+4}\ v_{i+3}\}$ of G with a shortest branch of length three, a contradiction. Then, we may assume that $\{v_{i+4}, v_{i+6}\} \cap \Lambda(G) \neq \emptyset$. Without loss of generality, we may suppose that $v_{i+4} \in \Lambda(G)$. If $v_{i+5} \notin \Lambda(G)$, then $v_{i+6} \notin \Lambda(G)$. Otherwise, by Claim 2, $G[E(v_iPv_{i+3}\overleftarrow{C}v_i) \cup E(v_{i+4}) \cup E(v_{i+6})]$ is a d-system of G, a contradiction. Then $(\{x_1, x_2\} \cup (V(C) \setminus \{v_i, v_{i+3}, v_{i+4}\})) \subseteq V_2(G)$. Therefore, G has an odd branch-bond $\{v_ix_1x_2v_{i+3}, v_iv_{i+1}v_{i+2}v_{i+3}, v_iv_{i+6}v_{i+5}v_{i+4}\}$ of G with a shortest branch of length three, a contradiction. Hence, we assume that $v_{i+5} \in \Lambda(G)$. Then, $d_G(v_{i+4}) = 3$ and $d_G(v_{i+5}) = 3$: Otherwise, without loss of generality, we may suppose that $d_G(v_{i+4}) > 3$, then, by Claim 2, $G[E(v_iPv_{i+3}\overleftarrow{C}v_{i1}) \cup (E(v_{i+4}) \setminus \{v_{i+4}v_{i+5}\}) \cup E(v_{i+5})]$ is a d-system of G, a contradiction. By Claim 10(1), $\{v_{i+3}, v_{i+6}\} \cap \Lambda(G) = \emptyset$. Hence, $G \in \mathcal{G}_0$.

This completes the proof of Theorem 7. $\square$

From the theorem above, the matching number of any graph in $\mathcal{G}_0$ is at least 5, so we can immediately obtain the following result.

Corollary 3. *Let G be a essentially 2-edge-connected graph with $g(G) \geq 6$, $\alpha'(G) \leq 4$ such that each odd branch-bond of G has a shortest branch of length at most 2. If the core G_0 of G is 2-connected, then G has a d-system.*

Theorem 8. *Let G be a connected graph with $g(G) \geq 6$. If $\kappa(L(G)) \geq 2$, $\alpha(L(G)) \leq 5$ and every odd branch-bond of $L(G)$ contains an edge branch, then $L(G)$ has a 2-factor if and only if G is not a member of $\mathcal{G}_0$.*

Proof of Theorem 8. Observe that a maximum independent set of $L(G)$ corresponds a maximum matching of G, then $\alpha'(G) = \alpha(L(G)) \leq 5$. Note that every member of $\mathcal{G}_0$ has no d-system, by Lemma 4, the line graph of every member of $\mathcal{G}_0$ has no 2-factor, the necessity of Theorem 8 clearly holds.

Suppose that $L(G)$ has no 2-factor, it suffices to show that $G \in \mathcal{G}_0$. By Lemma 4, G has no d-system. Since each odd branch-bond of $L(G)$ contains an edge branch, by Lemma 3, each odd branch-bond of G contains a shortest branch of length at most 2. Note that $L(G)$ is 2-connected if and only if G is essentially 2-edge-connected. Suppose, first, that the core G_0 of G is 2-connected. By Theorem 7, $G \in \mathcal{G}_0$.

Suppose, now, that $\kappa(G_0) = 1$.

Claim 12. *For any super-block H of G, it holds that $\alpha'(H) \geq 3$.*

Proof. Since $L(G)$ is 2-connected, each block of G_0 is not a tree. Therefore, by $g(G) \geq 6$, for any super-block H of G, it holds that $g(H) \geq 6$ and thus $\alpha'(H) \geq 3$. □

By $\kappa(G_0) = 1$, G has at least two super-blocks. We will prove that G has exactly two super-blocks. Otherwise, we assume that G has at least three super-blocks H_1, H_2 and H_3. By Claim 12, $\alpha'(H_i) \geq 3$ for all $i \in \{1,2,3\}$. If $H_1 \cap H_2 \cap H_3 \neq \emptyset$, then we may let $H_1 \cap H_2 \cap H_3 = \{v\}$. By Claim 12, $\alpha'(G) \geq \alpha'(H_1 \cup H_2 \cup H_3) \geq \alpha'(H_1) + \alpha'(H_2 - \{v\}) + \alpha'(H_3 - \{v\}) \geq 3 + 2 + 2 \geq 6$, contradicting $\alpha'(G) \leq 5$. Hence, there exists a pair of super-block H_i, H_j such that $H_i \cap H_j = \emptyset$ $(i, j \subseteq \{1,2,3\})$, then $\alpha'(G) \geq \alpha'(H_i \cup H_j) \geq \alpha'(H_i) + \alpha'(H_j) \geq 3 + 3 = 6$, a contradiction. Hence, G_0 has exactly two super-blocks, say H_1, H_2.

By $\kappa(G_0) = 1$, $V(H_1) \cap V(H_2) \neq \emptyset$, say $\{v\} = V(H_1) \cap V(H_2)$. Then, $\alpha'(G[V(H_1)]) = \alpha'(G[V(H_2)]) = 3$: Otherwise, there exists at least one super-block, say H_1 such that $\alpha'(H_1) \geq 4$, then, by Claim 12, $\alpha'(G) \geq \alpha'(G[V(H_1)]) + \alpha'(G[V(H_2) - \{v\}]) \geq 4 + 2 \geq 6$, a contradiction. Since every odd branch-bond of G contains a shortest branch of length at most 2, every odd branch-bond of H_i $(i \in \{1,2\})$ contains a shortest branch of length at most 2. By Corollary 3, H_i has d-system in H_i $(i \in \{1,2\})$. By the definition of H_i, G has a d-system in G, a contradiction.

This completes the proof of Theorem 8. □

Proof of Theorem 5. By Theorems $4(i)(iii)$, we may assume that $cl^{2f}(G) = L(H)$, where H satisfies Theorem $4(ii)$. As adding edge to a graph does not increase the independence number α and does not decrease the connectivity κ, both $\kappa(cl^{2f}(G)) \geq \kappa(G) \geq 2$ and $\alpha(cl^{2f}(G)) \leq \alpha(G) \leq 5$ hold. Since every odd branch-bond of G has an edge-branch, by Lemma 1, every odd branch-bond of $cl^{2f}(G)$ has an edge-branch. Therefore, by Theorem 8, $cl^{2f}(G) = L(H)$ has a 2-factor if and only if the closure $cl^{2f}(G)$ of G is not isomorphic to the line graph of a member of $\mathcal{G}_0$. □

5. The Proof of Theorem 6

For proving Theorem 6, it suffices to show the following theorem.

Theorem 9. *Let G be a connected graph with $g(G) \geq 6$. If $\kappa'(L(G)) \geq 2$, $\alpha(L(G)) \leq 4$ and every odd branch-bond of $L(G)$ contains an edge branch, then $L(G)$ has a 2-factor if and only if G is not a member of $\mathcal{F}_0 \cup \{F_1, F_2, F_3, F_4\}$.*

Proof of Theorem 9. Observe that a maximum independent set of $L(G)$ corresponds a maximum matching of G, then $\alpha'(G) = \alpha(L(G)) \leq 4$. By $\kappa'(L(G)) \geq 2$, then $\kappa(L(G)) \geq 1$. Note that every member of $\mathcal{F}_0 \cup \{F_1, F_2, F_3, F_4\}$ has no d-system, by Lemma 4, the line graph of every member of $\mathcal{F}_0 \cup \{F_1, F_2, F_3, F_4\}$ has no 2-factor, the necessity of Theorem 9 clearly holds.

Suppose that $L(G)$ has no 2-factor, it suffices to show that $G \in \mathcal{F}_0 \cup \{F_1, F_2, F_3, F_4\}$. By Lemma 4, G has no d-system. Since every odd branch-bond of $L(G)$ contains an edge branch, by Lemma 3, every odd branch-bond of G contains a shortest branch of length at most 2.

If $\kappa(L(G)) \geq 2$, then, by Corollary 1, $L(G)$ has a 2-factor, a contradiction. Therefore, we assume that G_0 has a cut edge. Let $B_1, B_2, \cdots, B_t$ be all the blocks of G_0. For any vertex v of B_i such that $B_i \cong P_2$ for all $i \in \{1, \cdots, t\}$, by $\kappa'(L(G)) \geq 2$, we have

$$d_G(v) \geq 3. \tag{1}$$

Claim 13. *G_0 is a tree.*

Proof. It suffices to show that every block in G_0 is isomorphic to P_2. By contradiction, suppose that there exists a block B_1 of G_0 such that B_1 is not isomorphic to P_2. Then, B_1 has a cycle. By $g(G) \geq 6$, we have $g(B_1) \geq 6$. Recall G_0 has a cut edge, we have $t \geq 2$. Then, B_i ($i \in \{2, \ldots, t\}$) is isomorphic to P_2. Otherwise, we assume that there exists a block B_2 (say) such that B_2 is not is isomorphic to P_2. Then, B_2 has cycle. Again by $g(G) \geq 6$, we have $g(B_2) \geq 6$, and thus $\alpha'(B_2) \geq 3$. Therefore, $\alpha'(G) \geq \alpha'(B_1 \cup B_2) \geq \alpha'(B_1) + \alpha'(B_2) - 1 \geq 3 + 3 - 1 = 5$, a contradiction.

Then, $t = 2$: Otherwise, we assume that $t \geq 3$. Note that B_i ($i \in \{2, \ldots, t\}$) is isomorphic to P_2, by (1), there exist two dependent edges e_1, e_2 incident with B_2 and B_3, respectively, and thus, $\alpha'(G) \geq \alpha'(B_1 \cup \{e_1, e_2\}) \geq 3 + 1 + 1 = 5$, a contradiction.

Let $B_2 = uv$ such that $u \in V(B_1)$. Then, by (1), we have $|E(v)| \geq 3$. Let $H_1' = B_1 \cup \{e : e$ is a pendent edge of G and e has one end in $V(B_1) \cap \Lambda(G)\}$. Hence, $\alpha'(H_1') = 3$: Otherwise, by $g(B_1) \geq 6$, we assume that $\alpha'(H_1') \geq 4$. Recall $B_2 = v_1v_2$, by (1), $\alpha'(G) \geq \alpha'(H_1' \cup G[E(v)]) \geq 4 + 1 \geq 5$, a contradiction. Since every odd branch-bond of G contains a shortest branch of length at most 2, every odd branch-bond of H_1' contains a shortest branch of length at most 2. By Corollary 3, H_1' has d-system $\mathcal{D}_4$ (say) that every edge of H_1' that is not in a star of $\mathcal{D}_4$ is dominated by a circuit in $\mathcal{D}_4$. Then, by (1), we can find a d-system $\mathcal{D}_4 \cup G[E(v)]$ in G, a contradiction. □

By Claim 13, we denote the length of a longest path of G_0 by l. If there exists a longest path of G_0 with $l \geq 4$, by (1), we can find $l + 1$ independent edges of G, contradicting $\alpha'(G) \leq 4$. Hence, $1 \leq l \leq 3$.

Suppose that $l = 1$. Then, by Claim 13, G_0 is a P_2, say v_1v_2. Hence, by (1), $\{v_1, v_2\} \subseteq V_{\geq 3}(G)$. Thus, $\{v_1, v_2\} \subseteq V_3(G)$: Otherwise, without loss of generality, we may suppose that $d_G(v_1) > 3$, then $G[E(v_1) \backslash \{v_1v_2\}] \cup G[E(v_2)]$ is a d-system of G, a contradiction. Therefore, $G \cong F_2$. Then, we assume that $2 \leq l \leq 3$. In the following, we need distinguish the following two cases.

Case 1. $l = 2$.

Note that $\alpha'(G) \leq 4$ and Claim 13, so G_0 has at most 4 leaves. Then $G_0 \in \{K_{1,4}, K_{1,3}, P_2\}$.

Suppose, first, that $G_0 \cong K_{1,4}$. We denoted by $G[x; y_1, y_2, y_3, y_4]$. By (1), $\{y_1, y_2, y_3, y_4\} \subseteq V_{\geq 3}(G)$ Therefore, by $\alpha'(G) \leq 4$, $N_G(x) = \{y_1, \cdots, y_4\}$ and thus $G[E(y_1) \cup E(y_2) \cup E(y_3) \cup E(y_4)]$ is a d-system of G, a contradiction.

Suppose, now, that $G_0 \cong K_{1,3}$. We denoted by $G[x; y_1, y_2, y_3]$. By (1), $\{y_1, y_2, y_3\} \subseteq V_{\geq 3}(G)$. Then $1 \leq |N_G(x) \setminus \{y_1, y_2, y_3\}| \leq 2$: Otherwise we can find a d-system

$$\mathcal{D}_5 = \begin{cases} G[E(y_1) \cup E(y_2) \cup E(y_3)], & \text{if } |N_G(x) \setminus \{y_1, y_2, y_3\}| = 0 \\ G[E(y_1) \cup E(y_2) \cup E(y_3) \cup (E(x) \setminus \{xy_1, xy_2, xy_3\})], & \text{if } |N_G(x) \setminus \{y_1, y_2, y_3\}| \geq 3. \end{cases}$$

of G, a contradiction. If $|N_G(x) \setminus \{y_1, y_2, y_3\}| = 1$, then there exist at least two vertices in $\{y_1, y_2, y_3\}$, say y_1, y_2 such that $\{y_1, y_2\} \subseteq V_3(G)$. Otherwise, there exists at most one vertex in $\{y_1, y_2, y_3\}$, say y_1 such that $d_G(y_1) = 3$, by (1), $\{y_2, y_3\} \subseteq V_{\geq 4}(G)$, then we can find a d-system $G[E(y_1) \cup (E(x) \setminus \{xy_1\}) \cup (E(y_2) \setminus \{xy_2\}) \cup (E(y_3) \setminus \{xy_3\})]$ of G, a contradiction. Hence, G is the graph obtained from $\mathcal{F}_0$ by Operation (1). If $|N_G(x) \setminus \{y_1, y_2, y_3\}| = 2$, then $\{y_1, y_2, y_3\} \subseteq V_3(G)$. Otherwise, there exists at least one vertex in $\{y_1, y_2, y_3\}$, say

y_1 such that $d_G(y_1) \geq 4$, then we can find a d-system $G[(E(y_1) \setminus \{xy_1\}) \cup E(y_2) \cup E(y_3) \cup (E(x) \setminus \{xy_2, xy_3\})]$ of G, a contradiction. Hence, $G \cong F_1$.

Finally, suppose that $G_0 \cong P_3$, say $P_3 = v_1v_2v_3$. By (1), $\{v_1, v_2, v_3\} \subseteq V_{\geq 3}(G)$. Then, $3 \leq d_G(v_2) \leq 4$: Otherwise we can find a d-system $G[E(v_1) \cup E(v_3) \cup (E(v_2) \setminus \{v_1v_2, v_2v_3\})]$ of G, a contradiction. If $d_G(v_2) = 3$, then there exists at least one vertex in $\{v_1, v_3\}$, say v_3 such that $d_G(v_3) = 3$. Otherwise, by $\{v_1, v_2, v_3\} \subseteq V_{\geq 3}(G)$, we have $\{v_1, v_3\} \subseteq V_{\geq 4}(G)$. Then, we can find a d-system $G[E(v_2) \cup (E(v_1) \setminus \{v_1v_2\}) \cup (E(v_3) \setminus \{v_2v_3\})]$ of G, a contradiction. Hence, G is the graph obtained from $\mathcal{F}_0$ by Operation (1). If $d_G(v_2) = 4$, then $\{v_1, v_3\} \subseteq V_3(G)$. Otherwise, there exists at least one vertex in $\{v_1, v_3\}$, say v_1 such that $d_G(v_1) \geq 4$, then we can find a d-system $G[(E(v_2) \setminus \{v_2v_3\}) \cup E(v_3) \cup (E(v_1) \setminus \{v_1v_2\})]$ of G, a contradiction. Hence, $G \cong F_3$.

Case 2. $l = 3$.

Let $P = v_1v_2 \cdots v_{l+1}$ be a longest path of G_0. G_0 has at most three leaves, otherwise, we assume that x_1, x_2, x_3, x_4 are four leaves of G_0, then, by the definition of G_0, we assume that $x_1z_1, x_2z_2, x_3z_3, x_4z_4$ are four pendent edges of G. Note that $l = 3$, so there exists an edge e of $G - \{x_1, x_2, x_3, x_4\}$, and thus $\{e, x_1z_1, x_2z_2, x_3z_3, x_4z_4\}$ is a matching of G with size 5, contradicting $\alpha'(G) \leq 4$. Now, we need distinguish the following two cases.

Case 2.1. G_0 has exactly three leaves.

Then, G_0 is isomorphic to the unique tree with a degree sequence 11123. Without loss of generality, we assume that $d_G(v_2) = 3$ and $N_G(v_2) \setminus \{v_1, v_3\} = \{v_5\}$. By (1), $\{v_1, v_3, v_4, v_5\} \subseteq V_{\geq 3}(G)$. Then, $d_G(v_3) = 3$: Otherwise we can find a d-system $G[E(v_1) \cup E(v_5) \cup (E(v_3) \setminus \{v_3v_4\}) \cup E(v_4)]$ of G, a contradiction. Hence, we also have $d_G(v_4) = 3$: Otherwise, we can find a d-system $G[E(v_1) \cup E(v_5) \cup E(v_3) \cup (E(v_4) \setminus \{v_3v_4\})]$ of G, a contradiction. Hence, G is the graph obtained from $\mathcal{F}_0$ by Operation (2).

Case 2.2. G_0 has exactly two leaves.

By $l = 3$, G_0 is a path of length 3 (say $v_1v_2 \cdots v_4$). By (1), $\{v_1, v_2, v_3, v_4\} \subseteq V_{\geq 3}(G)$.

Suppose, first, that $d_G(v_2) = 3$ or $d_G(v_3) = 3$. Without loss of generality, we assume $d_G(v_2) = 3$. If $d_G(v_3) = 3$, then, G is the graph obtained from $\mathcal{F}_0$ by Operation (2). Hence, we may assume that $d_G(v_3) \geq 4$. If $d_G(v_1) = 3$, then G is the graph obtained from $\mathcal{F}_0$ by Operation (2). If $d_G(v_1) \geq 4$, then $d_G(v_3) = 4$. Otherwise, we can find a d-system $G[(E(v_1) \setminus \{v_1v_2\}) \cup E(v_2) \cup (E(v_3) \setminus \{v_2v_3, v_3v_4\}) \cup E(v_4)]$ of G. Then, $d_G(v_4) = 3$: otherwise, we can find a d-system $G[(E(v_1) \setminus \{v_1v_2\}) \cup E(v_2) \cup (E(v_3) \setminus \{v_2v_3\}) \cup (E(v_4) \setminus \{v_3v_4\})]$ of G. Hence, G is the graph obtained from $\mathcal{F}_0$ by Operation (3).

Suppose, now, that $\{v_2, v_3\} \subseteq V_{\geq 4}(G)$. Then, $\{v_2, v_3\} \subseteq V_4(G)$: Otherwise, without loss of generality, we may assume that $d_G(v_2) \geq 5$. Then, we can find a d-system $G[(E(v_1) \cup (E(v_2) \setminus \{v_1v_2, v_2v_3\}) \cup (E(v_3) \setminus \{v_3v_4\}) \cup E(v_4)]$ of G, a contradiction. Thus, $\{v_1, v_4\} \subseteq V_3(G)$: Otherwise, without loss of generality, we may assume that $d_G(v_1) \geq 4$, we can find a d-system $G[(E(v_1) \setminus \{v_1v_2\}) \cup (E(v_2) \setminus \{v_2v_3\}) \cup (E(v_3) \setminus \{v_3v_4\}) \cup E(v_4)]$ of G, a contradiction. Hence, $G \cong F_4$.

This completes the proof of Theorem 9. □

Proof of Theorem 6. By Theorems $4(i)(iii)$, we may assume that $cl^{2f}(G) = L(H)$, where H satisfies Theorem $4(ii)$. As adding edges to a graph does not increase the independence number α and does not decrease the connectivity κ', both $\kappa'(cl^{2f}(G)) \geq \kappa'(G) \geq 2$ and $\alpha(cl^{2f}(G)) \leq \alpha(G) \leq 4$ hold. Since every odd branch-bond of G has an edge-branch, by Lemma 1, every odd branch-bond of $cl^{2f}(G)$ has an edge-branch. Therefore, by Theorem 9, $cl^{2f}(G) = L(H)$ has a 2-factor if and only if the closure $cl^{2f}(G)$ of G is not isomorphic to the line graph of a member of $\mathcal{F}_0 \cup \{F_1, F_2, F_3, F_4\}$. □

Remark 1. *We considered to repalce the condition $\alpha(G) \leq 5$ in Theorem 5, but the length of the proof is too long, and its readability is poor.*

6. Conclusions

In 2014, Xu et al. considered the independence number conditions for hamiltonicity of 2-connected claw-free graph. In this paper, we consider the problem of determining the largest possible independence number of 2-connected claw-free graph G such that every odd branch-bond of G has an edge branch to have a 2-factor, as well as other related problems. We also investigate the similar problem for 2-edge-connected graphs. It further reveals the profound connotation of graph keeping the (non)-existence of a 2-factor.

In the future, we can consider that $\alpha(G) \leq 6$ in Theorem 5. This work is meaningful and difficult, because the length of the proof is too long and complicated, and its readability is poor. We need to improve the proof technique and method.

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