axioms

Article

New Results about Radius of Convexity and Uniform Convexity of Bessel Functions

Luminiţa-Ioana Cotîrlă [1,*], Pál Aurel Kupán [2] and Róbert Szász [2]

1 Department of Mathematics, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania
2 Department of Mathematics and Informatics, Sapientia Hungarian University of Transylvania, 540485 Târgu Mureş, Romania; kupanp@ms.sapientia.ro (P.A.K.); rszasz@ms.sapientia.ro (R.S.)
* Correspondence: luminita.cotirla@math.utcluj.ro

Abstract: We determine in this paper new results about the radius of uniform convexity of two kinds of normalization of the Bessel function J_ν in the case $\nu \in (-2, -1)$, and provide an alternative proof regarding the radius of convexity of order alpha. We then compare results regarding the convexity and uniform convexity of the considered functions and determine interesting connections between them.

Keywords: Bessel function; convex function; uniformly convex functions; radius of convexity

MSC: 33C10

check for updates

Citation: Cotîrlă, L.-I.; Kupán, P.A.; Szász, R. New Results about Radius of Convexity and Uniform Convexity of Bessel Functions. *Axioms* **2022**, *11*, 380. https://doi.org/10.3390/axioms11080380

Academic Editor: Georgia Irina Oros

Received: 6 July 2022
Accepted: 30 July 2022
Published: 31 July 2022

1. Introduction

Let $U(r) = \{z \in \mathbb{C} : |z| < r\}$ be the disk, centered at zero, of radius r, where $r > 0$. We denote by $U(r) = U(0, r)$.

We say that a function f of the form

$$f(z) = z + a_2 z^2 + \dots \tag{1}$$

is convex on $U(r)$ if and only if $f(U(r))$ is a convex domain in the set $\mathbb{C}$ and the function f is univalent.

We know that the function f is convex on $U(r)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U(r).$$

We say that f is a convex function of order α on $U(r)$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad z \in U(r).$$

The radius of convexity of order α for f is defined by the equality

$$r_f^c(\alpha) = \sup\left\{r \in (0, \infty) :\ \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,\ \ z \in U(r)\right\}. \tag{2}$$

We say that f is uniformly convex in the disk $U(r)$ if the function f has the form in (1), it is a convex function, and it has the property that the arc $f(\gamma)$ is convex for every circular arc γ contained in the disk $U(r)$ with center ζ, also in $U(r)$. The function f is uniformly convex in the disk $U(r)$ if and only if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|,\ \ z \in U(r).$$

We know that the radius of uniform convexity is defined by

$$r_f^{uc}(\alpha) = \sup\left\{r \in (0,\infty) : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in U(r)\right\}. \tag{3}$$

The Bessel function of the first kind is defined by

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)}(z/2)^{2n+\nu}.$$

Consider the following normalized forms:

$$g_\nu(z) = 2^\nu\Gamma(1+\nu)z^{1-\nu}J_\nu(z) = z - \frac{1}{4(\nu+1)}z^3 + \ldots, \tag{4}$$

and

$$h_\nu(z) = 2^\nu\Gamma(1+\nu)z^{1-\nu/2}J_\nu(z^{\frac{1}{2}}) = z - \frac{1}{4(\nu+1)}z^2 + \ldots, \tag{5}$$

where ν is a real number and $-2 < \nu < -1$, and g_ν and h_ν are entire functions.

This article can be considered a continuation of previous papers [1,2] which dealt with geometric properties of Bessel functions.

For more details about the geometric properties of Bessel functions, interested readers are referred to the following papers: [1,3–13].

The aim of this work is to determine the radius of convexity of order α, $r_f^c(\alpha)$ for $f = g_\nu$ and $f = h_\nu$ and the radius of uniform convexity $r_f^{uc}(\alpha)$ for the case $\nu \in (-2,-1)$ and to derive an interesting connection between the convexity and uniform convexity.

In the next section, we provide several results which are necessary later in this work.

2. Preliminaries

Lemma 1 ([14], p. 483, Hurwitz)**.** *If $\nu \in (-2,-1)$, then $J_\nu(z)$ has exactly two purely imaginary conjugate complex zeros, and all the other zeros are real.*

The zeros $z^{-\nu}J_\nu(z)$ are taken to be $\pm j_{\nu,n}$, where $n \in \mathbb{N}^* = \{1,2,3,\ldots\}$. We may suppose, without restricting the generality, that $j_{\nu,1} = ia$, $a > 0$, and $0 < a < j_{\nu,2} < j_{\nu,3} < \cdots < j_{\nu,n} < \cdots$.

Lemma 2 ([14], p. 502)**.** *The following equality holds*

$$\sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}. \tag{6}$$

Lemma 3 ([8])**.** *In the notations of Lemma 2, we have*

$$\frac{zg_\nu'(z)}{g_\nu(z)} = 1 - 2\sum_{n=1}^{\infty} \frac{z^2}{j_{\nu,n}^2 - z^2}, \tag{7}$$

and

$$\frac{zh_\nu'(z)}{h_\nu(z)} = 1 - \sum_{n=1}^{\infty} \frac{z}{j_{\nu,n}^2 - z}. \tag{8}$$

The series are uniformly convergent on every compact subset of $\mathbb{C} \setminus \{\pm j_{\nu,n} : n \in \mathbb{N}^*\}$.

Lemma 4 ([9])**.** *If $v \in \mathbb{C},\ \delta \in \mathbb{R}$, and $\delta > \rho \geq |v|$, then*

$$\left|\frac{v}{\delta - v}\right| \leq \frac{\rho}{\delta - \rho} \quad \text{and} \quad \left|\frac{v}{(\delta - v)^2}\right| \leq \frac{\rho}{(\delta - \rho)^2}.$$

Proof. The following implications hold

$$|\delta - v| \geq \delta - \rho \Rightarrow \frac{1}{|\delta - v|} \leq \frac{1}{\delta - \rho} \Rightarrow \left|\frac{1}{(\delta - v)^2}\right| \leq \frac{1}{(\delta - \rho)^2}.$$

If the last two inequalities are multiplied by the inequality $|v| \leq \rho$, we obtain the desired results. □

Lemma 5. *If $v \in \mathbb{C},\ \delta, \gamma \in \mathbb{R}$, $\gamma \geq \delta > \rho \geq |v|$, then*

$$\left|\frac{v^2}{(\delta + v)(\gamma - v)}\right| \leq \frac{\rho^2}{(\delta - \rho)(\gamma + \rho)}. \tag{9}$$

Proof. We can prove the second inequality of the following equivalence:

$$\left|\frac{1}{(\delta + v)(\gamma - v)}\right| \leq \frac{1}{(\delta - \rho)(\gamma + \rho)} \Leftrightarrow (\delta - \rho)(\gamma + \rho) \leq \big|(\delta + v)(\gamma - v)\big|, \tag{10}$$

where $\gamma \geq \delta > \rho \geq |v|$.

We prove the inequality (10) in two steps.

Let $v = x + iy$; then, it is obvious that

$$\big|(\delta + v)(\gamma - v)\big| = \sqrt{[(\delta + x)^2 + y^2][(+y^2 + \gamma - x)^2]} \geq \big|(\gamma - x)(\delta + x)\big|, \tag{11}$$

where $\gamma \geq \delta > \rho \geq \sqrt{x^2 + y^2}$.

On the other hand, a simple calculation results in

$$(\delta + x)(\gamma - x) \geq (\delta - \rho)(\gamma + \rho),\ x \in [-\rho, \rho]. \tag{12}$$

It is easily seen that (11) and (12) imply the second inequality of (10). Finally, multiplying the inequality $\rho^2 \geq |v|^2$ by the first inequality of (10), we obtain (9) and the proof is complete. □

Lemma 6. *If $v \in \mathbb{C},\ \delta, \gamma \in \mathbb{R}$, and $\gamma \geq \delta > \rho \geq |v|$, then*

$$\left|\frac{2v^2[2\gamma\delta + (\gamma - \delta)v]}{(\gamma - v)^2(\delta + v)^2}\right| \leq \frac{2r^2[2\gamma\delta - (\gamma - \delta)\rho]}{(\gamma + \rho)^2(\delta - \rho)^2}. \tag{13}$$

Proof. The inequality obviously holds provided that $\gamma = \delta$ (see (10)), thus, we have to prove it in the case that $\gamma > \delta$.

We can then prove the following inequality:

$$\left|\frac{2\gamma\delta + (\gamma - \delta)v}{(\delta + v)(\gamma - v)}\right| \leq \frac{2\gamma\delta - (\gamma - \delta)\rho}{(\delta - \rho)(\gamma + \rho)},\ \ \gamma \geq \delta > \rho \geq |v|. \tag{14}$$

We define $z = x + iy$ and define the mapping

$$\phi : [-\rho, \rho] \to \mathbb{R},\ \phi(y) = \frac{(\omega + x)^2 + y^2}{[(\delta + x)^2 + y^2][(\gamma - x)^2 + y^2]},\ \omega = \frac{2\gamma\delta}{\gamma - \delta}.$$

Then, we have

$$\phi'(y) = 2y\frac{[(\delta+x)^2+y^2][(\gamma-x)^2+y^2]-[(\delta+x)^2+(\gamma-x)^2+2y^2][(\omega+x)^2+y^2]}{[(\delta+x)^2+y^2]^2[(\gamma-x)^2+y^2]^2}.$$

As $\phi'(y) < 0,\ y \in (0,\rho)$ and $\phi'(y) > 0,\ y \in (-\rho, 0)$, it follows that

$$\phi(y) \leq \phi(0) = \frac{(\omega+x)^2}{[(\delta+x)^2][(\gamma-x)^2]},\ y \in [-\rho,\rho]. \tag{15}$$

We can determine the maximum of the function

$$\varphi : [-\rho,\rho] \to \mathbb{R},\ \ \varphi(x) = \frac{\omega+x}{(\delta+x)(\gamma-x)}.$$

We have

$$\varphi'(x) = \frac{x^2+2\omega x-\gamma\delta}{(\delta+x)^2(\gamma-x)^2}$$

The derivative $\varphi'(x) = 0$ has one positive root, $x_1 = \sqrt{\omega^2+\gamma\delta}-\omega$, and one negative root, $x_2 = -\sqrt{\omega^2+\gamma\delta}-\omega$. As $x_2 < -r$ and $x_1 \in (-\rho,\rho)$, it follows the inequality

$$\frac{\omega+x}{(\delta+x)(\gamma-x)} = \varphi(x) \leq \max\{\varphi(-\rho),\varphi(\rho)\} = \varphi(-\rho) = \frac{\omega-\rho}{(\delta-\rho)(\gamma+\rho)} \tag{16}$$

for every $x \in [-\rho,\rho]$. From (15) and (16), we have (14). Finally, multiplying the inequalities (14), $|v^2| \leq \rho^2$ and the first inequality of (10), we infer (13). $\square$

Lemma 7. *If the functions g_ν and h_ν are defined by (4) and (5), respectively, then*

$$\frac{zg_\nu''(z)}{g_\nu'(z)} = z\frac{zJ_{\nu+2}(z)-3J_{\nu+1}(z)}{J_\nu(z)-zJ_{\nu+1}(z)}. \tag{17}$$

$$\frac{zh_\nu''(z)}{h_\nu'(z)} = \frac{zJ_{\nu+2}(z^{\frac{1}{2}})-4z^{\frac{1}{2}}J_{\nu+1}(z^{\frac{1}{2}})}{4J_\nu(z^{\frac{1}{2}})-2z^{\frac{1}{2}}J_{\nu+1}(z^{\frac{1}{2}})}. \tag{18}$$

Proof. We differentiate the equality (4), and at the second time we differentiate it logarithmically. After multiplying by z, we obtain the following equality:

$$\frac{zg_\nu''(z)}{g_\nu'(z)} = \frac{z^2J_\nu''(z)+2z(1-\nu)J_\nu'(z)+\nu(\nu-1)J_\nu(z)}{zJ_\nu'(z)+(1-\nu)J_\nu(z)}.$$

The function J_ν is a solution of the Bessel differential equation; thus, we can replace the function z^2J_ν'' using the equality $z^2J_\nu''(z) = (\nu^2-z^2)J_\nu(z)-zJ_\nu'(z)$, and it follows that

$$\frac{zg_\nu''(z)}{g_\nu'(z)} = \frac{z(1-2\nu)J_\nu'(z)+(2\nu^2-\nu-z^2)J_\nu(z)}{zJ_\nu'(z)+(1-\nu)J_\nu(z)}.$$

In the second step, we use the following well-known equality: $zJ_\nu'(z) = \nu J_\nu(z) - zJ_{\nu+1}(z)$, and infer

$$\frac{zg_\nu''(z)}{g_\nu'(z)} = \frac{z(2\nu-1)J_{\nu+1}(z)-z^2J_\nu(z)}{J_\nu(z)-zJ_{\nu+1}(z)}.$$

Finally, we replace $zJ_\nu(z)$ in the numerator by $zJ_\nu(z) = 2(\nu+1)J_{\nu+1}(z) - zJ_{\nu+2}(z)$, and obtain (17).

We differentiate equality (5) twice, similarly to the case of the function g_ν, and obtain

$$\frac{zh_\nu''(z)}{h_\nu'(z)}=\frac{\nu(\nu-2)J_\nu(z^{\frac{1}{2}})+(3-2\nu)z^{\frac{1}{2}}J_\nu'(z^{\frac{1}{2}})+zJ_\nu''(z^{\frac{1}{2}})}{2(2-\nu)J_\nu(z^{\frac{1}{2}})+2z^{\frac{1}{2}}J_\nu'(z^{\frac{1}{2}})}.$$

We use the equality $zJ_\nu''(z^{\frac{1}{2}})=(\nu^2-z)J_\nu(z^{\frac{1}{2}})-z^{\frac{1}{2}}J_\nu'(z^{\frac{1}{2}})$, and obtain

$$\frac{zh_\nu''(z)}{h_\nu'(z)}=\frac{(2\nu^2-2\nu-z)J_\nu(z^{\frac{1}{2}})+(2-2\nu)z^{\frac{1}{2}}J_\nu'(z^{\frac{1}{2}})}{2(2-\nu)J_\nu(z^{\frac{1}{2}})+2z^{\frac{1}{2}}J_\nu'(z^{\frac{1}{2}})}.$$

Now, using the equality $z^{\frac{1}{2}}J_\nu'(z^{\frac{1}{2}})=\nu J_\nu(z^{\frac{1}{2}})-z^{\frac{1}{2}}J_{\nu+1}(z^{\frac{1}{2}})$, we infer

$$\frac{zh_\nu''(z)}{h_\nu'(z)}=\frac{(2\nu-2)z^{\frac{1}{2}}J_{\nu+1}(z^{\frac{1}{2}})-zJ_\nu(z^{\frac{1}{2}})}{4J_\nu(z^{\frac{1}{2}})-2z^{\frac{1}{2}}J_{\nu+1}(z^{\frac{1}{2}})},$$

and combining this with the equality $z^{\frac{1}{2}}J_\nu(z^{\frac{1}{2}})=2(\nu+1)J_{\nu+1}(z^{\frac{1}{2}})-z^{\frac{1}{2}}J_{\nu+2}(z^{\frac{1}{2}})$, (18) follows. □

3. Main Results

Theorem 1. *If $\alpha\in[0,1)$ and $\nu\in(-2,-1)$, then the radius of convexity of order α for the mapping g_ν is $r_\nu^c(\alpha)=r_1$, where r_1 is the unique root of the equation*

$$1+r\frac{I_{\nu+2}(r)+3I_{\nu+1}(r)}{I_{\nu+1}(r)+rI_\nu(r)}=\alpha \tag{19}$$

in the interval $(0,a)$.

Proof. According to the proof of Theorem 1 [2], the equalities

$$\frac{zg_\nu'(z)}{g_\nu(z)}=1-2\sum_{n=1}^{\infty}\frac{z^2}{j_{\nu,n}^2-z^2},\ \sum_{n=1}^{\infty}\frac{1}{j_{\nu,n}^2}=\frac{1}{4(\nu+1)}.$$

imply

$$\frac{zg_\nu'(z)}{g_\nu(z)}=1-\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2}-2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}.$$

The logarithmic differentation of this equality leads to

$$\begin{aligned}1+\frac{zg_\nu''(z)}{g_\nu'(z)}&=\\ 1-\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2}-2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}&-\\ \frac{\frac{a^2}{1+\nu}\frac{a^2z^2}{(a^2+z^2)^2}+2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{2z^4[2a^2j_{\nu,n}^2+z^2(j_{\nu,n}^2-a^2)]}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}}{1-\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2+z^2}-2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}}&.\end{aligned} \tag{20}$$

It is proven in Theorem 1 [2] that the radius of starlikeness, $r_{g_\nu}^*$, for the function g_ν is the smallest root of the equation

$$\begin{aligned}1+\frac{a^2}{2(1+\nu)}\frac{r^2}{a^2-r^2}&\\ -2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)}&=ir\frac{g_\nu'(ir)}{g_\nu(ir)}=0,\end{aligned}$$

in the interval $(0, a)$. Thus, we have

$$0 < r^*_{g_\nu} < a < j_{\nu,2} < j_{\nu,3} < \cdots < j_{\nu,n} < \cdots . \tag{21}$$

Taking into account that $\nu + 1 < 0$, the equality (20) implies the following inequality:

$$\begin{aligned} &\operatorname{Re}\left(1+\frac{z g_\nu''(z)}{g_\nu'(z)}\right) \geq \\ &1+\frac{a^2}{2(1+\nu)}\left|\frac{z^2}{a^2+z^2}\right| - 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right| - \\ &\frac{-\frac{a^2}{1+\nu}\left|\frac{a^2z^2}{(a^2+z^2)^2}\right| + 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{2z^4[2a^2j_{\nu,n}^2+z^2(j_{\nu,n}^2-a^2)]}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}\right|}{1+\frac{a^2}{2(1+\nu)}\left|\frac{z^2}{a^2+z^2}\right| - 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right|} \end{aligned} \tag{22}$$

for every $z \in U(r^*_\nu)$.

Using $\delta = a^2$, $\rho = r^2$ and $v = z^2$ in Lemma 4, we obtain

$$\frac{a^2}{2(1+\nu)}\left|\frac{z^2}{a^2+z^2}\right| \geq \frac{a^2}{2(1+\nu)}\frac{r^2}{a^2-r^2} \text{ and } \frac{a^2}{2(1+\nu)}\left|\frac{z^2}{(a^2+z^2)^2}\right| \geq \frac{a^2}{(a^2-r^2)^2}\frac{r^2}{2(1+\nu)}. \tag{23}$$

In a similar manner, Lemma 5 and Lemma 6 imply that

$$\begin{aligned} \left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right| &\leq \frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)} \\ \left|\frac{2z^4[2a^2j_{\nu,n}^2+z^2(j_{\nu,n}^2-a^2)]}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}\right| &\leq \frac{2r^4[2a^2j_{\nu,n}^2-r^2(j_{\nu,n}^2-a^2)]}{(a^2-r^2)^2(j_{\nu,n}^2+r^2)^2}. \end{aligned} \tag{24}$$

Now, inequalities (22)–(24) imply the following inequality:

$$\begin{aligned} &\operatorname{Re}\left(1+\frac{z g_\nu''(z)}{g_\nu'(z)}\right) \geq \\ &1+\frac{a^2}{2(1+\nu)}\frac{r^2}{a^2-r^2} - 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)} - \\ &\frac{-\frac{a^2}{1+\nu}\frac{a^2r^2}{(a^2-r^2)^2} + 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{2r^4[2a^2j_{\nu,n}^2-r^2(j_{\nu,n}^2-a^2)]}{(a^2-r^2)^2(j_{\nu,n}^2+r^2)^2}}{1+\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2-r^2} - 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)}} = 1+\frac{irg_\nu''(ir)}{g_\nu'(ir)} = \Phi(r), \end{aligned} \tag{25}$$

provided that $a > r^*_{g_\nu} > |z|$, where $r^*_{g_\nu}$ verifies the inequalities (21).

The following equalities hold: $\Phi(0) = 1$ and $\lim_{r \nearrow r^*_{g_\nu}} \Phi(r) = -\infty$. Consequently, equation $1 + \frac{irg_\nu''(ir)}{g_\nu'(ir)} = \alpha$ has a real root in the interval $(0, r^*_{g_\nu})$. The smallest positive real root of the equation $1 + \frac{irg_\nu''(ir)}{g_\nu'(ir)} = \alpha$ is denoted by $r^c_{g_\nu}(\alpha)$, and this root is the radius of convexity of order α of the function g_ν. The first equality of Lemma 7 and the equality $J_\nu(iz) = i^\nu I_\nu(z)$ imply that the equation $1 + \frac{irg_\nu''(ir)}{g_\nu'(ir)} = \alpha$ is equivalent to (19).
□

We determine the radius of uniform convexity of the mapping g_ν in the next theorem.

Theorem 2. *If $\nu \in (-2,-1)$, then the radius of uniform convexity for the mapping g_ν is $r_\nu^*(\alpha) = r_2$, where r_2 is the smallest positive root of the equation*

$$\frac{1}{2} + r\frac{I_{\nu+2}(r) + 3I_{\nu+1}(r)}{I_{\nu+1}(r) + rI_\nu(r)} = 0 \tag{26}$$

in the interval $(0, r_\nu^*)$.

Proof. Equality (20) implies the following inequality:

$$\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \leq \frac{-\frac{a^2}{2(1+\nu)}\left|\frac{z^2}{a^2+z^2}\right| + 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right| + -\frac{a^2}{1+\nu}\left|\frac{a^2z^2}{(a^2+z^2)^2}\right| + 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{2z^4[2a^2j_{\nu,n}^2+z^2(j_{\nu,n}^2-a^2)]}{(a^2+z^2)^2(j_{\nu,n}^2-z^2)^2}\right|}{1+\frac{a^2}{2(1+\nu)}\left|\frac{z^2}{a^2+z^2}\right| - 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\left|\frac{z^4}{(a^2+z^2)(j_{\nu,n}^2-z^2)}\right|}. \tag{27}$$

We can again use inequalities (22) and (23), and in combination with (27), we have

$$\left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \leq \frac{-\frac{a^2}{2(1+\nu)}\frac{r^2}{a^2-r^2} + 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)} + -\frac{a^2}{1+\nu}\frac{a^2r^2}{(a^2-r^2)^2} + 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{2r^4[2a^2j_{\nu,n}^2-r^2(j_{\nu,n}^2-a^2)]}{(a^2-r^2)^2(j_{\nu,n}^2+r^2)^2}}{1+\frac{a^2}{2(1+\nu)}\frac{z^2}{a^2-r^2} - 2\sum_{n=2}^{\infty}\frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2}\frac{r^4}{(a^2-r^2)(j_{\nu,n}^2+r^2)}} = -\frac{irg_\nu''(ir)}{g_\nu'(ir)}.$$

Inequalities (25) and (27) imply

$$\mathrm{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right) - \left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| \geq 1 + 2\frac{irg_\nu''(ir)}{g_\nu'(ir)},\ z \in U(r_\nu^*). \tag{28}$$

The smallest positive root of the equation $1 + 2\frac{irg_\nu''(ir)}{g_\nu'(ir)} = 0$ in the interval $(0, r_\nu^*)$ is denoted by r_ν^{uc}. According to (28), the value r_ν^{uc} is the biggest with the property that

$$\mathrm{Re}\left(1+\frac{zg_\nu''(z)}{g_\nu'(z)}\right) - \left|\frac{zg_\nu''(z)}{g_\nu'(z)}\right| > 0,\ z \in U(r_\nu^{uc}).$$

Lemma 7 and the equality $J_\nu(iz) = i^\nu I_\nu(z)$ imply that the equation $1 + 2\frac{irg_\nu''(ir)}{g_\nu'(ir)} = 0$ is equivalent to (26), completing the proof. □

Theorems 1 and 2 imply the following result.

Corollary 1. *The mapping g_ν is uniformly convex in the disk $U(r)$ if and only if it is convex of order $\frac{1}{2}$.*

Theorem 3. *If $\alpha \in [0,1)$ and $\nu \in (-2,-1)$, then the radius of convexity of order α for the mapping h_ν is $r_{h_\nu}^c(\alpha) = r_3$, where r_3 is the smallest real root of the equation*

$$1 + \frac{rI_{\nu+2}(r^{\frac{1}{2}}) + 4r^{\frac{1}{2}}I_{\nu+1}(r^{\frac{1}{2}})}{4I_\nu(r^{\frac{1}{2}}) + 2r^{\frac{1}{2}}I_{\nu+1}(r^{\frac{1}{2}})} = \alpha \tag{29}$$

in the interval $(0, r_{h_\nu}^*)$.

Proof. According to the proof of Theorem 2 [2], the equalities

$$\frac{zh_\nu'(z)}{h_\nu(z)} = 1 - \sum_{n=1}^{\infty} \frac{z}{j_{\nu,n}^2 - z}, \ \sum_{n=1}^{\infty} \frac{1}{j_{\nu,n}^2} = \frac{1}{4(\nu+1)}$$

imply

$$\frac{zh_\nu'(z)}{h_\nu(z)} = 1 - \frac{a^2}{4(\nu+1)} \cdot \frac{z}{a^2+z} - \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{z^2}{(a^2+z)\left(j_{\nu,n}^2 - z\right)},$$

where $z \in U(0,r)$.

The logarithmic differentiation of the equality leads to

$$1 + \frac{zh_\nu''(z)}{h_\nu'(z)} = 1 - \frac{a^2}{4(\nu+1)} \cdot \frac{z}{a^2+z} - \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{z^2}{(a^2+z)\left(j_{\nu,n}^2 - z\right)} -$$

$$- \frac{\frac{a^2}{4(1+\nu)} \cdot \frac{a^2 z}{(a^2+z)^2} + \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{z^2\left[2a^2 j_{\nu,n}^2 + z\left(j_{\nu,n}^2 - a^2\right)\right]}{\left(j_{\nu,n}^2 - z\right)^2 (a^2+z)^2}}{1 - \frac{a^2}{4a^2+z} \cdot \frac{z}{(\nu+1)} - \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{z^2}{\left(j_{\nu,n}^2 - z\right)(a^2+z)}}. \tag{30}$$

It is proven in [2] that the radius of starlikeness, $r_{h_\nu}^*$, for function h_ν is the smallest root of the equation

$$\frac{-rh_\nu'(-r)}{h_\nu(-r)} = 0, \ r \in \left(0, a^2\right), \ z \in U(0,r).$$

However,

$$\frac{-rh_\nu'(-r)}{h_\nu(-r)} = 1 + \frac{a^2}{4(\nu+1)} \cdot \frac{r}{a^2 - r} -$$

$$- \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{r^2}{(a^2-r)\left(j_{\nu,n}^2 + r\right)} = 0, \ r \in \left(0, a^2\right).$$

Taking into the account that $\nu + 1 < 0$, we obtain from relation (30)

$$\operatorname{Re}\left(1 + \frac{zh_\nu''(z)}{h_\nu'(z)}\right) \geq \frac{a^2}{4(\nu+1)} \cdot \left|\frac{z}{a^2+z}\right| - \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \left|\frac{z^2}{(a^2+z)\left(j_{\nu,n}^2 - z\right)}\right| -$$

$$- \frac{\frac{-a^2}{4(\nu+1)} \cdot \left|\frac{a^2 z}{(a^2+z)^2}\right| + \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \left|\frac{z^2\left[2a^2 j_{\nu,n}^2 + z\left(j_{\nu,n}^2 - a^2\right)\right]}{(a^2+z)^2\left(j_{\nu,n}^2 - z\right)^2}\right|}{1 + \frac{a^2}{4(\nu+1)} \cdot \left|\frac{z}{a^2+z}\right| - \sum_{n}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \left|\frac{z^2}{(a^2+z)\left(j_{\nu,n}^2 - z\right)}\right|} \tag{31}$$

and $z \in U(0,r), r \in \left(0, r_{h_\nu}^*\right)$. We obtain from Lemmas 4 and 5 the following inequality:

$$\operatorname{Re}\left(1 + \frac{zh_\nu''(z)}{h_\nu'(z)}\right) \geq 1 + \frac{a^2}{4(\nu+1)} \cdot \frac{r}{a^2 - r} - \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{r^2}{(a^2-r)(a^2+r)} -$$

$$- \frac{-\frac{a^2}{4(\nu+1)} \cdot \frac{a^2 r}{(a^2-r)^2} + \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{r^2\left[2a^2 j_{\nu,n}^2 - r\left(j_{\nu,n}^2 - a^2\right)\right]}{(a^2-r)^2\left(j_{\nu,n}^2 + r\right)^2}}{1 + \frac{a^2}{4(\nu+1)} \cdot \frac{r}{a^2-r} - \sum_{n=2}^{\infty} \frac{a^2+j_{\nu,n}^2}{j_{\nu,n}^2} \cdot \frac{r^2}{(a^2-r)\left(j_{\nu,n}^2 - r\right)}} = \tag{32}$$

$$= 1 - \frac{rh_\nu''(-r)}{h_\nu'(-r)} = \psi(r), \ a > r_{h_\nu}^* > |z|,$$

similarly to the proof of Theorem 1. The mapping

$$\psi : \left(0, r^*_{h_\nu}\right) \to \mathbb{R},\ \psi(r) = 1 + \frac{-rh''_\nu(-r)}{h'_\nu(-r)},$$

is strictly decreasing, and $a > r^*_{h_\nu} > |z|$.

We then have $\lim\limits_{r \nearrow r^*_{h_\nu}} \psi(r) = -\infty$, $\psi(0) = 1$, and the equation

$$1 + \frac{-rh''_\nu(-r)}{h'_\nu(-r)} = \alpha$$

has at least one real root in the interval $\left(0, r^*_{h_\nu}\right)$.

The smallest positive real root of the equation $1 - \frac{rh''_\nu(-r)}{h'_\nu(-r)} = \alpha$ is denoted by $r^c_{h_\nu}(\alpha)$, and this root is the radius of convexity of order α of the function h_ν. The second equality of Lemma 7 and the equality $J_\nu(iz) = i^\nu I_\nu(z)$ imply that the equation $1 - \frac{rh''_\nu(-r)}{h'_\nu(-r)} = \alpha$ is equivalent to (29). □

Theorem 4. *If $\alpha \in [0,1)$ and $\nu \in (-2,-1)$, then the radius of uniform convexity of h_ν is $r^*_{h_\nu}(\alpha) = r_4$, where r_4 is the smallest positive root of the equation*

$$\frac{rI_{\nu+2}(r^{\frac{1}{2}}) + 4r^{\frac{1}{2}} I_{\nu+1}(r^{\frac{1}{2}})}{4I_\nu(r^{\frac{1}{2}}) + 2r^{\frac{1}{2}} I_{\nu+1}(r^{\frac{1}{2}})} = \frac{1}{2} \tag{33}$$

in the interval $(0, r^*_{h_\nu})$.

Proof. Equality (30) implies the following inequality:

$$\left|\frac{zh''_\nu(z)}{h'_\nu(z)}\right| \le -\frac{a^2}{4(\nu+1)} \cdot \left|\frac{z}{a^2+z}\right| + \sum_{n=2}^{\infty} \frac{a^2 + j^2_{\tilde{\nu},n}}{j^2_{\tilde{\nu},n}} \cdot \left|\frac{z^2}{(a^2+z)\left(j^2_{\tilde{\nu},n} - z\right)}\right| +$$
$$+ \frac{\frac{-a^2}{4(\nu+1)} \cdot \left|\frac{a^2 z}{(a^2+z)^2}\right| + \sum_{n=2}^{\infty} \frac{a^2+j^2_{\tilde{\nu},n}}{j^2_{\tilde{\nu},n}} \cdot \left|\frac{z^2\left[2a^2 j^2_{\tilde{\nu},n} + z\left(j^2_{\tilde{\nu},n} - a^2\right)\right]}{(a^2+z)^2\left(j^2_{\tilde{\nu},n} - z\right)^2}\right|}{1 + \frac{a^2}{4(\nu+1)} \cdot \left|\frac{z}{a^2+z}\right| - \sum_{n=2}^{\infty} \frac{a^2+j^2_{\tilde{\nu},n}}{j^2_{\tilde{\nu},n}} \cdot \left|\frac{z^2}{(a^2+z)\left(j^2_{\tilde{\nu},n} - z\right)}\right|}. \tag{34}$$

We obtain the following from the relation (31), Lemma 4, and the relation (34):

$$\left|\frac{zh''_\nu(z)}{h'_\nu(z)}\right| \le -\frac{a^2}{4(\nu+1)} \cdot \frac{r}{a^2 - r} + \sum_{n=2}^{\infty} \frac{a^2 + j^2_{\tilde{\nu},n}}{j^2_{\tilde{\nu},n}} \cdot \frac{r^2}{(a^2 - r)\left(j^2_{\tilde{\nu},n} + r\right)} +$$
$$+ \frac{\frac{a^2}{4(\nu+1)} \cdot \frac{a^2 r}{(a^2-r)^2} + \sum_{n=2}^{\infty} \frac{a^2+j^2_{\tilde{\nu},n}}{j^2_{\tilde{\nu},n}} \cdot \frac{r^2\left[2a^2 j^2_{\tilde{\nu},n} - r\left(j^2_{\tilde{\nu},n} - a^2\right)\right]}{(a^2-r)^2\left(j^2_{\tilde{\nu},n}+r\right)^2}}{1 - \frac{a^2}{4(a^2-r)} \cdot \frac{r}{\nu+1} - \sum_{n=2}^{\infty} \frac{a^2+j^2_{\tilde{\nu},n}}{j^2_{\tilde{\nu},n}} \cdot \frac{r^2}{(a^2-r)\left(j^2_{\tilde{\nu},n}+r\right)}} = \frac{rh''_\nu(-r)}{h'_\nu(-r)},\ |z| \le r < a^2.$$

Inequalities (32) and (34) imply

$$\mathrm{Re}\left(1 + \frac{zh''_\nu(z)}{h'_\nu(z)}\right) - \left|\frac{zh''_\nu(z)}{h'_\nu(z)}\right| \ge 1 - \frac{2rh''_\nu(-r)}{h'_\nu(-r)},\ z \in U(r^*_{h_\nu}). \tag{35}$$

The smallest positive root of the equation $1 - \frac{2rh''_\nu(-r)}{h'_\nu(-r)} = 0$ in the interval $(0, r^*_{h_\nu})$ is denoted by $r^{uc}_{h_\nu}$.

According to (35), the value $r^{uc}_{h_\nu}$ is the biggest with the property that

$$\mathrm{Re}\left(1+\frac{zh''_\nu(z)}{h'_\nu(z)}\right)-\left|\frac{zh''_\nu(z)}{h'_\nu(z)}\right|>0,\ z\in U(r^{uc}_{h_\nu}).$$

The equation $1-\frac{2rh''_\nu(-r)}{h'_\nu(-r)}=0$ is equivalent to (33), completing the proof. Lemma 7 and the equality $J_\nu(iz)=i^\nu I_\nu(z)$ imply that the equation $1-\frac{2rh''_\nu(-r)}{h'_\nu(-r)}=0$ is equivalent to (33). □

From Theorems 3 and 4, we obtain the following corollary.

Corollary 2. *The function h_ν is uniformly convex in the disk $U(r)$ if and only if it is convex of order $\frac{1}{2}$.*

Author Contributions: Conceptualization, L.-I.C., R.S. and P.A.K.; methodology, R.S.; software, L.-I.C., R.S. and P.A.K.; validation, R.S.; formal analysis, L.-I.C., R.S. and P.A.K.; investigation, L.-I.C., R.S. and P.A.K.; resources, L.-I.C., R.S. and P.A.K.; data curation, L.-I.C., R.S. and P.A.K.; writing—original draft preparation, L.-I.C., R.S. and P.A.K.; writing—review and editing, L.-I.C., R.S. and P.A.K.; visualization, L.-I.C., R.S. and P.A.K.; supervision, R.S.; project administration, L.-I.C., R.S. and P.A.K.; funding acquisition, L.-I.C., R.S. and P.A.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to express their sincere thanks to the referees for their valuable suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Deniz, E.; Szász, R. The radius of uniform convexity of Bessel functions. *J. Math. Anal. Appl.* **2017**, *453*, 572–588. [CrossRef]
2. Szász, R. About the radius of starlikeness of Bessel functions of the first kind. *Monatsh. Math.* **2015**, *176*, 323–330. [CrossRef]
3. Baricz, Á.; Szakál, A.; Szász, R.; Yagmur, N. Radii of starlikeness and convexity of a product and cross-product of Bessel functions. *Results Math.* **2018**, *73*, 62. [CrossRef]
4. Baricz, Á. Geometric properties of generalized Bessel functions. *Publ. Math. Debrecen* **2008**, *73*, 155–178.
5. Baricz, Á; *Generalized Bessel Functions of the First Kind*; Lecture Notes in Mathematics, 1994; Springer: Berlin/Heidelberg, Germany, 2010; Volume XIV, p. 206, ISBN: 978-3-642-12229-3.
6. Baricz, Á; Ponnusamy, S. Starlikeness and convexity of generalized Bessel functions. *Integral Transform. Spec. Funct.* **2010**, *21*, 641–653. [CrossRef]
7. Baricz, Á.; Szász, R. The radius of convexity of normalized Bessel functions. *Anal. Math.* **2015**, *41*, 141–151. [CrossRef]
8. Baricz Á.; Kupán, A.P.; Szász, R. The radius of starlikeness of normalized Bessel functions of the first kind. *Proc. AMS* **2014**, *142*, 2019–2025. [CrossRef]
9. Bulut, S.; Engel, O. The radius of starlikeness, Convexity and uniform convexity of the Legendre polynomials of odd degree. *Results Math.* **2019**, *74*, 48. [CrossRef]
10. Liang, Ch.Y.; Engel, O. Certain properties of the generalized Mittag-Leffler function.. *Electronic J. Math. Anal. Appl.* **2018**, *6*, 288–294.
11. Murat, Ç.; Deniz, E.; Szász, R. Radii of a-convexity of some normalized Bessel functions of the first kind. *Results Math.* **2017**, *72*, 2023–2035.
12. Engel, O.; Kupán, A.P.; Páll-Szabó, Á.O. About the radius of convexity of some analytic functions. *Creat. Math. Inform.* **2015**, *24*, 155–161. [CrossRef]
13. Engel, O.; Páll-Szabó, Á.O. The radius of convexity of particular functions and applications to the study of a second order differential inequality. *Izv. Nats. Akad. Nauk Armenii Mat.* **2017**, *52*, 127–132. Reprinted in *J. Contemp. Math. Anal.* **2017**, *52*, 118–127. [CrossRef]
14. Watson, G.N. *A Treatise of the Theory of Bessel Functions*; Cambridge University Press: Cambridge, UK, 1944.