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On the Chromatic Index of the Signed Generalized Petersen Graph $GP(n,2)$

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Abstract: Let G be a graph and $\sigma : E(G) \to \{+1, -1\}$ be a mapping. The pair (G, σ), denoted by G_σ, is called a signed graph. A (proper) l-edge coloring γ of G_σ is a mapping from each vertex–edge incidence of G_σ to M_q such that $\gamma(v,e) = -\sigma(e)\gamma(w,e)$ for each edge $e = vw$, and no two vertex–edge incidences have the same color; that is, $\gamma(v,e) \neq \gamma(v,f)$. The chromatic index is the minimal number q such that G_σ has a proper q-edge coloring, denoted by $\chi'(G_\sigma)$. In 2020, Behr proved that the chromatic index of a signed graph is its maximum degree or maximum plus one. In this paper, we considered the chromatic index of the signed generalized Petersen graph $GP(n,2)$ and show that its chromatic index is its maximum degree for most cases. In detail, we proved that (1) $\chi'(GP_\sigma(n,2)) = 3$ if $n \equiv 3 \bmod 6 (n \geq 9)$; (2) $\chi'(GP_\sigma(n,2)) = 3$ if $n = 2p (p \geq 4)$.

Keywords: chromatic index; signed graph; generalized Petersen graph

MSC: 05C15; 05C22

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1. Introduction

The graphs considered in this paper are finite and simple. The *Petersen graph* $GP(5,2)$ is a cubic graph with 10 vertices and 15 edges. The Petersen graph appears as a counterexample in many aspects of graph theory. It does not have a 3-edge-coloring proved by Naserasr et al. [1]. The Petersen graph has been widely studied in many aspects of graph theory. Bezrukov et al. [2] defined the nth Cartesian power of Petersen graph $GP(5,2)^n$ and got its cutwidth and wirelength. Then, they generalized these results to the Cartesian product of $GP(5,2)^n$ and m-dimensional binary hypercube. In 1969, Watkins gave the definition of generalized Petersen graph. Let n and k be positive integers with $k < n/2$; then the *generalized Petersen graph* $GP(n,k)$ has vertex set $\{u_i, v_i : 1 \leq i \leq n\}$ and three types of edges: (i) spoke edges: $\{u_i v_i : 1 \leq i \leq n\}$; (ii) outer-cycle edges: $\{u_i u_{i+1} : 1 \leq i \leq n\}$; (iii) inner-cycle edges: $\{v_i v_{i+k} : 1 \leq i \leq n\}$. All subscripts are modulo n. Obviously, the generalized Petersen graph is a 3-regular graph. The outer-cycle edges form the outer cycle, denoted c_0, and the inner-cycle edges form the inner cycles, denoted c_l, where $l \in \{1, 2, \cdots, \frac{n}{gcd(n,k)}\}$ and $|E(c_l)| = \frac{n}{gcd(n,k)}$. The spoke edges form a perfect matching, denoted M_s. For the edge $e = uv$, the pairs $(u,e), (v,e)$ are called the *incidences*. For terminology and notations not defined here, we refer to [3,4].

Given a graph G and a mapping $\sigma : E(G) \to \{+1, -1\}$, the pair (G, σ), denoted by G_σ, is called a signed graph. Here, σ is a *signature* of (G, σ) and $\sigma(e)$ is the sign of edge e. An edge e is positive if $\sigma(e) = +1$, and it is negative otherwise. We denote by $E^{\sigma-}(G_\sigma)$ the set of negative edges of G_σ. We call graph G the underlying graph of the signed graph G_σ. The sign of G_σ is the product of the signs of its edges. A cycle is *balanced* if its sign is $+1$ and *unbalanced* otherwise. *Switching* at a vertex v means negating the sign of every edge that has v as an endpoint. *Switching* at a vertex set X means switching each $v \in X$ in turn. If $G_{\sigma'}$ can be obtained from G_σ by switching at some vertices, then we say that they are *switching equivalent*, denoted by $G_{\sigma'} \sim G_\sigma$. We also call σ' and σ equivalent signatures of G_σ.

The coloring of signed graphs was first considered by Cartwright and Harry [5]. The coloring of G_σ is a mapping from $V(G_\sigma)$ to a color set such that every two vertices joined by a positive edge receive the same color and every two vertices joined by a negative edge receive different colors. They observed that G_σ has a 2-coloring if and only if G_σ is balanced. In 2016, Máčajová et al. [6] introduced the chromatic number of a signed graph and proved the Brooks' theorem for signed graph. The proper edge coloring of signed graphs was introduced by Behr [7], and independently, by Zhang et al. [8]. These two definitions are equivalent. Here, we use Behr's definition, which is based on the signed color set M_q; that is, $M_q = \{\pm1, \pm2, \cdots, \pm t\}$ if $q = 2t$, and $M_q = \{0, \pm1, \pm2, \cdots, \pm t\}$ if $q = 2t + 1$. A (proper) q-edge coloring γ of G_σ is a mapping from each vertex–edge incidence of G_σ to M_q such that $\gamma(v, e) = -\sigma(e)\gamma(w, e)$ for each edge $e = vw$, and no two vertex–edge incidences have the same color; that is, $\gamma(v, e) \neq \gamma(v, f)$. The chromatic index is the minimal number q such that G_σ has a proper n-edge coloring, denoted by $\chi'(G_\sigma)$. For an edge $e = uv$, if e is negative, (u, e) and (v, e) must be colored by the same color, say, c; we can also say that e is colored by c in this case. If e is positive, (u, e) and (v, e) must be colored by opposite colors, say, c and $-c$; we also say that e is colored by $\pm c$. Note that every edge can be colored by 0 if $0 \in M_q$.

Zhang et al. [8] mainly showed that $\chi'(G_\sigma) \leq \Delta + 1$ if $\Delta \leq 5$ or if G is a planar graph. Behr [7] proved the signed version of Vizing's theorem; that is, for any signed graph G_σ, $\Delta \leq \chi'(G_\sigma) \leq \Delta + 1$. The signed graph G_σ is type I (type II, respectively) if $\chi'(G_\sigma) = \Delta(G)$ ($\chi'(G_\sigma) = \Delta(G) + 1$, respectively). It is an interesting topic to determine whether a signed graph is type I or type II. The generalized Petersen graph is a famous and well-studied family of graphs. Steimle et al. [9] showed that if n is fixed, when $GP(n, k)$ and $GP(n, l)$ are isomorphic, there are several properties for the pair (k, l). Ralucca et al. [10] studied the spectrum of $GP(n, k)$ and added $GP(n, k)$ into the family of graphs with known spectra. Ebrahimi et al. [11] proved the necessary and sufficient condition for $GP(n, k)$ to have an efficient dominating set, and for several specific cases, authors gave the domination number of $GP(n, k)$. We would like to lay more attention on the coloring of $GP(n, k)$. A *Tait coloring* of a cubic graph is an edge-coloring in three colors such that each color is incident to each vertex. Castagna et al. [12] proved that all but the original Petersen graph have a Tait coloring. Watkins [13] gave another method to prove that generalized Petersen graphs, except the Petersen graph, have Tait coloring. Khennoufa et al. [14] studied the chromatic number of the edge coloring the total k-labeling of generalized Petersen graphs, and proved that for $n \geq 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, the edge coloring total k-labeling chromatic number of $GP(n, k)$ is 3 if n is odd or k is even, and the corresponding chromatic number is 2 if n is even and k is odd. Chen et al. [15] showed that if $k \geq 4$ and $n > 2k$, the strong chromatic index of each generalized Petersen graph $GP(n, k)$ is at most nine. Li et al. [16] studied the injective edge coloring numbers of $GP(n, 1)$ and $GP(n, 2)$. They got specific values for $GP(n, 1)$ with $n \geq 3$, and for $GP(n, 2)$ with $4 \leq n \leq 7$. Yang et al. [17] studied the strong chromatic index of $GP(n, k)$ when $1 \leq k \leq 3$. In [18], Cai et al. studied the edge coloring of the generalized Petersen graph and got the following results. When $n \geq 5$, the chromatic index of $GP(n, 1)$ is three. When $k = 5, 6$, for all signatures σ but some special cases, the chromatic index of $GP(n, 2)$ is three. In this paper, we mainly considered the edge coloring of signed generalized Petersen graph $GP_\sigma(n, k)$, where $k = 2$, and n satisfies certain conditions: (1) $n \equiv 3 \bmod 6 (n \geq 9)$; (2) $n = 2p (p \geq 4)$. The main aim in this paper is to show that deleting perfect matching of the signed generalized Petersen graph consists of balanced cycles. Note that the perfect matching can be colored by zero, and the left balanced cycles can be colored by ±1; then, we get a 3-coloring of the signed generalized Petersen graph.

2. Preliminaries

In this section, we introduce some existing properties and results, which can be used to prove our main theorems.

Firstly, we give some notation which will be used in the following paper. Obviously, the generalized Petersen graph has an outer-cycle, $gcd(n,k)$, inner-cycles and spokes. For convenience, we use $E^{\sigma-}_{c_0}(GP_\sigma(n,k))$ ($E^{\sigma-}_{c_l}(GP_\sigma(n,k)), l \in \{1,2,\cdots,gcd(n,k)\}$, respectively) to denote the set of negative edges on the outer cycle (inner cycles, respectively) of $GP_\sigma(n,k)$. We let $E^{\sigma-}_{c}(GP_\sigma(n,k)) = E^{\sigma-}_{c_0}(GP_\sigma(n,k)) \cup (\bigcup_{l=1}^{gcd(n,k)} E^{\sigma-}_{c_l}(GP_\sigma(n,k)))$. We denote negative edges in spokes set M_s of $GP_\sigma(n,k)$ by $E^{\sigma-}_{s}(GP_\sigma(n,k))$. We set $E^{\sigma-}_{s}(M) = E^{\sigma-}_{s}(GP_\sigma(n,2)) \cap E_s(M)$. In the same way, $E^{\sigma-}_{c}(M) = E^{\sigma-}_{c}(GP_\sigma(n,2)) \cap E_c(M)$, $E^{\sigma-}(M) = E^{\sigma-}(GP_\sigma(n,2)) \cap E(M)$. Let X be a subset of $E(G)$. We used $G - X$ to denote a graph obtained from G by deleting the subset X.

In [7], Behr gave several properties of edge coloring of some special structures of signed graphs.

Lemma 1 ([7])**.** *Every signed path can be properly edge colored with $\pm a$ (where $a \neq 0$). Furthermore, every signed path has exactly two different $\pm a$ colorings.*

Lemma 2 ([7])**.** *A signed cycle C can be properly edge colored with $\pm a$ (where $a \neq 0$) if and only if C is balanced. Furthermore, every balanced cycle has exactly two different $\pm a$ colorings.*

Behr also showed that switching does not affect the edge chromatic number of signed graphs.

Lemma 3 ([7])**.** *Suppose γ is a proper n-coloring of G_σ and $G_{\sigma'}$ is obtained from G_σ by switching a vertex set X. Define a new coloring γ' which is obtained from γ by negating all colors on all incidences involving vertices from X. Then, γ' is a proper n-coloring of $G_{\sigma'}$.*

Zaslavsky [19] gave the necessary and sufficient conditions of switching equivalence of two signed graphs G_σ and $G_{\sigma'}$.

Lemma 4 ([19])**.** *Two signed graphs G_σ and $G_{\sigma'}$ are switching equivalent if and only if they have the same set of unbalanced cycles.*

Since we will use matchings of the generalized Petersen graph to complete our proofs, we describe the structures of matchings of generalized Petersen graphs.

Proposition 1 ([20])**.** *Let $\mathbb{M}(GP(n,2))$ be the set of perfect matchings of $GP(n,2)$.*

(1) For $M \in \mathbb{M}(GP(n,2))$, if there are no spokes in M, then M should be one of the two perfect matchings illustrated in Figure 1(a,b); if there are spokes in M, then the number of spokes between any two consecutive spokes in M is even. Specifically, if there is only one spoke, it implies that n is odd.

(2) Let α denote the number of spokes between any two consecutive spokes in perfect matching M. $\mathbb{M}(GP(n,2))$ can be divided into two subsets:

$$\mathbb{M}_1(GP(n,2)) = \{M \in \mathbb{M}(GP(n,2)|\alpha \equiv 0\ (mod\ 4) \text{ or } M \text{ is illustrated in Figure 1(a)}\}, \quad (1)$$

$$\mathbb{M}_2(GP(n,2)) = \{M \in \mathbb{M}(GP(n,2))|\alpha \equiv 2\ (mod\ 4) \text{or } M \text{ is illustrated in Figure 1(b)}\}. \quad (2)$$

(3) Let a, b, c and d denote the numbers of A, B, C and D, respectively. Each perfect matching in $\mathbb{M}_1(GP(n,2))$ consists of a sequence of A and B with $4a + b = n$ in Figure 1(c). Each perfect matching in $\mathbb{M}_2(GP(n,2))$ consists of a sequence of C and D with $3c + 4d = n$ in Figure 1(d).

Refer to Proposition 1. We use $M_{B^bA^a}$ with $4a + b = n$ to denote perfect matching $M \in \mathbb{M}_1(GP(n,2))$, and use $M_{C^cD^d}$ with $3c + 4d = n$ to denote $M \in \mathbb{M}_2(P(n,2))$.

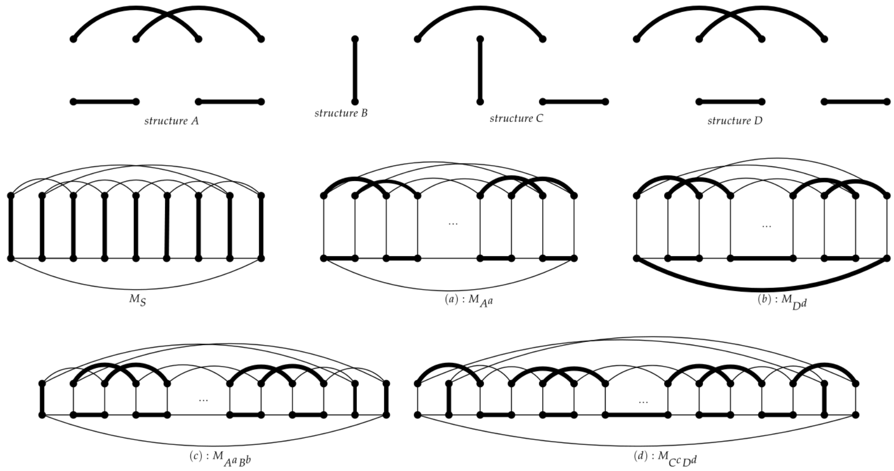

Figure 1. All types of perfect matchings of $GP(n,2)$. Here, bold lines denote the edges in the perfect matching.

Cai et al. gave the following lemma and proposition, which also play an important role in our proofs.

Lemma 5 ([18]). *If $GP_\sigma(n,k)$ has a perfect matching, denoted by M_0, and $(GP_\sigma(n,k) - M_0)$ is formed of balanced cycles, then $GP_\sigma(n,k)$ is 3-edge-colorable.*

Proposition 2 ([18]). *For $GP_\sigma(n,k)$ with any signature σ, let $\sigma' \in [\sigma]$ such that $|E^{\sigma'-}|$ is minimized; then all of the following results hold.*
(1) $|E^{\sigma'-}_{c_l}| \le 1$, where $l \in \{0,1,2,\cdots,gcd(n,k)\}$;
(2) $|E^{\sigma'-}_{s}| \le \lfloor \frac{n}{2} \rfloor$;
(3) $E^{\sigma'-}$ forms a matching of $GP_\sigma(n,k)$.

By [20], $GP_\sigma(n,2)$ has the four types of perfect matchings: (1) the matching consists of all spokes, denoted by M_s; (2) the matching is only formed of structure $A(C,D$, respectively), denoted by $M_{A^q}(M_{C^q}, M_{D^q}, respectively)$, where $n = 4q(n = 3q, n = 4q, respectively)$; (3) the matching consists of structures A and B, denoted by $M_{A^sB^t}$, where $n = 4s + t$ (here we let these structures A be consecutive; that is, $\overbrace{AA\cdots A}^{s}\overbrace{BB\cdots B}^{t}$); (4) the matching consists of structures C and D, denoted by $M_{C^sD^t}$, where $n = 3s + 4t$ (here we let these structures C be consecutive). If n takes some certain values, we can obtain several special perfect matchings of $GP(n,2)$. For proving later results, we make use of the following matchings.

Proposition 3. *There are several perfect matchings of $GP(n,2)$ to be used in our proof, which are*
(1) M_s and M_{C^p}, when $n \equiv 3 \bmod 6 (n \ge 9)$;
(2) M_s, M_{A^p}, M_{D^p} and $M_{A^sB^t}(4s + t = n)$, when $n = 4p$;
(3) M_s, $M_{C^2D^{p-1}}$ and $M_{A^sB^t}(4s + t = n)$, when $n = 4p + 2$.

By Proposition 3, it is easy to check that for $n \equiv 3 \bmod 6 (n \ge 9)$, $GP(n,2)$ has the perfect matching M_{C^p}. Moreover, the edge set of M_{C^p} has special properties. Thus, we give the specific edge set of the perfect matching M_{C^p}.

Definition 1. *For $n \equiv 3$ mod $6 (n \geq 9)$, let $i \in \{1,2,3\}$ and $E(M_{C_i^p}) = \{u_j v_j, u_{j+1} u_{j+2}, v_{j+2} v_{j+4} : j \equiv i (\text{mod } 3), 1 \leq j \leq n\}$. Furthermore, $M_{C_i^p}$'s is a partition of $GP(n,2)$ that is $\bigcup_i E_s(M_{C_i^p}) = E_s(GP(n,2))$ and $E_s(M_{C_h^p}) \cap E_s(M_{C_l^p}) = \emptyset$, where $h, l \in \{1,2,3\}$.*

In Figure 2, we give an example of the matching $M_{C_i^p}$ for $n = 9$.

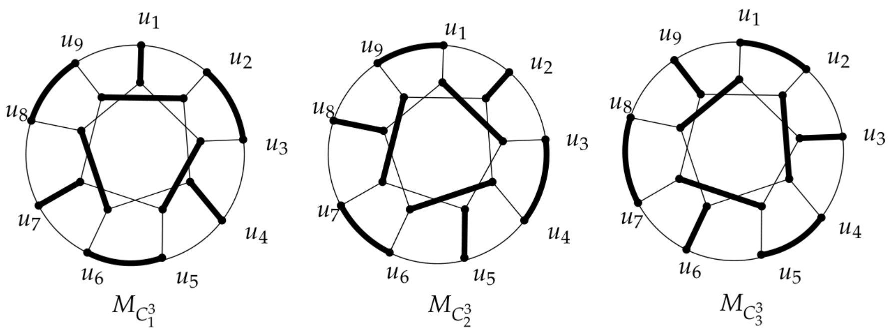

Figure 2. The matching $M_{C_i^3} (i \in \{1,2,3\})$ of $GP(9,2)$. Here bold lines denote the edges in the perfect matching.

Proposition 4. *If $n \equiv 3$ mod $6 (n \geq 9)$, $GP(n,2) - M_{C_i^p}$ is a Hamilton cycle for $i \in \{1,2,3\}$.*

Proof. Due to symmetry, we prove the case $i = 1$, and other cases can be proved using the same method. All subscripts are modulo n. Let $H = GP(n,2) - M_{C_1^p}$. Firstly, it is easy to check that H is a 2-regular graph. Moreover, H has three kinds of edge: (a) spokes: $\{u_j v_j, u_{j+1} v_{j+1} : j \equiv 2 (\text{mod } 3), 1 \leq j \leq n\}$; (b) outer-cycle edges: $\{u_{j-1} u_j, u_j u_{j+1} : j \equiv 1 (\text{mod } 3), 1 \leq j \leq n\}$; (c) inner-cycle edges: $\{v_j v_{j+2}, v_{j+2} v_{j+4} : j \equiv 2 (\text{mod } 3), 1 \leq j \leq n\}$.

To show $GP(n,2) - M_{C_i^p}$ is a Hamilton cycle, we use $\{u_{3t}, u_{3t+1}, u_{3t+2}, v_{3t+2}, v_{3t+4}, v_{3t+6} : 1 \leq t \leq p\}$ to denote p 6-paths. It is easy to check that $\{u_{3t} u_{3t+1}, u_{3t+1} u_{3t+2}\} \subseteq E_{c_0}(H)$ and $\{v_{3t+2} v_{3t+4}, v_{3t+4} v_{3t+6}\} \subseteq E_{c_r}(H)$. Then p 6-paths contain all edges of H. Therefore, H is a Hamilton cycle, that is $u_3, u_4, u_5, v_5, v_7, v_9, u_9, \cdots, u_{3t}, u_{3t+1}, u_{3t+2}, v_{3t+2}, v_{3t+4}, v_{3t+6}, \cdots, u_{3p-3}, u_{3p-2}, u_{3p-1}, v_{3p-1}, v_{3p+1}, v_3, u_3$, where $1 \leq t \leq p$. $\square$

According to Proposition 3, for $n = 2p (p \geq 4)$, there are several special perfect matchings of the generalized Petersen graph $GP(n,2)$. Next, we discuss the graph obtained from $GP(n,2)$ by deleting the above perfect matching. If $n = 2p$, the generalized Petersen graph $GP(n,2)$ has an outer cycle c_0 and two inner cycles denoted by c_1 and c_2. For convenience, we set $c_r = c_1 \cup c_2$. In the following, we characterize several matchings that play an important role in our proofs.

In Figure 3, we depict three kinds of perfect matching of $GP(12,2)$.

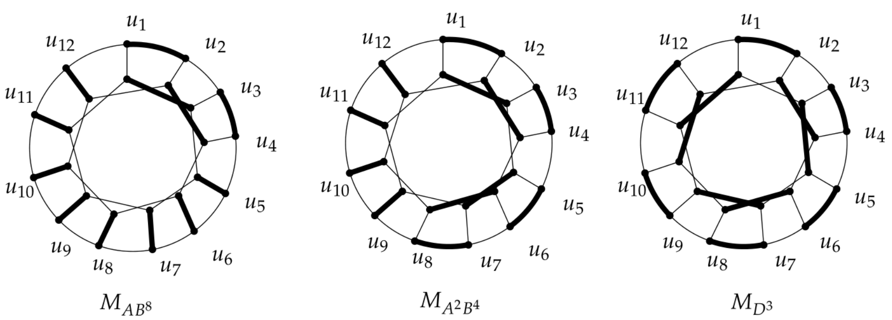

Figure 3. For $i = 1$, the perfect matchings $M_{A_i^h B^s} (h \in \{1,2\})$ and $M_{D_i^p}$ of $GP(12,2)$. Here, bold lines denote the edges in the perfect matching.

Definition 2. *Here we define the edge sets of three special perfect matchings.*

Case 1. *If $n = 4p(p \geq 2)$, for $i \in \{1, 2, 3, 4\}$, the perfect matching $M_{D_i^p}$ has the following edges:*

(i) $E_s(M_{D_i^p}) = \varnothing$;
(ii) $E_{c_0}(M_{D_i^p}) = \{u_{j-2}u_{j-1}, u_j u_{j+1} : j \equiv i (\text{mod } 4), 1 \leq j \leq n\}$;
(iii) $E_{c_r}(M_{D_i^p}) = \{v_{j-1}v_{j+1}, v_j v_{j+2} : j \equiv (i+2)(\text{mod } 4), 1 \leq j \leq n\}$.

Case 2. *If $n = 2p(p \geq 4)$, for $1 \leq i \leq n$, the perfect matching $M_{A_i^h B^s}$ has the following edges:*

(i) $E_s(M_{A_i^h B^s}) = \{u_j v_j : i + 4h \leq j \leq i + (n-1)\}$;
(ii) $E_{c_0}(M_{A_i^h B^s}) = \{u_{i+4t}u_{i+4t+1}, u_{i+4t+2}u_{i+4t+3} : 0 \leq t \leq h-1\}$;
(iii) $E_{c_r}(M_{A_i^h B^s}) = \{v_{i+4t}v_{i+4t+2}, v_{i+4t+1}v_{i+4t+3} : 0 \leq t \leq h-1\}$.

Case 3. *If $n = 4p + 2(p \geq 2)$, for $1 \leq i \leq n$, the perfect matching $M_{C_i^2 D^{p-1}}$ has the following edges:*

(i) $E_s(M_{C_i^2 D^{p-1}}) = \{u_i v_i, u_{i+3}v_{i+3}\}$;
(ii) $E_{c_0}(M_{C_i^2 D^{p-1}}) = \{u_{i+1}u_{i+2}, u_{i+2t}u_{i+2t+1} : 2 \leq t \leq \frac{n-i-1}{2}\}$;
(iii) $E_{c_r}(M_{C_i^2 D^{p-1}}) = \{v_{i+(n-1)}v_{i+1}, v_{i+2}v_{i+4}, v_{i+4t+1}v_{i+4t+3}, v_{i+4t+2}v_{i+4t+4} : 1 \leq t \leq p-1\}$.

Proposition 5. *For $n = 4p(p \geq 2)$ and $i \in \{1, 2, 3, 4\}$, $GP(n, 2) - M_{D_i^p}$ consists of p cycles.*

Proof. Without loss of generality, we prove the case $i = 1$. Let $R = GP(n, 2) - M_{D_1^p}$. It is easy to check that R is a 2-regular graph. Moreover, by Definition 2(case 1), R has three kinds of edge: (a) spokes: all; (b) outer-cycle edges: $E_{c_0}(R) = \{u_{j-2}u_{j-1}, u_j u_{j+1} : j \equiv 2(\text{mod } 4), 1 \leq j \leq n\}$; (c) inner-cycle edges: $E_{c_r}(R) = \{v_{j-1}v_{j+1}, v_j v_{j+2} : j \equiv 1(\text{mod } 4), 1 \leq j \leq n\}$.

R has p 8-cycles: $\{u_{4t}, u_{4t+1}, v_{4t+1}, v_{4t+3}, u_{4t+3}, u_{4t+2}, v_{4t+2}, v_{4t}, u_{4t} : 1 \leq t \leq p\}$. Each cycle contains two outer-cycle edges and two inner-cycle edges, and it is easy to check that $\{u_{4t}u_{4t+1}, u_{4t+2}u_{4t+3}\} \subseteq E_{c_0}(R)$ and $\{v_{4t}v_{4t+2}, v_{4t+1}v_{4t+3}\} \subseteq E_{c_r}(R)$. Then, p 8-cycles contain all edges of R. Thus, $GP(n, 2) - M_{D^p}$ consists of p 8-cycles, where $n = 4p$. □

Proposition 6. *For $n = 2p(p \geq 4)$ and $1 \leq i \leq n$, $GP_\sigma(n, 2) - M_{A_i^h B^s}$ is a Hamilton cycle, where $n = 4h + s$.*

Proof. Let $W_i^h = GP(n, 2) - M_{A_i^h B^s}$. Without loss of generality, we prove the case $i = 1$. It is easy to check that W_1^h is a 2-regular graph. All subscripts are modulo n.

(1) When $h = 1$, by Definition 2(case 2), W_1^1 has three kinds of edge: (a) spokes: $E_s(W_1^1) = \{u_1 v_1, u_2 v_2, u_3 v_3, u_4 v_4\}$; (b) outer-cycle edges: $E_{c_0}(W_1^1) = \{u_2 u_3, u_j u_{j+1} : 4 \leq j \leq n\}$; (c) inner-cycle edges: $E_{c_r}(W_1^1) = \{v_j v_{j+2} : 3 \leq j \leq n\}$.

Firstly, starting at u_2, it passes through u_3; then it passes the edge $u_3 v_3$ to the inner cycle. In the inner cycle it consecutively passes vertices $\{v_{2t+3} : 0 \leq t \leq (p-1)\}$. Here it goes through p inner-cycle vertices with odd subscripts.

Next, it passes the edge $u_1 v_1$ to the outer cycle. It will pass vertices $u_1, u_n, u_{n-1}, u_{n-2}, \cdots, u_5, u_4$. Then, it passes through $n-2$ outer-cycle vertices.

Then, it can pass the edge $u_4 v_4$ to the inner cycle. On the inner cycle it will pass $\{v_{2t} : 2 \leq t \leq (p+1)\}$. Here it passes through p inner-cycle vertices with even subscripts. Finally, it will pass the edge $v_2 u_2$ to the outer cycle. Now we obtain a cycle. Moreover, the cycle passes through all outer-cycle vertices and inner-cycle vertices. Thus, W_1^1 is a Hamilton cycle; that is, $u_2, u_3, v_3, v_5, \cdots, v_{n-1}, v_1, u_1, u_n, u_{n-1}, u_{n-2}, \cdots, u_5, u_4, v_4, v_6, \cdots, v_n, v_2$.

(2) When $h \geq 2$, by Definition 2(case 2), W_1^h has three kinds of edge: (a) spokes: $E_s(W_1^h) = \{u_j v_j : 1 \leq j \leq 4h\}$; (b) outer-cycle edges: $E_{c_0}(W_1^h) = \{u_{2t}u_{2t+1}, u_j u_{j+1} : 1 \leq$

$t \leq 2h-1, 4h \leq j \leq n\}$; (c) inner-cycle edges: $E_{c_r}(W_1^h) = \{v_{4t-1}v_{4t+1}, v_{4t}v_{4t+2}, v_jv_{j+2} : 1 \leq t \leq h-1, 4h \leq j \leq n\}$.

Firstly, starting at u_2, it goes through the following vertices: $\{u_2, u_3; v_3, v_5, u_5, u_4, v_4, v_6, u_6, u_7; v_7, v_9, u_9, u_8, v_8, v_{10}, u_{10}, u_{11}; \cdots ; v_{4h-5}, v_{4h-3}, u_{4h-3}, u_{4h-4}, v_{4h-4}, v_{4h-2}, u_{4h-2}, \quad u_{4h-1}\}$. Now it passes $4h-2$ outer-cycle vertices and $2(h-1)$ inner-cycle vertices with odd subscripts and even subscripts, respectively.

Secondly, it passes the edge $u_{4h-1}v_{4h-1}$ to the inner cycle. On the inner cycle it will passes vertices $\{v_{2t+1} : 2h-1 \leq t \leq p\}$. Here, it passes through $p-2h+2$ inner-cycle vertices with odd subscripts.

Thirdly, it passes the edge u_1v_1 to the outer cycle; it passes vertices: $\{u_1, u_n, u_{n-1}, \cdots, u_{4h}\}$. Here it goes through $n-4h+2$ outer-cycle vertices.

Next, it can pass the edge $u_{4h}v_{4h}$ to the inner cycle. In the inner cycle it will passes the following vertices: $\{v_{2t} : 2h \leq t \leq p+1\}$. Here it passes through $p-2h+2$ inner-cycle vertices with even subscripts.

Finally, it will pass the edge v_2u_2 to the outer cycle. Now, we obtain a cycle. Furthermore, the cycle passes through all inner-cycle vertices and outer-cycle vertices. Thus, W_1^h is Hamilton cycle; that is, $u_2, u_2, u_3, v_3, v_5, u_5, u_4, v_4, v_6, u_6, u_7, v_7, v_9, u_9, u_8, v_8, v_{10}, u_{10}, u_{11}, \cdots, v_{4h-5}, v_{4h-3}, u_{4h-3}, u_{4h-4}, v_{4h-4}, v_{4h-2}, u_{4h-2}, u_{4h-1}, v_{4h-1}, v_{4h+1}, \cdots, v_1, u_1, u_n, u_{n-1}, \cdots, u_{4h}, v_{4h}, v_{4h+2}, \cdots, v_n, v_2, u_2$. □

We portray three kinds of perfect matchings of $GP(14,2)$ in Figure 4.

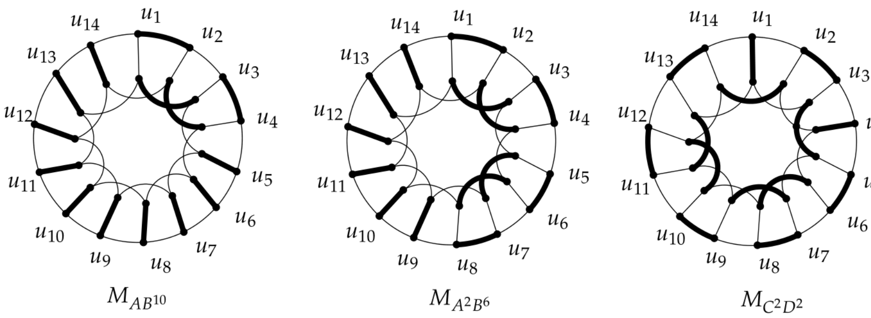

Figure 4. For $i=1$, the perfect matchings $M_{A_i^hB^s}(h \in \{1,2\})$ and $M_{C_i^2D^{p-1}}$ of $GP(14,2)$. Here, bold lines denote the edges in the perfect matching.

Proposition 7. *For $n = 4p+2$ and $i \in \{1, 2, \cdots, n\}$,*

(1) when $p = 2$, $GP(10,2) - M_{C_i^2D}$ is a Hamilton cycle;

(2) when $p > 2$, $GP(n,2) - M_{C_i^2D^{p-1}}$ consists of $p-2$ cycles of length 8 and a cycle of length 20.

Proof. In the following, we prove the case $i=1$. Due to symmetry, the other cases can be proved by the same method.

(1) Let $V = GP(10,2) - M_{C_1^2D}$. By Definition 2(case 3), then V has three kinds of edge: (a) spokes: $\{u_2v_2, u_3v_3, u_jv_j : 5 \leq j \leq 10\}$; (b) outer-cycle edges: $\{u_1u_2, u_3u_4, u_4u_5, u_6u_7, u_8u_9, u_{10}u_1\}$; (c) inner-cycle edges: $\{v_1v_3, v_2v_4, v_4v_6, v_5v_7, v_8v_{10}, v_9v_1\}$.

Starting at u_1, we can obtain a Hamilton cycle, which is $u_1, u_2, v_2, v_4, v_6, u_6, u_7, v_7, v_5, u_5, u_4, u_3, v_3, v_1, v_9, u_9, u_8, v_8, v_{10}, u_{10}, u_1$. Thus, V is a Hamilton cycle.

(2) Let $V' = GP(n,2) - M_{C_1^2D^{p-1}}$. By Definition 2(case 3), V' has three kinds of edge: (a) spokes: $\{u_2v_2, u_3v_3, u_tv_t : 5 \leq t \leq n\}$; (b) outer-cycle edges: $\{u_1u_2, u_3u_4, u_{4t}u_{4t+1}, u_{4t+2}u_{4t+3} : 1 \leq t \leq p\}$; (c) inner-cycle edges: $\{v_1v_3, v_2v_4, v_{4t}v_{4t+2}, v_{4t+1}v_{4t+3} : 1 \leq t \leq p\}$. The graph V' consists of $2p+2$ outer-cycle edges and inner-cycle edges, respectively, and $4p$ spokes.

Firstly, starting at point u_1, we obtain a 20-cycle T: $u_1, u_2, v_2, v_4, v_6, u_6, u_7, v_7, v_5, u_5, u_4, u_3, v_3, v_1, v_{n-1}, u_{n-1}, u_{n-2}, v_{n-2}, v_n, u_n, u_1$. The cycle T consists of six outer-cycle edges, six inner-cycle edges and eight spokes.

Furthermore, starting at points $u_{4l} : 2 \le l \le p-1$, we get $p-2$ cycles $\{u_{4l}, u_{4l+1}, v_{4l+1}, v_{4l+3}, u_{4l+3}, u_{4l+2}, v_{4l+2}, v_{4l}, u_{4l} : 2 \le l \le p-1\}$. It is easy to check that $\{u_{4l}v_{4l}, u_{4l+1}v_{4l+1}, u_{4l+2}v_{4l+2}, u_{4l+3}v_{4l+3}\} \subseteq E_s(V')$, $\{u_{4l}u_{4l+1}, u_{4l+2}u_{4l+3}\} \subseteq E_{c_0}(V')$ and $\{v_{4l}v_{4l+2}, v_{4l+1} v_{4l+3}\} \subseteq E_{c_r}(V')$. Moreover, the 20-cycle and $p-2$ 8-cycles contain all edges of V'. Thus, $GP(n,2) - M_{C^2D^{p-1}}$ consists of $p-2$ 8-cycles and a 20-cycle, where $n = 6+4p(p>2)$. □

Definition 3. *By Proposition 5, there is always an 8-cycle between two adjacent structures D. Let* $W_t = GP(n,2) - M_{A_tB^{n-4}}$*, and we denote the 8-cycles by* $O_t = \{u_t, u_{t+1}, v_{t+1}, v_{t+3}, u_{t+3}, u_{t+2}, v_{t+2}, v_t, u_t : 1 \le t \le n\}$*. Moreover,* $E_c(M_{A_tB^{n-4}}) = \{u_tu_{t+1}, u_{t+2}u_{t+3}, v_{t+1}v_{t+3}, v_tv_{t+2}\}$*,* $E_s(W_t) = \{u_tv_t, u_{t+1}v_{t+1}, u_{t+2}v_{t+2}, u_{t+3}v_{t+3}\}$*. It is easy to get that* $E(O_t) = E_c(M_{A_tB^{n-4}}) \cup E_s(W_t)$*.*

3. Main Results and Proof

In the following, we introduce our main results and proofs.

For $n \equiv 3 \bmod 6 (n \ge 9)$, the signed generalized Petersen graph $GP_\sigma(n,2)$ has perfect matchings $M_{C_i^p} (1 \le i \le 3)$. Next, we study the parity of $|E_s^{\sigma-}(GP_\sigma(n,2))|$ and $|E_s^{\sigma-}(M_{C_i^p})|$.

Lemma 6. *For any signature* σ*,* $n \equiv 3 \bmod 6 (n \ge 9)$*. Let* $i, j, k \in \{1,2,3\}$ *and* $j,k \ne i$*. If* $|E_s^{\sigma-}(GP_\sigma(n,2))|$ *and* $|E_s^{\sigma-}(M_{C_i^p})|$ *have opposite parity, then* $|E_s^{\sigma-}(M_{C_j^p})|$ *and* $|E_s^{\sigma-}(M_{C_k^p})|$ *have opposite parity.*

Proof. Without loss of generality, we assume that $|E_s^{\sigma-}(GP_\sigma(n,2))|$ is odd and $|E_s^{\sigma-}(M_{C_i^p})|$ is even. By the Definition 1, $|E_s^{\sigma-}(M_{C_j^p}) \cup E_s^{\sigma-}(M_{C_k^p})|$ is odd. Then, $|E_s^{\sigma-}(M_{C_j^p})|$ and $|E_s^{\sigma-}(M_{C_k^p})|$ have opposite parity, since $E_s(M_{C_j^p}) \cap E_s(M_{C_k^p}) = \varnothing$. □

By Proposition 2, each cycle has at most one negative edge. If $|E_c^{\sigma-}(GP(n,2))| = 1$, due to symmetry, we can assume that the negative edge is on the outer cycle.

Lemma 7. *Let* $E_c^{\sigma-}(GP_\sigma(n,2)) = \{u_1u_2\}$*. For* $i \in \{1,2\}$*, if* $|E_s^{\sigma-}(GP_\sigma(n,2))|$ *and* $|E_s^{\sigma-}(M_{C_i^p})|$ *have opposite parity, then* $GP_\sigma(n,2) - M_{C_i^p}$ *is a balanced Hamilton cycle.*

Proof. According to Proposition 4, we know that $GP_\sigma(n,2) - M_{C_i^p}$ is a Hamilton cycle. Without loss of generality, for $i \in \{1,2\}$, if we assume that $|E_s^{\sigma-}(GP_\sigma(n,2))|$ is odd and $|E_s^{\sigma-}(M_{C_i^p})|$ is even, then $|E_s^{\sigma-}(GP_\sigma(n,2))| - |E_s^{\sigma-}(M_{C_i^p})|$ is odd. For $i \in \{1,2\}$, it is easy to check that $u_1u_2 \notin E(M_{C_i^p})$. There are even negative edges in the Hamilton cycle. Thus, $GP_\sigma(n,2) - M_{C_i^p}$ is a balanced Hamilton cycle. □

In the following, we consider the case that there is a negative edge on each cycle.

Lemma 8. *Let* $E_c^{\sigma-}(GP_\sigma(n,2)) = \{u_1u_2, v_jv_{j+2}\}$*. For* $i \in \{1,2,3\}$*, if* $|E_s^{\sigma-}(GP_\sigma(n,2))|$ *and* $|E^{\sigma-}(M_{C_i^p})|$ *have the same parity, then* $GP_\sigma(n,2) - M_{C_i^p}$ *is a balanced Hamilton cycle.*

Proof. $|E^{\sigma-}(GP_\sigma(n,2))| = |E_s^{\sigma-}(GP_\sigma(n,2))| + |E_c^{\sigma-}(GP_\sigma(n,2))|$. It is easy to check that $|E_s^{\sigma-}(GP_\sigma(n,2))|$ and $|E^{\sigma-}(GP_\sigma(n,2))|$ have the same parity. If $|E_s^{\sigma-}(GP_\sigma(n,2))|$ and $|E^{\sigma-}(M_{C_i^p})|$ have the same parity, then $|E^{\sigma-}(GP_\sigma(n,2))|$ and $|E^{\sigma-}(M_{C_i^p})|$ have the same parity. Therefore, $GP_\sigma(n,2) - M_{C_i^p}$ is a balanced Hamilton cycle. □

The following theorem settles the issue for $k=2$, $n \equiv 3 \bmod 6 (n \ge 9)$.

Theorem 1. *For any* σ*,* $\chi'(GP_\sigma(n,2)) = 3$*, where* $n \equiv 3 \bmod 6 (n \ge 9)$*.*

Proof. Firstly, we show that there is always a perfect matching M such that $GP_\sigma(n,2) - M$ consists of balanced cycles.

Claim 1. *For $n \equiv 3 \bmod 6 (n \geq 9)$, there must be a perfect matching M such that $GP_\sigma(n,2) - M$ consists of balanced cycles.*

Proof of Claim 1. When $n \equiv 3 \bmod 6 (n \geq 9)$, the generalized Petersen graph $GP(n,2)$ has an outer cycle c_0 and an inner cycle c_1 since $gcd(n,2) = 1$. Let $\sigma' \in [\sigma]$ such that $E^{\sigma'-}$ is minimized. According to Proposition 2, $|E^{\sigma'-}_{c_0}(GP(n,2))| \leq 1$, $|E^{\sigma'-}_{c_1}(GP(n,2))| \leq 1$. In the following, we continue the proof according to the number of negative edges on the outer cycle and the inner cycle.

Case 1. $|E^{\sigma'-}_{c_0}(GP_{\sigma'}(n,2))| = 0$ and $|E^{\sigma'-}_{c_1}(GP_{\sigma'}(n,2))| = 0$.

In this case, $E^{\sigma'-}(GP(n,2)) \subseteq E(M_s)$; then, $GP_{\sigma'}(n,2) - M_s$ consists of two balanced cycles.

If $|E^{\sigma'-}_{c}(GP_{\sigma'}(n,2))| = 1$, then the negative edge may be on the outer cycle or the inner cycle. Due to symmetry, we assume that the negative edge is on the outer cycle. By Proposition 4, $GP_{\sigma'}(n,2) - M_{c_i^p} (i \in \{1,2,3\})$ is a Hamilton cycle.

Case 2. $|E^{\sigma'-}_{c_0}(GP_{\sigma'}(n,2))| = 1$ and $|E^{\sigma'-}_{c_1}(GP_{\sigma'}(n,2))| = 0$.

Without loss of generality, let $\sigma'(u_1u_2) = -1$. In this case, $u_1u_2 \notin E(M_{C_i^p})$ for $i \in \{1,2\}$, $u_1u_2 \in E(M_{C_3^p})$. If $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_3^p})|$ have the same parity, then $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))| - |E^{\sigma'-}_s(M_{C_3^p})|$ is even. Thus, $GP_{\sigma'}(n,2) - M_{C_3^p}$ is a balanced Hamilton cycle, since $u_1u_2 \in E(M_{C_3^p})$. If $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_3^p})|$ have opposite parity, then by Lemma 6, $|E^{\sigma'-}_s(M_{C_1^p})|$ and $|E^{\sigma'-}_s(M_{C_2^p})|$ have opposite parity. There must be a matching $M_{C_j^p}$ such that $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_j^p})|$ have opposite parity for $j \in \{1,2\}$. By Lemma 7, $GP_{\sigma'}(n,2) - M_{C_j^p}$ is a balanced Hamilton cycle.

Case 3. $|E^{\sigma'-}_{c_0}(GP_{\sigma'}(n,2))| = 1$ and $|E^{\sigma'-}_{c_1}(GP_{\sigma'}(n,2))| = 1$.

Let $E^{\sigma'-}_c(GP_{\sigma'}(n,2)) = \{u_1u_2, v_jv_{j+2}\}$. We continue our proof according to whether the edge v_jv_{j+2} is on the matching $M_{C_3^p}$.

Subcase 3.1. $v_jv_{j+2} \in E(M_{C_3^p})$.

In this case, $|E^{\sigma'-}_c(M_{C_i^p})|$ is even for $i \in \{1,2,3\}$. It is not difficult to get that $|E^{\sigma'-}(M_{C_i^p})| = |E^{\sigma'-}_c(M_{C_i^p})| + |E^{\sigma'-}_s(M_{C_i^p})|$. If $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_3^p})|$ have the same parity, then $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}(M_{C_3^p})|$ have the same parity. By Lemma 8, $GP_{\sigma'}(n,2) - M_{C_3^p}$ is a balanced Hamilton cycle. If $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_3^p})|$ have opposite parity, then by Lemma 6, $|E^{\sigma'-}_s(M_{C_1^p})|$ and $|E^{\sigma'-}_s(M_{C_2^p})|$ have opposite parity. There must be a matching $M_{C_t^p}$ such that $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_t^p})|$ have the same parity for $t \in \{1,2\}$, so $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}(M_{C_t^p})|$ have the same parity. By Lemma 8, $GP_{\sigma'}(n,2) - M_{C_t^p}$ is a balanced Hamilton cycle.

Subcase 3.2. $v_jv_{j+2} \notin E(M_{C_3^p})$.

By Definition 1, $M_{C_i^p}$'s are a partition of $GP_{\sigma'}(n,2)$. As $v_jv_{j+2} \notin E(M_{C_3^p})$, $v_jv_{j+2} \in E(M_{C_i^p})$ for $i \in \{1,2\}$. Due to symmetry, we only consider the case $v_jv_{j+2} \in E(M_{C_1^p})$. In this case, $|E^{\sigma'-}_c(M_{C_i^p})|$ is odd for $i \in \{1,3\}$ and $|E^{\sigma'-}_c(M_{C_2^p})|$ is even. It is not difficult to get that $|E^{\sigma'-}(M_{C_i^p})| = |E^{\sigma'-}_c(M_{C_i^p})| + |E^{\sigma'-}_s(M_{C_i^p})|$. If $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_2^p})|$ have the same parity, then $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}(M_{C_2^p})|$ have the same parity. By Lemma 8, $GP_{\sigma'}(n,2) - M_{C_2^p}$ is a balanced Hamilton cycle. If $|E^{\sigma'-}_s(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}_s(M_{C_2^p})|$ have opposite parity, then by Lemma 6, $|E^{\sigma'-}_s(M_{C_1^p})|$ and $|E^{\sigma'-}_s(M_{C_3^p})|$ have opposite parity. There must be a matching $M_{C_t^p}$ such that $|E_s^{\sigma'-}(GP_{\sigma'}(n,2))|$ and $|E_s^{\sigma'-}(M_{C_t^p})|$ have opposite parity for $t \in \{1,3\}$, so $|E_s^{\sigma'-}(GP_{\sigma'}(n,2))|$ and $|E^{\sigma'-}(M_{C_t^p})|$ have the same parity. By Lemma 8, $GP_{\sigma'}(n,2) - M_{C_t^p}$ is a balanced Hamilton cycle. □

By Claim 1, there is always a matching M such that $GP_{\sigma'}(n,2) - M$ consists of balanced cycles. Then, $\chi'(GP_{\sigma'}(n,2)) = 3$ by Lemma 5, where $n \equiv 3 \bmod 6 (n \geq 9)$. □

In the following, we study the edge coloring of the generalized Petersen graph $GP_\sigma(n,2)$ for any signature σ and $n = 2p(p \geq 4)$.

Lemma 9. *Let $W_i^h = GP_\sigma(n,2) - M_{A_i^h B^s}$. For $n = 2p$, if we can find a matching $M_{A_i^h B^s}$ satisfying one of the following conditions:*

(i) $|E_c^{\sigma-}(GP_\sigma(n,2))|$ is odd. $|E_c^{\sigma-}(M_{A_i^h B^s})|$ and $|E_s^{\sigma-}(W_i^h)|$ have opposite parity.

(ii) $|E_c^{\sigma-}(GP_\sigma(n,2))|$ is even. $|E_c^{\sigma-}(M_{A_i^h B^s})|$ and $|E_s^{\sigma-}(W_i^h)|$ have the same parity.

Then W_i^h is a balanced Hamilton cycle.

Proof. By Proposition 6, W_i^h is a Hamilton cycle.

(i) It is easy to check that $|E_c^{\sigma-}(GP_\sigma(n,2))| = |E_c^{\sigma-}(W_i^h)| + |E_c^{\sigma-}(M_{A_i^h B^s})|$. When $|E_c^{\sigma-}(GP_\sigma(n,2))|$ is odd, $|E_c^{\sigma-}(W_i^h)|$ and $|E_c^{\sigma-}(M_{A_i^h B^s})|$ have opposite parity. Since $|E_s^{\sigma-}(W_i^h)|$ and $|E_c^{\sigma-}(M_{A_i^h B^s})|$ have opposite parity, $|E_c^{\sigma-}(W_i^h)|$ and $|E_s^{\sigma-}(W_i^h)|$ have the same parity. Moreover, $|E^{\sigma'-}(W_i^h)| = |E_c^{\sigma'-}(W_i^h)| + |E_s^{\sigma'-}(W_i^h)|$, so W_i^h contains even negative edges. Therefore, W_i^h is a balanced Hamilton cycle.

(ii) It is easy to check that $|E_c^{\sigma-}(GP_\sigma(n,2))| = |E_c^{\sigma-}(W_i^h)| + |E_c^{\sigma-}(M_{A_i^h B^s})|$. When $|E_c^{\sigma-}(GP_\sigma(n,2))|$ is even, $|E_c^{\sigma-}(W_i^h)|$ and $|E_c^{\sigma-}(M_{A_i^h B^s})|$ have the same parity. Since $|E_s^{\sigma-}(W_i^h)|$ and $|E_c^{\sigma-}(M_{A_i^h B^s})|$ have the same parity, $|E_c^{\sigma-}(W_i^h)|$ and $|E_s^{\sigma-}(W_i^h)|$ have the same parity. Moreover, $|E^{\sigma'-}(W_i^h)| = |E_c^{\sigma'-}(W_i^h)| + |E_s^{\sigma'-}(W_i^h)|$, so W_i^h contains even negative edges. Therefore, W_i^h is a balanced Hamilton cycle. □

Theorem 2. *For any σ, $\chi'(GP_\sigma(n,2)) = 3$, where $n = 2p(p \geq 4)$.*

Proof. Firstly, we show that there is always a perfect matching M such that $GP_\sigma(n,2) - M$ consists of balanced cycles.

Claim 2. *For $n = 2p(p \geq 4)$, there must be a perfect matching M such that $GP_\sigma(n,2) - M$ consists of balanced cycles.*

Proof of Claim 2. If $n = 2p(p \geq 4)$, then the generalized Petersen graph $GP(n,2)$ has an outer cycle c_0 and two inner cycles c_1, c_2. Let $\sigma' \in [\sigma]$ such that $E^{\sigma'-}$ is minimized. According to Proposition 2, each cycle has at most one negative edge. In the following, we continue the proof according to the number of negative edges on cycles.

Case 1. $|E_c^{\sigma'-}(GP_{\sigma'}(n,2))|$ is odd.

In the following, we continue our discussion according to the parity of p.

Subcase 1.1. p is an odd integer.

Let $W_i = GP_{\sigma'}(n,2) - M_{A_i B^{n-4}}$. It is easy to check that W_i is a Hamilton cycle for $i \in \{1,2,\cdots,n\}$. By Lemma 9(i), if we can find a matching $M_{A_i B^{n-4}}$ satisfying the condition, then $GP_{\sigma'}(n,2) - M_{A_i B^{n-4}}$ is a balanced Hamilton cycle. Otherwise, for arbitrary i, $|E_c^{\sigma'-}(M_{A_i B^{n-4}})|$ and $E_s^{\sigma-}(W_i)$ have the same parity.

If $p = 5$, by Proposition 7, $GP_{\sigma'}(n,2) - M_{C_i^2 D}$ is a Hamilton cycle. We denote the Hamilton cycle by T. We set $E(T_l) = \{u_l u_{l+1}, u_{l+2} u_{l+3}, v_l v_{l+2}, v_{l+1} v_{l+3}, u_l v_l, u_{l+1} v_{l+1}, u_{l+2} v_{l+2}, u_{l+3} v_{l+3}\}$, where $l \in \{i-3, i, i+3\}$. It is easy to check that $E(T_l) = E_c(M_{A_l B^{n-4}}) \cup E_s(W_l)$ for $l \in \{i-3, i, i+3\}$ by Definition 3. Since $|E_c^{\sigma'-}(M_{A_l B^{n-4}})|$ and $E_s^{\sigma'-}(W_l)$ have the same parity, $|E^{\sigma'-}(T_l)|$ is even. Furthermore, $E(T) = (E(T_{i-3}) - u_iv_i) \cup (E(T_i) - u_iv_i - u_{i+3}v_{i+3}) \cup (E(T_{i+3}) - u_{i+3}v_{i+3})$, so $|E^{\sigma'-}(T)|$ is even. Therefore, T is a balanced Hamilton cycle.

If $p \neq 5$, by Proposition 7, $GP_{\sigma'}(n,2) - M_{C_i^2D^{p-1}}$ consists of a 20-cycle and $p-2$ 8-cycles. Firstly, we denote the 20-cycle by T. Let $E(T_l) = \{u_lu_{l+1}, u_{l+2}u_{l+3}, v_lv_{l+2}, v_{l+1}v_{l+3}, u_lv_l, u_{l+1}v_{l+1}, u_{l+2}v_{l+2}, u_{l+3}v_{l+3}\}$, where $l \in \{i-3, i, i+3\}$. By Definition 3, it is easy to check that $E(T_l) = E_c(M_{A_lB^{n-4}}) \cup E_s(W_l)$ for $l \in \{i-3, i, i+3\}$, where $W_l = GP_{\sigma'}(n,2) - M_{A_lB^{n-4}}$. Since $|E_c^{\sigma'-}(M_{A_lB^{n-4}})|$ and $E_s^{\sigma-}(W_l)$ have the same parity, $|E^{\sigma'-}(T_l)|$ is even. Furthermore, $E(T) = (E(T_{i-3}) - u_iv_i) \cup (E(T_i) - u_iv_i - u_{i+3}v_{i+3}) \cup (E(T_{i+3}) - u_{i+3}v_{i+3})$, so $|E^{\sigma'-}(T)|$ is even. Therefore, T is a balanced cycle. Next, by Definition 3, we denote $p-2$ 8-cycles by $\{O_t : t = i+7+4q, 0 \leq q \leq p-3\}$. Then, $E(O_t) = E_c(M_{A_tB^{n-4}}) \cup E_s(W_t)$, where $W_t = GP_{\sigma'}(n,2) - M_{A_tB^{n-4}}$. Since $E_c(M_{A_tB^{n-4}}) \cap E_s(W_t) = \varnothing$, $|E^{\sigma'-}(O_t)| = |E_c^{\sigma'-}(M_{A_tB^{n-4}})| + |E_s^{\sigma'-}(W_t)|$. Since $|E_c^{\sigma'-}(M_{A_tB^{n-4}})|$ and $|E_s^{\sigma'-}(W_t)|$ have the same parity, $|E^{\sigma'-}(O_t)|$ is even. Therefore, $p-2$ 8-cycles are balanced.

Subcase 1.2. p is an even integer.

In this case, we continue our proof by using the matching $M_{D_i^p}$. By Proposition 5, $GP_{\sigma'}(n,2) - M_{D_i^p}$ consists of p 8-cycles. By Definition 3, we denote the p 8-cycles by $\{O_j : j \equiv (i+3)(\text{mod } 4)\}$. Then, $E(O_j) = E_c(M_{A_jB^{n-4}}) \cup E_s(W_j)$, where $W_j = GP_{\sigma'}(n,2) - M_{A_jB^{n-4}}$. Since $E_c(M_{A_jB^{n-4}}) \cap E_s(W_j) = \varnothing$, $|E^{\sigma'-}(O_j)| = |E_c^{\sigma'-}(M_{A_jB^{n-4}})| + |E_s^{\sigma'-}(W_j)|$. Since $|E_c^{\sigma'-}(M_{A_jB^{n-4}})|$ and $E_s^{\sigma-}(W_j)$ have the same parity, every 8-cycle contains even negative edges. Therefore, $GP_{\sigma'}(n,2) - M_{D_i^p}$ consists of p balanced cycles.

Case 2. $|E_c^{\sigma'-}(GP'_{\sigma}(n,2))|$ is even.

By Lemma 9(ii), if we can find a matching $M_{A_iB^{n-4}}$ satisfying the condition, then $GP_{\sigma'}(n,2) - M_{A_iB^{n-4}}$ is a balanced Hamilton cycle. Otherwise, for arbitrary i, $|E_c^{\sigma'-}(M_{A_iB^{n-4}})|$ and $|E_s^{\sigma'-}(W_i)|$ have opposite parity.

Next, we continue our proof by using the matching $M_{A_i^2B^{n-8}}$. Let $W_i^2 = GP_{\sigma'}(n,2) - M_{A_i^2B^{n-8}}$. By Proposition 6, it is easy to know that W_i^2 is a Hamilton cycle. For arbitrary i, $E_c^{\sigma'-}(M_{A_i^2B^{n-8}}) = E_c^{\sigma'-}(M_{A_iB^{n-4}}) \cup E_c^{\sigma'-}(M_{A_{i+4}B^{n-4}})$, $E_s^{\sigma'-}(W_i^2) = E_s^{\sigma'-}(W_i) \cup E_s^{\sigma'-}(W_{i+4})$. If $|E_c^{\sigma'-}(M_{A_iB^{n-4}})|$ and $|E_c^{\sigma'-}(M_{A_{i+4}B^{n-4}})|$ have the same parity, then $|E_s^{\sigma'-}(W_{i+4})|$ and $|E_s^{\sigma'-}(W_i)|$ have the same parity. If $|E_c^{\sigma'-}(M_{A_iB^{n-4}})|$ and $|E_c^{\sigma'-}(M_{A_{i+4}B^{n-4}})|$ have opposite parity, then $|E_s^{\sigma'-}(W_i)|$ and $|E_s^{\sigma'-}(W_{i+4})|$ have opposite parity. Thus, $|E_c^{\sigma'-}(M_{A_i^2B^{n-8}})|$ and $|E_s^{\sigma'-}(W_i^2)|$ have the same parity. By Lemma 9, W_i^2 is a balanced Hamilton cycle. □

According to Claim 2, for $n = 2p (p \geq 4)$, there is always a perfect matching M such that $GP_{\sigma'}(n,2) - M$ consists of balanced cycles. Thus, according to Lemma 5, $\chi'(GP_{\sigma'}(n,2)) = 3$ for $n = 2p (p \geq 4)$. □

4. Conclusions

In this paper, we proved that (1) $\chi'(GP_\sigma(n,2)) = 3$ if $n \equiv 3 \text{ mod } 6 (n \geq 9)$; (2) if $n = 2p$ $(p \geq 4)$, $\chi'(GP_\sigma(n,2)) = 3$. For $k = 2$, there are still two unsolved cases, which are $n \equiv 1$ mod 6 and $n \equiv 5$ mod 6. When $n \equiv 1$ mod 6, $GP(n,2) - M_{A^sB^t} (4s + t = n)$ is not a Hamilton cycle, and when $n \equiv 5$ mod 6, $GP(n,2)$ itself is not Hamiltonian. Thus, our method in this paper is not suitable for these two cases, which is one subject of our future work. In addition, it is also an interesting and challenging problem to consider $GP(n,k)$ for $k \geq 3$, since there is no characterization of perfect matchings of $GP(n,k)$.

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