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Cauchy Integral and Boundary Value for Vector-Valued Tempered Distributions

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Abstract: Using the historically general growth condition on scalar-valued analytic functions, which have tempered distributions as boundary values, we show that vector-valued analytic functions in tubes $T^C = \mathbb{R}^n + iC$ obtain vector-valued tempered distributions as boundary values. In a certain vector-valued case, we study the structure of this boundary value, which is shown to be the Fourier transform of the distributional derivative of a vector-valued continuous function of polynomial growth. A set of vector-valued functions used to show the structure of the boundary value is shown to have a one–one and onto relationship with a set of vector-valued distributions, which generalize the Schwartz space $\mathcal{D}'_{L^2}(\mathbb{R}^n)$; the tempered distribution Fourier transform defines the relationship between these two sets. By combining the previously stated results, we obtain a Cauchy integral representation of the vector-valued analytic functions in terms of the boundary value.

Keywords: analytic functions; vector-valued tempered distributions; boundary value; Cauchy integral

MSC: 32A26; 32A40; 46F12; 46F20

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1. Introduction

Tillmann [1] introduced the analysis of analytic functions, which obtain tempered distributional boundary values in $\mathcal{S}'(\mathbb{R}^n)$. In [1], Tillmann worked with scalar-valued analytic functions in tubes $T^{C_\mu} = \mathbb{R}^n + iC_\mu$, where the $C_\mu = \{y \in \mathbb{R}^n : (-1)^{\mu_j} y_j > 0,\ j = 1, \ldots, n\}$ with $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ being any of the 2^n n-tuples, whose components are either 0 or 1 and characterize the growth conditions on the analytic functions, which obtain the $\mathcal{S}'(\mathbb{R}^n)$ boundary values. This analysis by Tillmann was motivated by the work by Köthe in [2,3].

Using a more restrictive growth on the analytic functions, we showed in [4] that vector-valued analytic functions in tubes $T^C = \mathbb{R}^n + iC$, where C is an open convex cone, having this more restrictive growth obtain vector-valued tempered distributions in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, with $\mathcal{X}$ being a specified topological vector space. In this paper, our first objective is to generalize this result of [4] to the general growth form of Tillmann for the vector-valued analytic functions. We obtain this boundary value generalization in Section 4 of this paper.

Moreover, in Section 4, we study the structure of this boundary value in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. To do this, we first restrict the topological vector space $\mathcal{X}$ by imposing certain conditions on it to ensure that the boundary value is the Fourier transform of a distributional derivative of a continuous vector-valued function $\mathbf{g}$, which has polynomial growth in the norm of the space $\mathcal{X}$. By further restricting $\mathcal{X}$ to be a Hilbert space, we show that function $\mathbf{g}$ is in a set that has a one–one and onto relationship with a set of vector-valued distributions, which generalize the $\mathcal{D}'_{L^2}(\mathbb{R}^n)$ distributions of Schwartz. The relationship between these two sets is obtained using the tempered distribution Fourier transform; the proof of this relationship is proved in Section 3 of this paper. Using the relationships of these noted two sets, we are able to obtain an additional structure of the tempered distribution boundary value of the analytic functions in Section 4.

A few papers have been written concerning the construction of a Cauchy integral for tempered distributions. All of these papers concern scalar-valued analytic functions and scalar-valued tempered distributions. The first paper known to this author was by J. Sebastião e Silva [5] (Section 5) and concerned scalar-valued analytic functions and tempered distributions in one dimension. An associated analysis by Sebastião e Silva is contained in [6]. Carmichael [7] defined a Cauchy integral for tempered distributions in the $\mathbb{C}^n$ setting corresponding to analytic functions in each of the 2^n quadrant tubes $T^{C_\mu} \subset \mathbb{C}^n$ and showed that the analytic functions with growth, such as that of Tillmann in $(\mathbb{C} - \mathbb{R})^n$ could be recovered as the defined Cauchy integral of the tempered distribution boundary value; the results of [7] can be extended to the vector-valued analytic functions in T^{C_μ} and the tempered distribution setting considered in this paper by the same techniques as those of [7]. The Cauchy integrals introduced by Sebastião e Silva in [5] and by Carmichael in [7] are in fact equivalence classes of analytic functions defined by an integral involving the Cauchy kernel.

Vladimirov [8–10] defined a Cauchy integral for tempered distributions associated with analytic functions in general tubes $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ corresponding to open convex cones C with the functions satisfying a growth condition similar to that of Tillmann. Vladimirov has shown that the analytic functions that he has considered can be recovered by a Cauchy integral involving the tempered distribution boundary values of the analytic functions. An associated analysis by Vladimirov is contained in [11,12]. The works mentioned in this paragraph and the previous paragraph all concern scalar-valued analytic functions and scalar-valued tempered distributions.

In Section 5 of this paper, we build on our analysis of Sections 3 and 4 to obtain a Cauchy integral representation of the vector-valued analytic functions, which are shown to have tempered vector-valued distributions as the boundary values in Section 4. The proof of our result here and the form of the Cauchy integral representation are substantially different from any of the previous results concerning Cauchy integral representation of the analytic functions having tempered distribution boundary values.

2. Definitions and Notation

Throughout, $\mathcal{X}$ will denote a topological vector space with the stated appropriate properties corresponding to the results that we wish to prove. For $\mathcal{X}$ being a normed space, we denote the norm by $\mathcal{N}$. Θ will denote the zero element of $\mathcal{X}$; and if $\mathcal{X}$ is a Hilbert space, we denote the space by $\mathcal{H}$. For integration of the vector-valued functions and vector-valued analytic functions, we refer to Dunford and Schwartz [13]. For foundational information concerning vector-valued distributions, we refer to Schwartz [14,15].

The n-dimensional notation to be used in this paper will be the same as in [16,17]. Note $\overline{0} = (0, 0, \ldots, 0)$ is the origin in $\mathbb{R}^n$. The information concerning cones $C \subset \mathbb{R}^n$ needed is explicitly stated in [16] (Section 2) and [17] (Chapter 1). We do not repeat the definitions and notations concerning cones as stated in [16] (Section 2), and we ask the reader to refer to this reference.

The $L^p(\mathbb{R}^n, \mathcal{X})$ functions, $1 \leq p \leq \infty$, with values in a Banach space $\mathcal{X}$ and their norm $|\mathbf{h}|_p$ [13] (p. 119) are noted in [13] (Chapter III). The Fourier transform on $L^1(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n, \mathcal{X})$ is given in [17] (p. 3). All Fourier (inverse Fourier) transforms on scalar or vector-valued functions will be denoted $\hat{\phi}(x) = \mathcal{F}[\phi(t); x]$ $(\mathcal{F}^{-1}[\phi(t); x])$. As stated in [18,19], the Plancherel theory is not true for vector-valued functions, except when $\mathcal{X} = \mathcal{H}$, a Hilbert space. The Plancherel theory is complete in the $L^2(\mathbb{R}^n, \mathcal{H})$ setting in that the inverse Fourier transform is the inverse mapping of the Fourier transform with $\mathcal{F}^{-1}\mathcal{F} = I = \mathcal{F}\mathcal{F}^{-1}$ with I being the identity mapping.

We denote $\mathcal{S}(\mathbb{R}^n)$ as the tempered functions with associated distributions being $\mathcal{S}'(\mathbb{R}^n)$ or associated vector-valued distributions being $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. The Fourier (inverse Fourier) transform on $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ is the usual definition and is given in [14] (p. 73).

3. Fourier and Inverse Fourier Transform on a Function Subset of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$

Let $\mathcal{X}$ be a Banach space. We defined the space $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X}), 1 \leq p < \infty,$ in [16]. We repeat the definition here because of the importance of these functions for our results in this paper.

Definition 1. *For a Banach space $\mathcal{X}$, $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X}), 1 \leq p < \infty,$ is the set of all measurable functions $g(t), t \in \mathbb{R}^n$, with values in $\mathcal{X}$ such that there exists a real number $m \geq 0$ for which $(1 + |t|^p)^{-m} g(t) \in L^p(\mathbb{R}^n, \mathcal{X})$.*

Note that m can be taken as a nonnegative integer in Definition 1. As noted in [16], $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, $1 \leq p < \infty$. The spaces $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$ will be important in this paper.

Throughout this paper, the differential operator D_t, $t \in \mathbb{R}^n$ will take the form

$$D_t = \frac{-1}{2\pi i}\left(\frac{\partial}{\partial t_1}, \frac{\partial}{\partial t_2}, \ldots, \frac{\partial}{\partial t_n}\right).$$

Thus, for α being any n-tuple of nonnegative integers,

$$D_t^\alpha = \left(\frac{-1}{2\pi i}\right)^{|\alpha|}\left(\frac{\partial^{\alpha_1}}{\partial t_1^{\alpha_1}}, \frac{\partial^{\alpha_2}}{\partial t_2^{\alpha_2}}, \ldots, \frac{\partial^{\alpha_n}}{\partial t_n^{\alpha_n}}\right).$$

The goal of this section is to show a one–one and onto relationship between the set of functions $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$ and another subset of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, where $\mathcal{H}$ is a Hilbert space. This relationship is obtained by both the Fourier and inverse Fourier transforms in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. We define the space that has this stated relationship to $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$, as follows.

Definition 2. *Let m be any nonnegative integer. The set of Hilbert space $\mathcal{H}$-valued generalized functions in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ of the form*

$$V_t = \sum_{|\alpha| \leq m} D_t^\alpha g_\alpha(t)$$

where $g_\alpha \in L^2(\mathbb{R}^n, \mathcal{H})$, $|\alpha| \leq m$, will be denoted as $L2(\mathbb{R}^n, \mathcal{H})$.

We emphasize that $L2(\mathbb{R}^n, \mathcal{H}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. When $\mathcal{H} = \mathbb{C}^1$, note that $L2(\mathbb{R}^n, \mathbb{C}^1) = \mathcal{D}'_{L^2}(\mathbb{R}^n)$, the Schwartz space of distributions contained in $\mathcal{S}'(\mathbb{R}^n)$ of the form of finite sums of distributional derivatives of $L^2(\mathbb{R}^n)$ functions. For $\phi \in \mathcal{D}_{L^2}(\mathbb{R}^n)$, the Schwartz space that is the set of test functions for $\mathcal{D}'_{L^2}(\mathbb{R}^n)$, the application $\langle V, \phi \rangle$, $V \in \mathcal{D}'_{L^2}(\mathbb{R}^n)$, yields a complex number. In exactly the same way, for $V \in L2(\mathbb{R}^n, \mathcal{H})$ and $\phi \in \mathcal{D}_{L^2}(\mathbb{R}^n)$, the application $\langle V, \phi \rangle$ yields an element of $\mathcal{H}$; and the algebraic and differentiation calculations on the form $\langle V, \phi \rangle$ hold for $V \in L2(\mathbb{R}^n, \mathcal{H})$, as usual, just as these calculations hold on the form $\langle V, \phi \rangle$ for $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. This is an important note in relation to our construction of the Cauchy integral (later in this paper).

We now obtain the relationship between $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$ and $L2(\mathbb{R}^n, \mathcal{H})$ for any Hilbert space $\mathcal{H}$.

Lemma 1. *The $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ Fourier transform maps $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$ one-one and onto $L2(\mathbb{R}^n, \mathcal{H})$. The $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ inverse Fourier transform maps $L2(\mathbb{R}^n, \mathcal{H})$ one-one and onto $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$.*

Proof. Let the function $\mathbf{g} \in \mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$. From Definition 1, there is a real number $m \geq 0$ for which $(1 + |t|^2)^{-m}\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$, and m can be taken as a nonnegative integer. Since $\mathbf{g} \in \mathcal{S}'_2(\mathbb{R}^n, \mathcal{H}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, the Fourier transform of $\mathbf{g}$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ is an element of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$; we put $V_x = \mathcal{F}[\mathbf{g}]_x$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$, and let Δ denote the Laplace operator in the variable $x \in \mathbb{R}^n$. Using integration by parts, we have

$$
\begin{aligned}
\langle V_x, \phi(x) \rangle &= \langle \mathbf{g}(t), \mathcal{F}[\phi(x); t] \rangle \\
&= \langle \frac{\mathbf{g}(t)}{(1+|t|^2)^m}, \int_{\mathbb{R}^n} \phi(x)(1+|t|^2)^m e^{2\pi i \langle x,t \rangle} dx \rangle \\
&= \langle \frac{\mathbf{g}(t)}{(1+|t|^2)^m}, \mathcal{F}[(1-(4\pi^2)^{-1}\Delta)^m \phi(x); t] \rangle \\
&= \langle \mathcal{F}[\frac{\mathbf{g}(t)}{(1+|t|^2)^m}; x], (1-(4\pi^2)^{-1}\Delta)^m \phi(x) \rangle.
\end{aligned}
\tag{1}
$$

Since $(1+|t|^2)^{-m}\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$, then $\mathbf{h}(x) = \mathcal{F}[(1+|t|^2)^{-m}\mathbf{g}(t); x] \in L^2(\mathbb{R}^n, \mathcal{H})$. From (1), we have

$$\langle V_x, \phi(x) \rangle = \langle (1-(4\pi^2)^{-1}\Delta)^m \mathbf{h}(x), \phi(x) \rangle,$$

and $V_x = \mathcal{F}[\mathbf{g}]_x = (1-(4\pi^2)^{-1}\Delta)^m \mathbf{h}(x) \in L2(\mathbb{R}^n, \mathcal{H})$. Thus, the $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ Fourier transform maps $\mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ to $L2(\mathbb{R}^n, \mathcal{H})$.

We now desire to prove that any element of $L2(\mathbb{R}^n, \mathcal{H})$ is the $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ Fourier transform of an element in $\mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$. Let $V \in L2(\mathbb{R}^n, \mathcal{H})$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. By Definition 2, there is a nonnegative integer m, such that

$$V_t = \sum_{|\alpha| \le m} D_t^\alpha \mathbf{g}_\alpha(t)$$

with $\mathbf{g}_\alpha(t) \in L^2(\mathbb{R}^n, \mathcal{H})$, $|\alpha| \le m$. Since $L2(\mathbb{R}^n, \mathcal{H}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, $\mathcal{F}^{-1}[V]_x$ exists in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, and we have for the nonnegative integer m

$$
\begin{aligned}
\langle \mathcal{F}^{-1}[V]_x, \phi(x) \rangle &= \sum_{|\alpha| \le m} \langle D_t^\alpha \mathbf{g}_\alpha(t), \mathcal{F}^{-1}[\phi(x); t] \rangle \\
&= \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle \mathbf{g}_\alpha(t), D_t^\alpha \int_{\mathbb{R}^n} \phi(x) e^{-2\pi i \langle x,t \rangle} dx \rangle \\
&= \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle \mathbf{g}_\alpha(t), (-1/2\pi i)^{|\alpha|} \int_{\mathbb{R}^n} \phi(x)(-2\pi i)^{|\alpha|} x^\alpha e^{-2\pi i \langle x,t \rangle} dx \rangle \\
&= \sum_{|\alpha| \le m} \langle (-1)^{|\alpha|} \mathbf{g}_\alpha(t), \int_{\mathbb{R}^n} x^\alpha \phi(x) e^{-2\pi i \langle x,t \rangle} dx \rangle \\
&= \sum_{|\alpha| \le m} \langle (-1)^{|\alpha|} \mathbf{g}_\alpha(t), \mathcal{F}^{-1}[x^\alpha \phi(x); t] \rangle \\
&= \sum_{|\alpha| \le m} \langle \mathcal{F}^{-1}[(-1)^{|\alpha|} \mathbf{g}_\alpha(t); x], x^\alpha \phi(x) \rangle.
\end{aligned}
$$

For each α, $|\alpha| \le m$, put $\mathbf{h}_\alpha(x) = \mathcal{F}^{-1}[(-1)^{|\alpha|}\mathbf{g}_\alpha(t); x]$. We have $\mathbf{h}_\alpha(x) \in L^2(\mathbb{R}^n, \mathcal{H})$, $|\alpha| \le m$, since each $\mathbf{g}_\alpha(t) \in L^2(\mathbb{R}^n, \mathcal{H})$; moreover, $\sum_{|\alpha| \le m} \mathbf{h}_\alpha(x) \in L^2(\mathbb{R}^n, \mathcal{H})$. Thus, we have

$$
\begin{aligned}
\langle \mathcal{F}^{-1}[V]_x, \phi(x) \rangle &= \sum_{|\alpha| \le m} \langle \mathbf{h}_\alpha(x), x^\alpha \phi(x) \rangle \\
&= \langle \sum_{|\alpha| \le m} x^\alpha \mathbf{h}_\alpha(x), \phi(x) \rangle,
\end{aligned}
\tag{2}
$$

and $\mathcal{F}^{-1}[V]_x = \sum_{|\alpha| \le m} x^\alpha \mathbf{h}_\alpha(x)$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. For the $L^2(\mathbb{R}^n, \mathcal{H})$ norm $|\cdot|_2$ and the order m of the summation defining V, we consider

$$|(1+|x|^2)^{-m-2} \sum_{|\alpha| \le m} x^\alpha \mathbf{h}_\alpha(x)|_2. \tag{3}$$

For $|\alpha| \leq m$, note that $|x^\alpha| \leq |x|^{|\alpha|} \leq (1+|x|)^{|\alpha|} \leq (1+|x|)^m$. Since $(1+|x|)^m \leq 2^m$ if $|x| \leq 1$ and $(1+|x|)^m \leq (1+|x|^2)^m$ if $|x| \geq 1$, then

$$\begin{aligned}
&|x^\alpha(1+|x|^2)^{-m-2}| \leq (1+|x|)^m(1+|x|^2)^{-m-2} \\
&\leq \max_{x\in\mathbb{R}^n}\{2^m, (1+|x|^2)^m\}(1+|x|^2)^{-m-2} \\
&\leq \max\{2^m, 1\} = 2^m
\end{aligned}$$

for $|\alpha| \leq m$ since $m \geq 0$ is a nonnegative integer. Thus, for the $L^2(\mathbb{R}^n, \mathcal{H})$ norm in (3), we have

$$|(1+|x|^2)^{-m-2} \sum_{|\alpha|\leq m} x^\alpha \mathbf{h}_\alpha(x)|_2 \leq 2^m \left| \sum_{|\alpha|\leq m} \mathbf{h}_\alpha(x) \right|_2 < \infty \tag{4}$$

since $\sum_{|\alpha|\leq m} \mathbf{h}_\alpha(x) \in L^2(\mathbb{R}^n, \mathcal{H})$. Recalling (2), we have by (4) that $\mathcal{F}^{-1}[V]_x = \sum_{|\alpha|\leq m} x^\alpha \mathbf{h}_\alpha(x) \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ for any $V \in L2(\mathbb{R}^n, \mathcal{H})$; and $V_t = \mathcal{F}[\sum_{|\alpha|\leq m} x^\alpha \mathbf{h}_\alpha(x)]_t$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. Thus, the $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ Fourier transform maps $\mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ onto $L2(\mathbb{R}^n, \mathcal{H})$; the fact that this mapping is one–one follows directly from the fact that the Fourier transform is a one–one mapping on $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. The same statements and proofs as in this proof of Lemma 1 for the Fourier transform hold in exactly the same way for the inverse Fourier transform on $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$; and we have that the $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ inverse Fourier transform maps $L2(\mathbb{R}^n, \mathcal{H})$ one–one and onto $\mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$. The proof of Lemma 1 is complete. □

Let C be a regular cone in $\mathbb{R}^n$; that is, C is an open convex cone in $\mathbb{R}^n$, which does not contain any entire straight line. $C^* = \{t \in \mathbb{R}^n : \langle t, y\rangle \geq 0 \text{ for all } y \in C\}$ is the dual cone of C. We consider now the Cauchy kernel

$$K(z-t) = \int_{C^*} e^{2\pi i\langle z-t,u\rangle} du,\ z \in T^C = \mathbb{R}^n + iC,\ t \in \mathbb{R}^n.$$

The ultradistributional test function spaces $\mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}(\mathbb{R}^n), 1 < p \leq \infty$, where $*$ is Beurling (M_p) or Roumieu $\{M_p\}$, defined in [17] (Section 2.3, p. 21). For C being a regular cone, we proved in [17] (Section 4.1, Theorem 4.1.1) that $K(z-\cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}(\mathbb{R}^n)$ for $z \in T^C, 1 < p \leq \infty$, under specified conditions on the sequence M_p of positive numbers, which we assume here. (See [17] (pp. 13–14, Theorem 4.1.1) for assumptions on the sequence M_p.) The Schwartz space $\mathcal{D}_{L^2}'(\mathbb{R}^n)$ consists of finite sums of distributional derivatives of $L^2(\mathbb{R}^n)$ functions; thus, the space $L2(\mathbb{R}^n, \mathcal{H}|)$ is the extension of $\mathcal{D}_{L^2}'(\mathbb{R}^n)$ to vector-valued distributions with values in $\mathcal{H}$. Thus, for $p = 2$, we emphasize that the form $\langle V_t, K(z-t)\rangle,\ z \in T^C,$ is well defined for $V \in L2(\mathbb{R}^n, \mathcal{H})$, and yields an element of $\mathcal{H}$; the algebraic and differentiation calculations on the form $\langle V, \phi\rangle$ hold for $V \in L2(\mathbb{R}^n, \mathcal{H})$ and $\phi \in \mathcal{D}_{L^2}(\mathbb{R}^n)$, as usual, just as these calculations hold for the form $\langle V, \phi\rangle$ for $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$. We use this information in Section 5 of this paper.

4. Boundary Values in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$

Let C be an open convex cone in $\mathbb{R}^n$. In [4] (Theorem 8), we proved that an analytic function $\mathbf{f}(z),\ z \in T^C$, with values in a specified topological vector space $\mathcal{X}$ and satisfying a certain norm growth obtained a vector-value-tempered distributional boundary value, as $y \to \bar{0},\ y \in C' \subset\subset C$, for any compact subcone C' of C. The norm growth used in [4] (Theorem 8) was not as general as the growth of Tillmann [1] in which the original tempered distributional boundary value results in the scalar-valued case were obtained. In this section, we extend the result [4] (Theorem 8) by assuming a norm growth on the analytic function equivalent to that of Tillmann [1]; our result here also contains new information concerning the boundary value. As a corollary of our result, we obtain a precise representation of the boundary value when the conditions on the topological vector space $\mathcal{X}$ are restricted.

Following Vladimirov [11] (p. 230), we shall use the term "spectral function" but will extend the definition of this term to the vector-valued case. For an analytic function

$\mathbf{f}(z),\ z \in T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$, with values in a topological vector space $\mathcal{X}$, the spectral function of $\mathbf{f}(z)$ is that vector-valued distribution $V \in \mathcal{D}'(\mathbb{R}^n, \mathcal{X})$, such that $e^{-2\pi\langle y,t\rangle}V_t \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X}),\ y \in C$; and $\mathbf{f}(x+iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}V_t]_x$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ for $z = x+iy \in T^C$.

We begin by assuming that the topological vector space $\mathcal{X}$ is locally convex, separable, and quasi-complete where quasi-complete is in the sense of Schwartz [15] (p. 198). We further assume that $\mathcal{X}$ is a normed space with norm $\mathcal{N}$. These stated assumptions on $\mathcal{X}$ were the assumptions under which we obtained [4] (Theorem 8) and are the assumptions on the topological vector space $\mathcal{X}$ under which we obtain Theorem 1 below.

Throughout the paper, by $y \to \overline{0},\ y \in C$, we mean that $y \to \overline{0},\ y \in C' \subset\subset C$ for every compact subcone $C' \subset\subset C$.

The following theorem generalizes and extends [4] (Theorem 8) for $\mathcal{X}$, satisfying the properties noted above.

Theorem 1. *Let C be an open convex cone. Let $\boldsymbol{f}(z)$ be analytic in T^C and have values in $\mathcal{X}$. Let*

$$\mathcal{N}(\boldsymbol{f}(x+iy)) \leq M(1+|z|)^q|y|^{-r},\ z = x+iy \in T^C, \tag{5}$$

where $M > 0$ is a real constant, q is a nonnegative integer, $r > 1$ is an integer, and M, q, and r are independent of $z = x+iy \in T^C$. There exists an element $U \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, such that

$$\lim_{y \to \overline{0}, y \in C} \boldsymbol{f}(x+iy) = U \tag{6}$$

in the weak and strong topologies of $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. Further, $U = \mathcal{F}[V]$ with $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ being the spectral function of $\boldsymbol{f}(z),\ z \in T^C$, , such that $supp(V) \subseteq C^*$.

Proof. We apply the proofs of [4] (Theorems 3 and 8). Note that in the second sentence of the proof of [4] (Theorem 8) that the value of $\eta \geq 1$ is arbitrary but fixed; in the present proof, we simply take $\eta = 1$, where it is appropriate to use $\eta = 1$. Let $\lambda > 0$; put $\rho = \sigma + i\lambda,\ \sigma \in \mathbb{R}^1$; and define $\mathbf{f}'(\rho; x, y) = \mathbf{f}(x + \rho y),\ y \in pr(C)$, where $pr(C)$ denotes the projection of C, which is the intersection of C with the unit sphere in $\mathbb{R}^n$. (Thus, $|y| = 1$ if $y \in pr(C)$.) $\mathbf{f}'(\rho; x, y)$ is an analytic function of ρ in the half plane $\lambda = \text{Im}(\rho) > 0$ and has values in $\mathcal{X}$. We have $\mathbf{f}'(\rho; x, y) = \mathbf{f}(x+\rho y) = \mathbf{f}((x+\sigma y) + i\lambda y),\ \lambda > 0$, for $z = x+iy$ with $y \in pr(C)$; and note that $\lambda y \in C$ for $\lambda > 0$ and $y \in pr(C)$. Now for $y = \text{Im}(z) \in pr(C)$ and $0 < \lambda \leq \eta = 1$ we have

$$\begin{aligned}
&\mathcal{N}(\mathbf{f}'(\rho; x, y)) \leq M(1 + |(x+\sigma y) + i\lambda y|)^q |\lambda y|^{-r} \\
&= M(1 + (\lambda^2 + |x+\sigma y|^2)^{1/2})^q \lambda^{-r} \\
&\leq M(1 + (1 + (|x| + |\sigma|)^2)^{1/2})^q \lambda^{-r} \\
&\leq M(1 + ((1 + |x| + |\sigma|)^2)^{1/2})^q \lambda^{-r} \\
&= M(2 + |x| + |\sigma|)^q \lambda^{-r}
\end{aligned} \tag{7}$$

which is of the form, with norm $\mathcal{N}$ replacing the absolute value, of [4] (15), which is used in exactly the same way in the proof of [4] (Theorem 8) as in the proof of [4] (Theorem 3). Thus, for $y = \text{Im}(z) \in pr(C)$ and $0 < \lambda \leq \eta = 1$ the bound on $\mathcal{N}(\mathbf{f}'(\rho; x, y))$ is in the proper form to proceed with the proof of this present Theorem 1 exactly as in the form of the proofs of [4] (Theorems 3 and 8). We obtain the structured function of the form $\Lambda^{(-r-1)}\mathbf{f}'(\rho; x, y),\ y \in pr(C)$, which satisfies the growth (similar to [4] (37))

$$\mathcal{N}(\Lambda^{(-r-1)}\mathbf{f}'(\rho; x, y)) \leq M^{(r+1)}(2 + |x| + |\sigma|)^q (2 + |\sigma|)^{r+1}$$

for $0 < \lambda \leq \eta = 1$ where $M^{(r+1)}$ is a positive constant, and obtains the representation (similar to [4] (38))

$$\mathbf{f}(x+\rho y) = \mathbf{f}'(\rho; x, y) = \frac{\partial^{r+1}(\Lambda^{(-r-1)}\mathbf{f}'(\rho; x, y))}{\partial \sigma^{r+1}},\ \sigma = Re(\rho).$$

Now, we proceed in our proof of Theorem 1 in exactly the same way as in [4] (Theorem 8) (p. 328) to obtain the desired boundedness properties leading to the existence of an element $V \in \mathcal{D}'(\mathbb{R}^n, \mathcal{X})$, such that $e^{-2\pi\langle y,t\rangle}V_t \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, $y \in C$, and $\mathbf{f}(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}V_t]_x$, $z = x + iy \in T^C$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ from the results of Schwartz [14] (Prop. 22, p. 76). (These results of Schwartz [14] (Prop. 22, p. 76) were obtained in their original scalar-valued case in [20]; the related results were then obtained by Lions [21]). Thus, $V \in \mathcal{D}'(\mathbb{R}^n, \mathcal{X})$ is the spectral function of $\mathbf{f}(z)$, $z \in T^C$. The remainder of the proof of [4] (Theorem 8, pp. 329–330) and the succeeding discussion after the conclusion of the proof of [4] (Theorem 8) can be applied to the present proof of Theorem 1 in the same way to yield that, in fact, $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ and that

$$\lim_{y \to \bar{0}, y \in C} \mathbf{f}(x + iy) = \lim_{y \to \bar{0}, y \in C} \mathcal{F}[e^{-2\pi\langle y,t\rangle}V_t] = \mathcal{F}[V] = U \tag{8}$$

in the weak topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. However, $\mathcal{S}(\mathbb{R}^n)$ is a Montel space; thus, the convergence in (8) is in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ as well. We emphasize that $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ and that $U = \mathcal{F}[V] \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ is the desired boundary value in (6) as obtained in (8).

We now prove that $\text{supp}(V) \subseteq C^*$. Let $t_o \in C_* = \mathbb{R}^n \setminus C^*$; C_* is an open set in $\mathbb{R}^n$ since C^* is a closed set. From the definition of C^*, for $t_o \in C_*$, there is a point $y_o \in C$, such that $\langle y_o, t_o\rangle < 0$. Using the fact that C_* is open and the continuity of $\langle t, y_o\rangle$ at $t_o \in C_*$ as a function of t, there is a fixed $\tau > 0$ and a fixed neighborhood $N(t_o;\gamma) = \{t \in \mathbb{R}^n : |t - t_o| < \gamma,\ \gamma > 0\} \subset C_*$, such that $\langle t, y_o\rangle < -\tau < 0$ for all $t \in N(t_o;\gamma)$. Let $\phi \in \mathcal{D}(\mathbb{R}^n)$, such that $\text{supp}(\phi) \subset N(t_o,\gamma)$. Recall that $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, such that $e^{-2\pi\langle y,t\rangle}V_t \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, $y \in C$, and $\mathbf{f}(x + iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}V_t]_x$, $z = x + iy \in T^C$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. Thus $e^{-2\pi\langle y,t\rangle}V_t = \mathcal{F}^{-1}[\mathbf{f}(x + iy)]_t$, $x + iy \in T^C$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$; or $V_t = e^{2\pi\langle y,t\rangle}\mathcal{F}^{-1}[(x + iy)]_t$, $x + iy \in T^C$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. Let $y = \beta y_o$, $y_o \in C$, $\beta > 0$, now. We have $y = \beta y_o \in C$ and

$$\begin{aligned}
\langle V, \phi\rangle &= \langle e^{2\pi\langle \beta y_o, t\rangle}\mathcal{F}^{-1}[\mathbf{f}(x + i\beta y_o]_t, \phi(t)\rangle \\
&= \langle \mathcal{F}^{-1}[\mathbf{f}(x + i\beta y_o]_t, e^{2\pi\langle \beta y_o, t\rangle}\phi(t)\rangle \\
&= \langle \mathbf{f}(x + i\beta y_o), \mathcal{F}^{-1}[e^{2\pi\langle \beta y_o, t\rangle}\phi(t); x]\rangle \\
&= \int_{\mathbb{R}^n} \mathbf{f}(x + i\beta y_o) \int_{supp(\phi)} e^{2\pi\langle \beta y_o, t\rangle}\phi(t)e^{-2\pi i\langle x,t\rangle}dtdx
\end{aligned} \tag{9}$$

for the function $\phi \in \mathcal{D}(\mathbb{R}^n)$ chosen above. Using integration by parts and letting Δ denote the Laplacian in the $t \in \mathbb{R}^n$ variable, we have for any positive integer m

$$\begin{aligned}
&\mathcal{N}\left(\int_{\mathbb{R}^n} \mathbf{f}(x + i\beta y_o) \int_{supp(\phi)} e^{2\pi\langle \beta y_o, t\rangle}\phi(t)e^{-2\pi i\langle x,t\rangle}dtdx\right) \\
&= \mathcal{N}\left(\int_{\mathbb{R}^n} \frac{\mathbf{f}(x + i\beta y_o)}{(1 + |x|^2)^m} \int_{supp(\phi)} e^{2\pi\langle \beta y_o, t\rangle}\phi(t)(1 + |x|^2)^m e^{-2\pi i\langle x,t\rangle}dtdx\right) \\
&= \mathcal{N}\left(\int_{\mathbb{R}^n} \frac{\mathbf{f}(x + i\beta y_o)}{(1 + |x|^2)^m} \int_{supp(\phi)} (1 - \frac{\Delta}{4\pi^2})^m (e^{2\pi\langle \beta y_o, t\rangle}\phi(t))e^{-2\pi i\langle x,t\rangle}dtdx\right).
\end{aligned} \tag{10}$$

(For the present, the positive integer m is arbitrary; later, we explicitly choose m to obtain the desired convergence of all integrals through Equation (15) below). For the interior integral over $\text{supp}(\phi)$ in (10), we note that by applying $(1 - (\Delta/4\pi^2))^m$ to the product $e^{2\pi\langle \beta y_o, t\rangle}\phi(t)$ and then bounding the terms in the resulting sum, including the terms involving 2π or it powers, we obtain a finite sum of terms involving powers of $\beta(y_o)_j$, $j = 1, \ldots, n$, multiplied by $e^{2\pi\langle \beta y_o, t\rangle}$, where $(y_o)_j$ is the jth component of y_o, $j = 1, \ldots, n$, and multiplied by bounds on $\phi(t)$ or one of its partial derivatives with $e^{2\pi\langle \beta y_o, t\rangle}$ in each term of the sum. Of course, the boundedness of $\phi(t)$ and any of its partial derivatives are valid because of the compact support of $\phi(t)$. Moreover, note that $|\beta(y_o)_j| \leq \beta|y_o|$, $j = 1, \ldots, n$. Thus, since the interior integral in (10) is over $\text{supp}(\phi) \subset N(t_o;\gamma)$, we obtain the following bound on this interior integral:

$$\begin{aligned} &\left|\int_{supp(\phi)} (1-\frac{\Delta}{4\pi^2})^m (e^{2\pi\langle \beta y_o, t\rangle}\phi(t))e^{-2\pi i\langle x,t\rangle}dt\right| \\ &\le \int_{supp(\phi)} |(1-\frac{\Delta}{4\pi^2})^m (e^{2\pi\langle \beta y_o, t\rangle}\phi(t))|dt \\ &\le T_{supp(\phi)}(1+\beta|y_o|)^{4(m+1)} \sup_{t\in supp(\phi)} e^{2\pi\langle \beta y_o, t\rangle} \end{aligned} \tag{11}$$

where $T_{supp(\phi)}$ is a positive constant depending only on $\text{supp}(\phi)$. Using (11) in (10), we have

$$\begin{aligned} &\mathcal{N}\left(\int_{\mathbb{R}^n} \frac{\mathbf{f}(x+i\beta y_o)}{(1+|x|^2)^m} \int_{supp(\phi)} (1-\frac{\Delta}{4\pi^2})^m (e^{2\pi\langle \beta y_o, t\rangle}\phi(t))e^{-2\pi i\langle x,t\rangle}dtdx\right) \\ &\le T_{supp(\phi)}(1+\beta|y_o|)^{4(m+1)} \sup_{t\in supp(\phi)} e^{2\pi\langle \beta y_o, t\rangle}\int_{\mathbb{R}^n} \frac{\mathcal{N}(\mathbf{f}(x+i\beta y_o))}{(1+|x|^2)^m}dx \end{aligned} \tag{12}$$

where $y_o \in C,\ \beta > 0$ is arbitrary, and $\text{supp}(\phi) \subset N(t_o;\gamma) \subset C_*,\ t_o \in C_*,\ \gamma > 0$ and fixed. As noted before, since $\langle y_o, t_o\rangle < 0$ and C_* is open, by the continuity of $\langle t, y_o\rangle$ at $t_o \in C_*$ as a function of $t \in \mathbb{R}^n$, the fixed $\tau > 0$ is chosen and the fixed $N(t_o;\gamma) \subset C_*$ is chosen, such that $\langle t, y_o\rangle < -\tau < 0$ for all $t \in N(t_o;\gamma) \subset C_*$. Since $\text{supp}(\phi) \subset N(t_o;\gamma)$, we have

$$\sup_{t\in supp(\phi)} e^{2\pi\langle \beta y_o, t\rangle} \le e^{-2\pi\tau\beta},$$

which yields from (12)

$$\begin{aligned} &\mathcal{N}\left(\int_{\mathbb{R}^n} \frac{\mathbf{f}(x+i\beta y_o)}{(1+|x|^2)^m} \int_{supp(\phi)} (1-\frac{\Delta}{4\pi^2})^m (e^{2\pi\langle \beta y_o, t\rangle}\phi(t))e^{-2\pi i\langle x,t\rangle}dtdx\right) \\ &\le T_{supp(\phi)}e^{-2\pi\tau\beta}(1+\beta|y_o|)^{4(m+1)} \int_{\mathbb{R}^n} \frac{\mathcal{N}(\mathbf{f}(x+i\beta y_o))}{(1+|x|^2)^m}dx \end{aligned} \tag{13}$$

where $y_o \in C,\ \tau > 0,$ and $\gamma > 0$ are fixed and are independent of the arbitrary $\beta > 0$. We now bound the integral on the right of the inequality in (13) using the assumed growth (5) on $\mathbf{f}(z),\ z \in T^C$; (13) holds for all $\beta > 0$. To obtain the $\text{supp}(V)$ containment result, we are going to let $\beta \to \infty$ in (13); thus, we may assume that $\beta > 1$ in the remainder of this proof. By simple calculations and for $\beta > 1$, we have

$$1 + |x + i\beta y_o| = \beta(\frac{1}{\beta} + ((\frac{|x|}{\beta})^2 + |y_o|^2)^{1/2}) \le \beta(1 + (|x|^2 + |y_o|^2)^{1/2})$$

and

$$(1 + |x + i\beta y_o|)^q \le \beta^q(1 + (|x|^2 + |y_o|^2)^{1/2})^q \le \beta^q(1 + |y_o| + |x|)^q.$$

Hence, from (5),

$$\mathcal{N}(\mathbf{f}(x + i\beta y_o)) \le M\beta^q(1 + |y_o| + |x|)^q|\beta y_o|^{-r}$$

and

$$\int_{\mathbb{R}^n} \frac{\mathcal{N}(\mathrm{f}(x+i\beta y_o))}{(1+|x|^2)^m}dx \le M\beta^{q-r}|y_o|^{-r}\int_{\mathbb{R}^n} \frac{(1+|y_o|+|x|)^q}{(1+|x|^2)^m}dx. \tag{14}$$

Combining (10), (12), (13), and (14) yields

$$\mathcal{N}\left(\int_{\mathbb{R}^n} \mathbf{f}(x+i\beta y_o) \int_{supp(\phi)} e^{2\pi\langle\beta y_o,t\rangle}\phi(t)e^{-2\pi i\langle x,t\rangle}dtdx\right) \tag{15}$$
$$\leq MT_{supp(\phi)}(1+\beta|y_o|)^{4(m+1)}\beta^{q-r}|y_o|^{-r}e^{-2\pi\tau\beta}\int_{\mathbb{R}^n}\frac{(1+|y_o|+|x|)^q}{(1+|x|^2)^m}dx.$$

The positive integer m in (15) was introduced in (10), and at that point in the proof, m was arbitrary. We now choose m, such that $m > 2(q+n+1)$. With this choice of m, the integral in (15) converges where $y_o \in C$ is a fixed point in C; further, with this choice of m, all calculations from (10) leading to (15) are valid and the integrals converge. Because of the exponential term $e^{-2\pi\tau\beta}$, where $\tau > 0$ is fixed and now $\beta > 1$ is arbitrary, the right side of (15) has limit 0 as $\beta \to \infty$. Thus, from (9) $\langle V, \phi\rangle = \Theta$ for $\phi \in \mathcal{D}(\mathbb{R}^n)$, such that $\text{supp}(\phi) \subset N(t_o, \gamma) \subset C_*$ for t_o being an arbitrary but fixed point in the open set $C_* = \mathbb{R}^n \setminus C^*$. That is, for each fixed point, $t_o \in C_* = \mathbb{R}^n \setminus C^*$, with C_* being an open set, there is a neighborhood $N(t_o;\gamma) \subset C_*$ of t_o, such that for all $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $\text{supp}(\phi) \subset N(t_o;\gamma)$, we have $\langle V, \phi\rangle = \Theta$. Thus, V vanishes on a neighborhood of each point of C_*; this proves that V vanishes on the open set $C_* = \mathbb{R}^n \setminus C^*$. Thus, $\text{supp}(V) \subseteq C^*$, which is a closed set in $\mathbb{R}^n$. The proof of Theorem 1 is complete. □

Yoshinaga [22] (Proposition 3) provides a representation of the tempered vector-valued distributions in the case of the topological vector space $\mathcal{X}$ being a complete space of type (DF). Yoshinaga's result is as follows for $\mathcal{X}$, being a complete space of type (DF): $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, if and only if there exists a continuous function $\mathbf{g}$ on $\mathbb{R}^n$ with values in $\mathcal{X}$, an integer $k \geq 0$, and a n-tuple α of nonnegative integers, such that $V = D^\alpha \mathbf{g}$ and $\{\mathbf{g}(t)/(1+|t|^2)^{kn}; t \in \mathbb{R}^n\}$ is a bounded subset of $\mathcal{X}$. (In fact, in Yoshinaga's symbolism, $\alpha = (k, k, ..., k)$.)

The functions $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{X})$ of Definition 1 are an integral part of the following corollary to Theorem 1; recall that these functions are defined by the necessity for $\mathcal{X}$ being a Banach space. We know that a Banach space satisfies all of the conditions on $\mathcal{X}$ stated prior to Theorem 1 and also is a complete norm space of type (DF); since a Hilbert space is a Banach space, a Hilbert space also satisfies all of these stated conditions on $\mathcal{X}$. Thus, the above-stated result of Yoshinaga and Theorem 1 of this paper both hold for $\mathcal{X}$ being a Banach or Hilbert space.

We obtain a corollary of Theorem 1 now in which more precise information is obtained concerning the spectral function V and the boundary value U of Theorem 1.

Corollary 1. *Let C be an open convex cone and $\mathcal{X}$ be a Banach space. Let $\boldsymbol{f}(z)$ be analytic in $T^C = \mathbb{R}^n + iC$, have values in $\mathcal{X}$, and satisfy* (5)*. There is a continuous function $\boldsymbol{g} \in \mathcal{S}'_2(\mathbb{R}^n, \mathcal{X})$ with supp$(\boldsymbol{g}) \subseteq C^*$ a.e. and an n-tuple α of nonnegative integers, such that the spectral function $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ of Theorem 1 has the form $V_t = D_t^\alpha \boldsymbol{g}(t)$, and there is $U = \mathcal{F}[V] \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ such that*

$$\lim_{y\to\bar{0}, y\in C} \boldsymbol{f}(x+iy) = U$$

in the weak and strong topologies of $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. Further, for $\mathcal{X} = \mathcal{H}$ being a Hilbert space, we have $\mathcal{F}[\boldsymbol{g}] \in L2(\mathbb{R}^n, \mathcal{H})$; and the boundary value $U \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ has the form

$$U_x = x^\alpha \mathcal{F}[\boldsymbol{g}]_x = x^\alpha (1 - \frac{\Delta}{4\pi^2})^m \boldsymbol{h}(x) \tag{16}$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ where $\boldsymbol{h} \in L^2(\mathbb{R}^n, \mathcal{X})$, α is an n-tuple of nonnegative integers, and $m \geq 0$ is a real number that can be taken to be a nonnegative integer.

Proof. We apply the results of Theorem 1 and consider the spectral function $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ obtained in Theorem 1 where $\mathcal{X}$ is a Banach space in this corollary. As per the result of Yoshinaga [22] (Proposition 3) stated above, there is a continuous function $\mathbf{g}$ on $\mathbb{R}^n$

with values in $\mathcal{X}$, an n-tuple α of nonnegative integers, and an integer $k \geq 0$, such that $V_t = D_t^{\alpha}\mathbf{g}(t)$ and $\{\frac{\mathbf{g}(t)}{(1+|t|^2)^{kn}}; t \in \mathbb{R}^n\}$ is a bounded subset of $\mathcal{X}$. (In Yoshinaga's symbolism, α is the n-tuple with all components being k.) Thus, there is a real constant $R > 0$, such that

$$\mathcal{N}\left(\frac{\mathbf{g}(t)}{(1+|t|^2)^{kn}}\right) = \frac{\mathcal{N}(\mathbf{g}(t))}{(1+|t|^2)^{kn}} \leq R,\ t \in \mathbb{R}^n.$$

For the integer $k \geq 0$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} (\mathcal{N}\left(\frac{\mathbf{g}(t)}{(1+|t|^2)^{(k+2)n}}\right))^2 dt \\
&= \int_{\mathbb{R}^n} \left(\frac{1}{(1+|t|^2)^{2n}}\right)^2 (\mathcal{N}\left(\frac{\mathbf{g}(t)}{(1+|t|^2)^{kn}}\right))^2 dt \\
&\leq R^2 \int_{\mathbb{R}^n} \frac{1}{(1+|t|^2)^{4n}} dt < \infty
\end{aligned}$$

which proves that $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{X})$. Further, $\text{supp}(\mathbf{g}) \subseteq C^*$ a.e. since $\text{supp}(V) \subseteq C^*$. From Theorem 1, the boundary value $U \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ in (6) is $U = \mathcal{F}[V]$, the Fourier transform of the spectral function $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. Moreover, from Theorem 1, the boundary value U is obtained in both the weak and strong topologies of $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$.

Now, let $\mathcal{X} = \mathcal{H}$, a Hilbert space, in this Corollary 1. Since $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$, then $\mathcal{F}[\mathbf{g}] \in L2(\mathbb{R}^n, \mathcal{H})$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ by Lemma 1. We know from the above that the boundary value $U \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ is $U = \mathcal{F}[V]$, and $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ has the form $V_t = D_t^{\alpha}\mathbf{g}(t)$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$. We have

$$\begin{aligned}
&\langle U, \phi \rangle = \langle \mathcal{F}[V], \phi \rangle = \langle V, \hat{\phi} \rangle = \langle D_t^{\alpha}\mathbf{g}(t), \hat{\phi}(t) \rangle \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \mathbf{g}(t) D_t^{\alpha} \int_{\mathbb{R}^n} \phi(x) e^{2\pi i \langle x,t \rangle} dx dt \\
&= (-1)^{|\alpha|} \int_{\mathbb{R}^n} \mathbf{g}(t) \int_{\mathbb{R}^n} \phi(x)(-1/2\pi i)^{|\alpha|}(2\pi i)^{|\alpha|} x^{\alpha} e^{2\pi i \langle x,t \rangle} dx dt \\
&= \langle \mathbf{g}(t), \mathcal{F}[x^{\alpha}\phi(x); t] \rangle = \langle \mathcal{F}[\mathbf{g}]_x, x^{\alpha}\phi(x) \rangle = \langle x^{\alpha}\mathcal{F}[\mathbf{g}]_x, \phi(x) \rangle.
\end{aligned}$$

Thus, $U_x = x^{\alpha}\mathcal{F}[\mathbf{g}]_x$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ with $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$. Since $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$, by definition there is a real number $m \geq 0$, such that $\mathbf{g}(t)/(1+|t|^2)^m \in L^2(\mathbb{R}^n, \mathcal{H})$, and m can be taken to be a nonnegative integer. We have—by the proof of Lemma 1—that $\mathcal{F}[\mathbf{g}]_x = (1-(4\pi^2)^{-1}\Delta)^m \mathbf{h}(x) \in L2(\mathbb{R}^n, \mathcal{H})$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, where $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$ and Δ is the Laplace operator in the $x \in \mathbb{R}^n$ variable. Combining equalities, we have

$$U_x = x^{\alpha}\mathcal{F}[\mathbf{g}]_x = x^{\alpha}(1 - \frac{\Delta}{4\pi^2})^m \mathbf{h}(x)$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ with $\mathbf{h} \in L^2(\mathbb{R}^n, \mathcal{H})$, which is (16). The proof is complete. □

5. Cauchy Integral

A Cauchy integral of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ has been defined in one and many dimensions. Of course, the main problem in making such a definition is that the Cauchy kernel is not a tempered function in $\mathcal{S}(\mathbb{R}^n)$; an arbitrary element of $\mathcal{S}'(\mathbb{R}^n)$ applied to the Cauchy kernel is not well defined.

Let C be a regular cone in $\mathbb{R}^n$; that is, C is an open convex cone that does not contain an entirely straight line. With C^* being the dual cone of C, the Cauchy kernel function is

$$K(z-t) = \int_{C^*} e^{2\pi i \langle z-t,u \rangle} du,\ z \in T^C,\ t \in \mathbb{R}^n,$$

as defined in Section 3. For the tube T^C being the upper or lower half-planes in $\mathbb{C}^1$ or the tube defined by one of the 2^n quadrant cones $C_{\mu} = \{y \in \mathbb{R}^n : (-1)^{\mu_j} y_j > 0,\ j = 1, \ldots, n\}$

where μ is any of the 2^n n-tuples whose components are either 0 or 1, the Cauchy kernel takes the usual form. In order to generate an element of $\mathcal{S}(\mathbb{R}^n)$ from the Cauchy kernel in the half plane setting in $\mathbb{C}^1$ and the tube defined by a quadrant cone, one divides the Cauchy kernel by a certain specifically chosen polynomial.

Sebastião e Silva [5] introduced a Cauchy integral for tempered distributions in the half-plane setting. Carmichael [7] defined a Cauchy integral for tempered distributions in the $\mathbb{C}^n$ setting corresponding to analytic functions in the quadrant cone setting T^{C_μ} in $\mathbb{C}^n$ and showed that the analytic functions in $(\mathbb{C} - \mathbb{R})^n$, which have distributional boundary values in $\mathcal{S}'(\mathbb{R}^n)$, can be recovered as the Cauchy integral of the boundary value; the results of [7] can be extended to the vector-valued tempered distributions considered in this paper by the same techniques as those in [7]. The Cauchy integrals introduced by both Sebastião e Silva and Carmichael are in fact equivalence classes of analytic functions defined by an integral involving the Cauchy kernel. Vladimirov [8–10] has defined a Cauchy integral for tempered distributions associated with analytic functions in general tubes $T^C = \mathbb{R}^n + iC \subset \mathbb{C}^n$ corresponding to regular cones C similar to the analytic functions we considered in this paper. Vladimirov showed that the analytic functions that he considered can be recovered by a Cauchy integral involving the tempered distributional boundary values of the analytic functions. The papers mentioned in this paragraph all concern scalar-valued analytic functions and distributions.

In this section, we build on our analyses of Sections 3 and 4 to obtain a Cauchy integral representation of the vector-valued analytic functions, which we considered in Theorem 1 and in Corollary 1. The proof of our results here—and the forms of our results—are different from any of the previous results concerning the Cauchy integral of the tempered distribution representation of the analytic functions. By our technique here, we do not need to divide the Cauchy kernel or the boundary value in (16) by a specified form of the polynomial and do not need to apply other special features of proof previously used by the authors in order to obtain that our Cauchy integral is well defined and that the analytic function considered is represented by a Cauchy integral involving the boundary value.

The Cauchy integral representation of the analytic functions that we considered in this paper follows. Note that cone C in the following result is assumed to be a regular cone. In Theorem 1 and Corollary 1, we assumed that cone C was an open convex cone. However, an open convex cone could contain an entirely straight line; in this case, the dual cone has measure 0 and $K(z-t) = 0,\ z \in T^C,\ t \in \mathbb{R}^n$. To avoid this triviality, we assume that cone C in the following Cauchy integral representation is a regular cone.

Theorem 2. *Let C be a regular cone in $\mathbb{R}^n$ and $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{f}(z)$ be analytic in $T^C = \mathbb{R}^n + iC$, have values in $\mathcal{H}$, and satisfy (5). There is a continuous function $\boldsymbol{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ with supp$(\boldsymbol{g}) \subseteq C^*$ a.e. and an n-tuple α of nonnegative integers, such that*

$$\boldsymbol{f}(z) = z^\alpha \langle \mathcal{F}[\boldsymbol{g}]_\nu, K(z-\nu)\rangle,\ z \in T^C, \tag{17}$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. Further,

$$\langle \mathcal{F}[\boldsymbol{g}]_\nu, K(z-\nu)\rangle = \Theta,\ z \in T^{-C}, \tag{18}$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$.

Proof. From Theorem 1, there is an element $V \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, the spectral function of $\mathbf{f}(z),\ z \in T^C$, such that $e^{-2\pi\langle y,t\rangle} V_t \in \mathcal{S}'(\mathbb{R}^n, \mathcal{H}),\ y \in C$; supp$(V) \subseteq C^*$; and $\mathbf{f}(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle} V_t]_x,\ y \in C,$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. Further, by Corollary 1, there is a continuous function $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ with supp$(\mathbf{g}) \subseteq C^*$ a.e. and an n-tuple α of nonnegative integers, such that $V_t = D_t^\alpha \mathbf{g}(t),\ t \in \mathbb{R}^n$. Now, let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $z = x + iy \in T^C$. Recall that we have defined the differential operator D to be $D_t = (-1/2\pi i)(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_n})$. We have

$$
\begin{aligned}
&\langle \mathbf{f}(x+iy), \phi(x)\rangle = \langle \mathcal{F}[e^{-2\pi\langle y,t\rangle}V_t]_x, \phi(x)\rangle \\
&= \langle e^{-2\pi\langle y,t\rangle}V_t, \hat{\phi}(t)\rangle = \langle V_t, \int_{\mathbb{R}} \phi(x)e^{2\pi i\langle z,t\rangle}dx\rangle \\
&= \langle D_t^{\alpha}\mathbf{g}(t), \int_{\mathbb{R}^n} \phi(x)e^{2\pi i\langle z,t\rangle}dx\rangle \\
&= (-1)^{|\alpha|}\int_{C^*}\mathbf{g}(t)\int_{\mathbb{R}^n}\phi(x)(-1/2\pi i)^{|\alpha|}(2\pi i)^{|\alpha|}z^{\alpha}e^{2\pi i\langle z,t\rangle}dxdt \\
&= \int_{C^*}\mathbf{g}(t)\int_{\mathbb{R}^n}\phi(x)z^{\alpha}e^{-2\pi\langle y,t\rangle}e^{2\pi i\langle x,t\rangle}dxdt \\
&= \int_{C^*}e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)\mathcal{F}[z^{\alpha}\phi(x);t]dt \\
&= \langle z^{\alpha}\mathcal{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)]_x, \phi(x)\rangle
\end{aligned} \tag{19}
$$

where $I_{C^*}(t)$ is the characteristic function of C^*. We have proven in [17] (Lemma 4.2.1, p. 62) that $I_{C^*}(t)e^{-2\pi\langle y,t\rangle} \in L^p$, $y \in C$, for all p, $1 \le p \le \infty$. Since $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$, then $\mathcal{F}[\mathbf{g}]_x \in L2(\mathbb{R}^n, \mathcal{H})$ in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ by Lemma 1. Recall also from Section 3 that the Cauchy kernel $K(z-\cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p}(\mathbb{R}^n)$, $1 < p \le \infty$, for $z \in T^C$ with C being a regular cone and that an element of $L2(\mathbb{R}^n, \mathcal{H})$ applied to $K(z-\cdot)$, $z \in T^C$, is a well-defined function of $z \in T^C$. Continuing (19) and using convolution, we now have

$$
\begin{aligned}
&\langle \mathbf{f}(x+iy), \phi(x)\rangle = \langle z^{\alpha}(\mathcal{F}[\mathbf{g}] * \mathcal{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle}])_x, \phi(x)\rangle \\
&= \langle z^{\alpha}\langle \mathcal{F}[\mathbf{g}]_{\nu}, \mathcal{F}[I_{C^*}(t)e^{-2\pi\langle y,t\rangle}]_{(x-\nu)}\rangle, \phi(x)\rangle \\
&= \langle z^{\alpha}\langle \mathcal{F}[\mathbf{g}]_{\nu}, \int_{C^*}e^{2\pi i\langle z-\nu,t\rangle}dt\rangle, \phi(x)\rangle \\
&= \langle z^{\alpha}\langle \mathcal{F}[\mathbf{g}]_{\nu}, K(z-\nu)\rangle, \phi(x)\rangle
\end{aligned} \tag{20}
$$

where $I_{C^*}(t)$ is the characteristic function of C^*. Since $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$, then $\mathcal{F}[\mathbf{g}] \in L2(\mathbb{R}^n, \mathcal{H})$ by Lemma 1; and as previously noted, $\mathcal{F}[\mathbf{g}]$ applied to the Cauchy kernel is a well-defined function of $z \in T^C$ and is an analytic function of $z \in T^C$ with values in $\mathcal{H}$. Thus, from (20) we have obtained

$$\mathbf{f}(z) = z^{\alpha}\langle \mathcal{F}[\mathbf{g}]_{\nu}, K(z-\nu)\rangle,\ z \in T^C,$$

in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, and (17) is obtained.

To prove (18), first note that for a regular cone, C, $-C$ is also a regular cone; and $(-C)^*$ = $-C^*$. Thus, for $z \in T^{-C}$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$
\begin{aligned}
&\langle\langle \mathcal{F}[\mathbf{g}]_{\nu}, K(z-\nu)\rangle, \phi(x)\rangle = \langle\langle \mathcal{F}[\mathbf{g}]_{\nu}, \int_{-C^*}e^{-2\pi\langle y,t\rangle}e^{2\pi i\langle x-\nu,t\rangle}dt\rangle, \phi(x)\rangle \\
&= \langle\langle \mathcal{F}[\mathbf{g}]_{\nu}, \mathcal{F}[I_{-C^*}(t)e^{-2\pi\langle y,t\rangle}]_{(x-\nu)}\rangle, \phi(x)\rangle \\
&= \langle\langle(\mathcal{F}[\mathbf{g}] * \mathcal{F}[I_{-C^*}(t)e^{-2\pi\langle y,t\rangle}])_x\rangle, \phi(x)\rangle \\
&= \langle \mathcal{F}[I_{-C^*}(t)e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)]_x, \phi(x)\rangle.
\end{aligned} \tag{21}
$$

Now $I_{-C^*}(t) = 0$ if $t \notin -C^*$ and, hence, if $t \in C^*$. This fact coupled with the fact that $\mathrm{supp}(\mathbf{g}) \subseteq C^*$ a.e. yields $I_{-C^*}(t)e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) = \Theta$ a.e. for $t \in \mathbb{R}^n$ and $y \in -C$. Hence $\mathcal{F}[I_{-C^*}(t)e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)]_x = \Theta$, $x \in \mathbb{R}^n$, $y \in -C$, in (21). Thus, from (21), we have $\langle \mathcal{F}[\mathbf{g}]_{\nu}, K(z-\nu)\rangle = \Theta$, $z \in T^{-C}$, in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$; and (18) is obtained. □

6. Conclusions

Tillmann [1] obtained the original analysis concerning the scalar-valued tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ as boundary values of analytic functions. We proved a boundary value result concerning vector-valued tempered distributions $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ as boundary values of vector-valued analytic functions in [4] (Theorem 8) but used a norm growth condition on the analytic functions, which was a special case for the growth of Tillmann. We desired to obtain a result, such as [4] (Theorem 8), but under the general norm growth on the analytic function, which was equivalent to the growth of Tillmann. We achieved this first goal of this paper in Theorem 1 for vector-valued analytic functions $\mathbf{f}(z)$ on tubes $T^C = \mathbb{R}^n + iC$ with C being an open convex cone. The values of the analytic functions and the tempered distributions were in a very general type of topological vector space. We achieved additional information in Theorem 1 concerning the spectral function of the analytic function.

We asked if additional information concerning the spectral function and the boundary value could be obtained if the topological vector space $\mathcal{X}$ was restricted somewhat. We obtained the desired information in Corollary 1 by restricting $\mathcal{X}$ to be a Banach space and then a Hilbert space; we showed the structure of the spectral function and the boundary value in these cases for $\mathcal{X}$. Integral to this analysis was the Lemma 1 result, which proved the relation under the Fourier transform between two important subsets of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ for our results in Corollary 1. It is important to note that the reason to restrict to Hilbert space $\mathcal{H}$ (which we do in our results) is that the Plancherel theory for the Fourier transform of the functions holds if and only if the functions have value in the Hilbert space.

The second principal goal of this paper was to obtain a Cauchy integral representation of the analytic functions considered in Theorem 1 and Corollary 1. Sebastião e Silva, Carmichael, and Vladimirov have obtained and studied the Cauchy integral of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$ in the scalar-valued case and in one and several dimensions; see the papers of these authors in the references. Their analyses basically concerned dividing the Cauchy kernel or the boundary value by a suitable polynomial whose order was large enough to make the quotient when evaluated by the tempered distribution to be well defined, or used other special features of proof that we do not use here.

In Section 5 of this paper, we constructed our Cauchy integral used in the representation of the assumed analytic function in a different manner by using the general known structure of the spectral function and our proven structure of the tempered distributional boundary value in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ for $\mathcal{H}$ being a Hilbert space. The analytic function obtaining the boundary value in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ was shown to be equated to the product of a polynomial and the constructed Cauchy integral.

This paper concerns theoretical mathematics, yet the topics considered find applications in mathematical physics and in mathematics that are applied to physical problems. We survey historically some areas of application in the scalar-valued case. We recall the work of Streater and Wightman [23] in studying quantum field theory. In a field theory, the "vacuum expectation values" are tempered distributions, which are boundary values in the tempered distribution topology of analytic functions with the analytic functions being Fourier–Laplace transforms. In addition, a field theory can be recovered from its "vacuum expectation values"; see [23] (Chapter 3). A similar field theory analysis using boundary values of analytic functions is contained in the work by Simon [24]. We also reference Raina [25] concerning "form factor bounds" in particle physics in which tempered distributional boundary values, which are of a special form, imply that the analytic functions that obtain these boundary values are Hardy H^p functions; this fact is then used in the analysis of the "form factor bounds". See also the associated papers listed in the references of [25].

As noted in Vladimirov [8], scalar-valued analytic functions of the type that we considered in this paper can arise in applying the Fourier–Laplace transform to convolution equations, which describe linear homogeneous processes with causality that find application in the quantum field theory, theory of electrical circuits, scattering of electromagnetic waves, and linear thermodynamic systems; refer to the list of references in [8]. We also note paper [26] by Vladimirov, concerning the linear conjugacy of scalar-valued analytic functions of several complex variables, which are again of the type that we considered in this paper with respect to growth. The linear conjugacy analysis involves scalar-valued tempered distributional boundary values of analytic functions represented as Fourier–

Laplace integrals. Vladimirov [26] (p. 207) states that many problems arising in mathematical physics reduce to the problem of linear conjugacy involving tempered distributions; Vladimirov [26] provides examples of such problems.

The survey of applications above (concerning the type of analysis used in this paper) involve scalar-valued functions and distributions. Yet, a close consideration of the linear conjugacy problem of [26], together with the vector-valued analysis of this paper, leads one to believe that the linear conjugacy problem can be extended to the vector-valued case. Further, in an analysis of the stated applications above, one must sometimes obtain a distributional solution of a partial differential equation; such calculations can be extended to the vector-valued case. We suggest that the considerable related analyses to the results of this paper and the results of related references in this paper can be achieved in the vector-valued case and will work toward this end in the future.

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