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The Influence of Multiplicative Noise and Fractional Derivative on the Solutions of the Stochastic Fractional Hirota–Maccari System

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Abstract: We address here the space-fractional stochastic Hirota–Maccari system (SFSHMs) derived by the multiplicative Brownian motion in the Stratonovich sense. To acquire innovative elliptic, trigonometric and rational stochastic fractional solutions, we employ the Jacobi elliptic functions method. The attained solutions are useful in describing certain fascinating physical phenomena due to the significance of the Hirota–Maccari system in optical fibers. We use MATLAB programm to draw our figures and exhibit several 3D graphs in order to demonstrate how the multiplicative Brownian motion and fractional derivative affect the exact solutions of the SFSHMs. We prove that the solutions of SFSHMs are stabilized by the multiplicative Brownian motion around zero.

Keywords: fractional Hirota–Maccari system; stochastic Hirota–Maccari system; Jacobi elliptic functions method

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1. Introduction

Recently, numerous significant phenomena have been represented by fractional derivatives, including electro-magnetic, image processing, acoustics, electrochemistry and anomalous diffusion phenomena [1–6]. One benefit of fractional models is that they may be stated more specifically than integer models, which encourages us to construct a number of significant and practical fractional models. On the other hand, the advantages of taking random influences into account in the analysis, simulation, prediction and modeling of complex processes have been highlighted in several fields including chemistry, geophysics, fluid mechanics, biology, atmosphere, physics, climate dynamics, engineering and other fields [7–10]. Since noise may produce statistical features and significant phenomena, it cannot be ignored. In general, it is more difficult to obtain exact solutions to fractional PDEs forced by a stochastic term than to classical ones.

Recently, finding approximate and exact solutions to PDEs using a variety of approaches has become the main objective for many scientists. Many effective methods, including the sine-Gordon expansion method [11], the trial equation method [12], (G'/G)-expansion [13,14], semi-inverse variational principle [15], the ansatz approach [16], perturbation methods [17,18], Darboux transformation [19], tanh-sech [20,21], $exp(-\phi(\varsigma))$-expansion [22] and the Jacobi elliptic function [23,24], have been devised to obtain exact solutions to PDEs.

As a result, we study here the following stochastic fractional-space Hirota–Maccari system (SFSHMs) with multiplicative noise in the Stratonovich sense:

$$i\Phi_t + \mathcal{D}^{\alpha}_{xy}\Phi + i\mathcal{D}^{\alpha}_{xxx}\Phi + \Phi\Psi - i\Phi\mathcal{D}^{\alpha}_{x}(|\Phi|^2) + i\sigma\Phi \circ W_t = 0, \quad (1)$$

$$3\mathcal{D}^{\alpha}_{x}\Psi + \mathcal{D}^{\alpha}_{y}(|\Phi|^2) = 0, \quad (2)$$

where $\Psi(x, y, t)$ denotes the real field of scalars and $\Phi(x, y, t)$ is the complex scalar field, x, y are independent spatial variables and t is the temporal variable. $\mathcal{D}^{\alpha}_{x}$ is the conformable derivative (CD) for $\alpha \in (0, 1]$ [25]. $W_t = \frac{dW}{dt}$ is the time derivative of Brownian motion $W(t)$ and σ is a noise strength.

The stochastic integral $\int_0^t \Phi(s)dW(s)$ is called the Stratonovich stochastic integral (denoted by $\int_0^t \Phi(s) \circ dW(s)$), if we calculate the stochastic integral at the middle, while the stochastic integral $\int_0^t \Phi(s)dW(s)$ is called Itô (denoted by $\int_0^t \Phi(s)dW(s)$) when we calculate it at the left end [26]. The relation between the Stratonovich integral and Itô integral is:

$$\int_0^t \Phi(s, Z_s)dW(s) = \int_0^t \Phi(s, Z_s) \circ dW(s) - \frac{1}{2}\int_0^t \Phi(s, Z_s)\frac{\partial\Phi(s, Z_s)}{\partial z}ds. \quad (3)$$

The conformable derivative for the function $\phi : (0, \infty) \to \mathbb{R}$ is defined for $\alpha \in (0, 1]$ as

$$\mathcal{D}^{\alpha}_{x}\phi(x) = \lim_{\kappa \to 0} \frac{\phi(x + \kappa x^{1-\alpha}) - \phi(x)}{\kappa}. \quad (4)$$

The important property of CD is the following chain rule:

$$\mathcal{D}^{\alpha}_{x}(\phi_1 \circ \phi_2)(x) = x^{1-\alpha}\phi_2'(x)\phi_1'(\phi_2(x)).$$

The Hirota–Maccari system (1-2), with $\sigma = 0$ and $\alpha = 1$, was derived by Maccari [27]. There are several physical applications of the integrable Hirota–Maccari system including the transmission of optical pulses across nematic liquid crystal waveguides and for a certain parameter regime, the transmission of femtosecond pulses through optical fibers. Due to the importance of the Hirota–Maccari system, many researchers have examined a lot of techniques in order to find the exact solutions for this system, such as the extended trial equation and the generalized Kudryashov [28], tanh-coth, sec-tan, rational sinh-cosh and sech-csch methods [29], (G'/G)-expansion [30], Hirota bilinear method [31], Weierstrass elliptic function expansion [32], Painleve approach [33], Painleve test [34], general projective Riccati equation and improved $tan(\frac{\phi(\theta)}{2})$-expansion method [35] and complex hyperbolic-function [36]. While the exact solutions of stochastic Hirota–Maccari system have been studied in [37] in the Itô sense by using three different methods: Riccati–Bernoulli sub-ODE, sine-cosine and He's semi-inverse.

The originality of this paper is to acquire the analytical solutions of the SFSHMs (1-2). This work is the first to attain the exact solutions of the SFSHMs (1-2). We employ the Jacobi elliptic functions approach to obtain a broad range of solutions, including hyperbolic, trigonometric and rational functions. Moreover, to study the effects of Brownian motion on the solutions of the SFSHMs (1-2), we build 3D graphs for some of the developed solutions by using MATLAB tools.

This is how the paper is organized: We use a suitable wave transformation in Section 2 to provide the wave equation of SFSHMs. We employ the Jacobi elliptic functions approach in Section 3 to obtain the analytical solutions of the SFSHMs (1-2). In Section 4, we look at how the Brownian motion affects the generated solutions. Finally, we state the conclusions of this paper.

2. Wave Equation for SFSHMs

To get the wave equation of the SFSHMs (1-2), let us utilize the following transformation:

$$\Phi(x,y,t)=Q(\zeta)e^{i\theta-\sigma W(t)-\sigma^2 t},\ \ \Psi(x,y,t)=P(\zeta)e^{-2\sigma W(t)-2\sigma^2 t}, \tag{5}$$

with

$$\zeta=(\frac{\zeta_1}{\alpha}x^\alpha+\frac{\zeta_2}{\alpha}y^\alpha+\zeta_3 t),\ \theta=\frac{\theta_1}{\alpha}x^\alpha+\frac{\theta_2}{\alpha}y^\alpha+\theta_3 t,$$

where θ_k, ζ_k for $k=1,2,3$ are nonzero constants. We substitute Equation (5) into Equations (1-2), and use

$$\begin{aligned}\frac{d\Phi}{dt} &= (\zeta_3 Q'+i\theta_3 Q-\sigma QW_t+\frac{1}{2}\sigma^2 Q-\sigma^2 Q)e^{i\theta-\sigma W(t)-\sigma^2 t},\\ &= (\zeta_3 Q'+i\theta_3 Q-\sigma QW_t-\frac{1}{2}\sigma^2 Q)e^{i\theta-\sigma W(t)-\sigma^2 t},\\ &= (\zeta_3 Q'+i\theta_3 Q-\sigma Q\circ W_t)e^{i\theta-\sigma W(t)-\sigma^2 t},\end{aligned}$$

and

$$\begin{aligned}\mathcal{D}_x^\alpha\Phi &= (\zeta_1 Q'+i\theta_1 Q)e^{i\theta-\sigma W(t)-\sigma^2 t},\ \ \mathcal{D}_y^\alpha\Phi(|\Phi|^2)=\zeta_2(Q^2)'e^{-2\sigma W(t)-2\sigma^2 t},\\ \mathcal{D}_{xxx}^\alpha\Phi &= (\zeta_1^3 Q'''+3i\theta_1\zeta_1^2 Q''-2\theta_1^2\zeta_1 Q'-\theta_1^2\zeta_1 Q'-i\theta_1^3 Q)e^{i\theta-\sigma W(t)-\sigma^2 t},\\ \mathcal{D}_{xy}^\alpha\Phi &= (\zeta_1\zeta_2 Q''+i\zeta_1\theta_2 Q'+i\zeta_2\theta_1 Q'-\theta_1\theta_2 Q)e^{i\theta-\sigma W(t)-\sigma^2 t},\end{aligned}$$

to obtain for the real part

$$(\zeta_1\zeta_2-\theta_1\zeta_1^2)Q''-(\theta_3+\theta_1\theta_2-\theta_1^3)Q+QPe^{-2\sigma W(t)-2\sigma^2 t} = 0, \tag{6}$$

$$3\zeta_1 P'+\zeta_2(Q^2)' = 0. \tag{7}$$

Integrating Equation (7), we have

$$P=\frac{-\zeta_2}{3\zeta_1}Q^2. \tag{8}$$

Setting Equation (8) into Equation (6) we obtain

$$Q''-A_1 Q^3 e^{-2\sigma W(t)-2\sigma^2 t}-A_2 Q=0, \tag{9}$$

where

$$A_1=\frac{\zeta_2}{3\zeta_1(\zeta_1\zeta_2-\theta_1\zeta_1^2)}\quad\text{and}\quad A_2=\frac{\theta_3+\theta_1\theta_2-\theta_1^3}{\zeta_1\zeta_2-\theta_1\zeta_1^2}. \tag{10}$$

Taking expectation $\mathbb{E}(\cdot)$ on both sides for Equation (9), we attain

$$Q''-A_1 Q^3 e^{-2\sigma^2 t}\mathbb{E}(e^{-2\sigma W(t)})-A_2 Q=0. \tag{11}$$

Since $W(t)$ is a normal process, then $\mathbb{E}(e^{-2\sigma W(t)})=e^{2\sigma^2 t}$. Therefore Equation (11) becomes

$$Q''-A_1 Q^3-A_2 Q=0. \tag{12}$$

3. The Analytical Solutions of the SFSHMs

In this section, we use the Jacobi elliptic functions method [38] to acquire the solutions to Equation (12). Consequently, we obtain the analytical solutions of the SFSHMs (1-2).

3.1. Method Description

Let the solutions of Equation (12) have the form

$$Q(\zeta) = \sum_{i=1}^{\mathsf{N}} a_i \mathcal{Z}^i(\zeta), \tag{13}$$

where $\mathcal{Z}$ solves

$$\mathcal{Z}' = \sqrt{\frac{1}{2}\ell_1 \mathcal{Z}^4 + \ell_2 \mathcal{Z}^2 + \ell_3}, \tag{14}$$

where ℓ_1, ℓ_2 and ℓ_3 are real parameters and N is a positive integer number.

We notice that Equation (14) has a variety of solutions depending on ℓ_1, ℓ_2 and ℓ_3 as in the following Table 1 :

Table 1. All possible solutions for Equation (14) for different values of ℓ_1, ℓ_2 and ℓ_3.

Case	ℓ_1	ℓ_2	ℓ_3	$\mathcal{Z}(\zeta)$
1	$2\mathbf{m}^2$	$-(1+\mathbf{m}^2)$	1	$sn(\zeta)$
2	2	$2\mathbf{m}^2-1$	$-\mathbf{m}^2(1-\mathbf{m}^2)$	$ds(\zeta)$
3	2	$2-\mathbf{m}^2$	$(1-\mathbf{m}^2)$	$cs(\zeta)$
4	$-2\mathbf{m}^2$	$2\mathbf{m}^2-1$	$(1-\mathbf{m}^2)$	$cn(\zeta)$
5	-2	$2-\mathbf{m}^2$	$(\mathbf{m}^2-1)$	$dn(\zeta)$
6	$\frac{\mathbf{m}^2}{2}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{1}{4}$	$\frac{sn(\zeta)}{1\pm dn(\zeta)}$
7	$\frac{\mathbf{m}^2}{2}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{\mathbf{m}^2}{4}$	$\frac{sn(\zeta)}{1\pm dn(\zeta)}$
8	$\frac{-1}{2}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{-(1-\mathbf{m}^2)^2}{4}$	$\mathbf{m}cn(\zeta)\pm dn(\zeta)$
9	$\frac{\mathbf{m}^2-1}{2}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{(\mathbf{m}^2-1)}{4}$	$\frac{dn(\zeta)}{1\pm sn(\zeta)}$
10	$\frac{1-\mathbf{m}^2}{2}$	$\frac{(1-\mathbf{m}^2)}{2}$	$\frac{(1-\mathbf{m}^2)}{4}$	$\frac{cn(\zeta)}{1\pm sn(\zeta)}$
11	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{1}{4}$	$\frac{sn(\zeta)}{dn\pm cn(\zeta)}$
12	2	0	0	$\frac{c}{\zeta}$
13	0	1	0	ce^{ζ}

Where $dn(\zeta) = dn(\zeta, \mathbf{m})$, $cn(\zeta) = cn(\zeta, \mathbf{m})$, $sn(\zeta) = sn(\zeta, \mathbf{m})$ are the Jacobi elliptic functions (JEFs) for $0 < \mathbf{m} < 1$. If $\mathbf{m} \to 1$, then the JEFs are transformed into the following hyperbolic functions:

$$\begin{aligned} cs(\zeta) &\to \operatorname{csch}(\zeta), \quad sn(\zeta) \to \tanh(\zeta), \quad cn(\zeta) \to \operatorname{sech}(\zeta), \\ dn(\zeta) &\to \operatorname{sech}(\zeta), \quad ds \to \operatorname{csch}(\zeta). \end{aligned}$$

3.2. Solutions of SFSHMs

Let us balance Q'' with Q^3 in Equation (12) to define N as follows:

$$\mathsf{N} + 2 = 3\mathsf{N} \Longrightarrow \mathsf{N} = 1. \tag{15}$$

Equation (14) is rewritten with $\mathsf{N} = 1$ as

$$Q(\zeta) = a_0 + a_1 \mathcal{Z}(\zeta). \tag{16}$$

Differentiating Equation (16) twice, we have, by using (14),

$$Q'' = a_1 \ell_2 \mathcal{Z} + a_1 \ell_1 \mathcal{Z}^3. \tag{17}$$

Plugging Equation (16) and Equation (17) into Equation (12) we have

$$(a_1\ell_1 - A_1a_1^3)\mathcal{Z}^3 - 3a_0a_1^2A_1\mathcal{Z}^2 + (a_1\ell_2 - 3A_1a_0^2a_1 + A_2a_1)\mathcal{Z} - (A_1a_0^3 - A_2a_0) = 0.$$

Setting each coefficient of $\mathcal{Z}^k$ for $k = 0,1,2,3$ equal to zero, we attain

$$a_1\ell_1 - A_1a_1^3 = 0,$$

$$3a_0a_1^2A_1 = 0,$$

$$a_1\ell_2 - 3A_1a_0^2a_1 + A_2a_1 = 0,$$

and

$$A_1a_0^3 - A_2a_0 = 0.$$

We obtain by solving these equations

$$a_0 = 0,\ \ a_1 = \pm\sqrt{\frac{\ell_1}{A_1}},\ \ \ell_2 = -A_2.$$

Thus, Equation (12) has the following solution

$$Q(\zeta) = \pm\sqrt{\frac{\ell_1}{A_1}}\mathcal{Z}(\zeta),\ \ \text{for}\ \ \frac{\ell_1}{A_1} > 0. \tag{18}$$

The following are two sets that depend on ℓ_1 and A_1 :

First set: If $\ell_1 > 0$ (from Table 1)and $A_1 > 0$, then the wave Equation (12) has the solution $Q(\zeta)$ as in the following Table 2:

Table 2. All possible solutions for wave Equation (12) when $\ell_1 > 0$.

Case	ℓ_1	ℓ_2	ℓ_3	$\mathcal{Z}(\zeta)$	$Q(\zeta)$
1	$2m^2$	$-(1+\mathbf{m}^2)$	1	$sn(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}sn(\zeta)$
2	2	$2\mathbf{m}^2-1$	$-\mathbf{m}^2(1-\mathbf{m}^2)$	$ds(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}ds(\zeta)$
3	2	$2-\mathbf{m}^2$	$(1-\mathbf{m}^2)$	$cs(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}cs(\zeta)$
4	$\frac{\mathbf{m}^2}{2}$	$\frac{(\mathbf{m}^2-2)}{2}$	$\frac{1}{4}$ or $\frac{\mathbf{m}^2}{4}$	$\frac{sn(\zeta)}{1\pm dn(\zeta)}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{sn(\zeta)}{1\pm dn(\zeta)}$
5	$\frac{1-\mathbf{m}^2}{2}$	$\frac{(1-\mathbf{m}^2)}{2}$	$\frac{(1-\mathbf{m}^2)}{4}$	$\frac{cn(\zeta)}{1\pm sn(\zeta)}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{cn(\zeta)}{1\pm sn(\zeta)}$
6	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{(1-\mathbf{m}^2)^2}{2}$	$\frac{1}{4}$	$\frac{sn(\zeta)}{dn\pm cn(\zeta)}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{sn(\zeta)}{dn\pm cn(\zeta)}$
7	2	0	0	$\frac{c}{\zeta}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{c}{\zeta}$

If $\mathbf{m} \to 1$, then the previous Table 2 becomes

Table 3. All possible solutions for wave Equation (12) when $\ell_1 > 0$ and $\mathbf{m} \to 1$.

Case	ℓ_1	ℓ_2	ℓ_3	$\mathcal{Z}(\zeta)$	$Q(\zeta)$
1	2	-2	1	$\tanh(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}\tanh(\zeta)$
2	2	1	0	$\text{sech}(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}\text{sech}(\zeta)$
3	2	1	0	$\text{csch}(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}\text{csch}(\zeta)$
4	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{1}{4}$	$\frac{\tanh(\zeta)}{1\pm\text{sech}(\zeta)}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{\tanh(\zeta)}{1\pm\text{sech}(\zeta)}$
5	2	0	0	$\frac{c}{\zeta}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{c}{\zeta}$

Now, using the previous Table 2 (or Table 3 when $\mathbf{m} \to 1$) and Equations (5) and (18), we obtain the exact solutions of the SFSHMs (1-2), for $\frac{\ell_1}{A_1} > 0$, as follows:

$$\Phi(x,y,t) = Q(\zeta)e^{(i\theta-\sigma W(t)-\sigma^2 t)}, \tag{19}$$

$$\Psi(x,y,t) = \frac{-\zeta_2}{3\zeta_1}Q^2(\zeta)e^{(-2\sigma W(t)-2\sigma^2 t)}, \tag{20}$$

where $\zeta = (\frac{\zeta_1}{\alpha}x^\alpha + \frac{\zeta_2}{\alpha}y^\alpha + \zeta_3 t)$, $\theta = \frac{\theta_1}{\alpha}x^\alpha + \frac{\theta_2}{\alpha}y^\alpha + \theta_3 t$.

Second set: If $\ell_1 < 0$ and $A_1 < 0$, then the solutions $Q(\zeta)$ of the wave Equation (12) are

Table 4. All possible solutions for wave Equation (12) when $\ell_1 < 0$.

Case	ℓ_1	ℓ_2	ℓ_3	$\mathcal{Z}(\zeta)$	$Q(\zeta)$
1	$-2\mathbf{m}^2$	$2\mathbf{m}^2 - 1$	$(1-\mathbf{m}^2)$	$cn(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}cn(\zeta)$
2	-2	$2-\mathbf{m}^2$	$(\mathbf{m}^2-1)$	$dn(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}dn(\zeta)$
3	$\frac{-1}{2}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{-(1-\mathbf{m}^2)^2}{4}$	$\mathbf{m}cn(\zeta) \pm dn(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}[\mathbf{m}cn(\zeta) \pm dn(\zeta)]$
4	$\frac{\mathbf{m}^2-1}{2}$	$\frac{(\mathbf{m}^2+1)}{2}$	$\frac{(\mathbf{m}^2-1)}{4}$	$\frac{dn(\zeta)}{1\pm sn(\zeta)}$	$\pm\sqrt{\frac{\ell_1}{A_1}}\frac{dn(\zeta)}{1\pm sn(\zeta)}$

If $\mathbf{m} \to 1$, then the previous Table 4 becomes

Table 5. All possible solutions for wave Equation (12) when $\ell_1 < 0$ and $\mathbf{m} \to 1$.

Case	ℓ_1	ℓ_2	ℓ_3	$\mathcal{Z}(\zeta)$	$Q(\zeta)$
1	-2	1	0	$\text{sech}(\zeta)$	$\pm\sqrt{\frac{\ell_1}{A_1}}\text{sech}(\zeta)$
2	$\frac{-1}{2}$	2	0	$2\text{sech}(\zeta)$	$\pm 2\sqrt{\frac{\ell_1}{A_1}}\text{sech}(\zeta)$

In this situation, we may obtain the analytical solutions of the SFSHMs (1-2) as reported in Equations (19) and (20) by utilizing the previous Table 4 (or Table 5 when $\mathbf{m} \to 1$).

4. The Effect of Noise and Fractional Derivative on Solutions

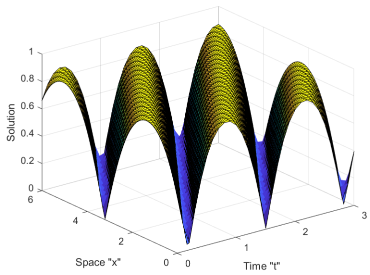
In this article, the impact of noise and fractional derivative on the acquired solutions of the SFSHMs (1-2) is discussed. We utilize the MATLAB tools to create some graphs, for various noise strength σ, for the following solutions:

$$\Phi(x,y,t) = \sqrt{\frac{\ell_1}{A_1}}sn(\frac{\zeta_1}{\alpha}x^\alpha + \frac{\zeta_2}{\alpha}y^\alpha + \zeta_3 t)e^{(i\theta-\sigma W(t)-\sigma^2 t)}, \tag{21}$$

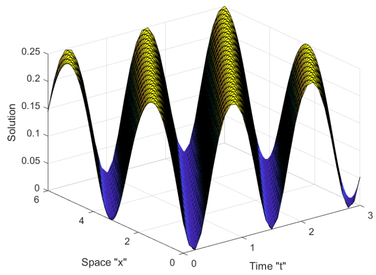
$$\Psi(x,y,t) = \frac{-\zeta_2 \ell_1}{3\zeta_1 A_1} sn^2\left(\frac{\zeta_1}{\alpha}x^\alpha + \frac{\zeta_2}{\alpha}y^\alpha + \zeta_3 t\right)e^{-2\sigma W(t) - 2\sigma^2 t}. \tag{22}$$

Fixing the following parameters: $\zeta_1 = \zeta_2 = \theta_2 = 1$, $\theta_1 = 0.5$, $\theta_3 = 0.4$, and $y = 0.5$, then $\zeta_3 = -2$, and $A_1 = \frac{2}{3}$. In this case $\mathbf{m} = 0.5$, $\ell_1 = 0.5$ and $\zeta = \frac{1}{\alpha}x^\alpha + \frac{1}{\alpha}(0.5)^\alpha - 2t$.

Firstly the effect of noise: In the next Figure 1, when $\sigma = 0$, we observe that the surface fluctuates



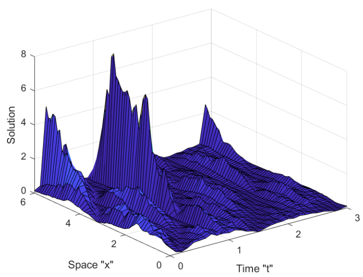
Equation (21) with $\sigma = 0$



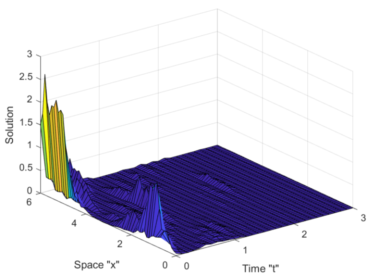
Equation (22) with $\sigma = 0$

Figure 1. 3D profile of Equations (21) and (22) with $\sigma = 0$.

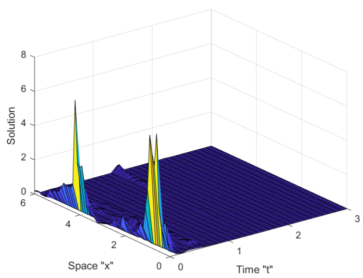
Furthermore, in Figure 2, if the noise intensity is raised, the surface becomes more planar after small transit behaviors as follows:



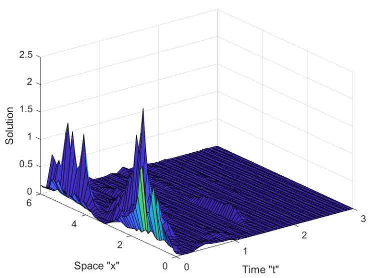
Equation (21) with $\sigma = 1$



Equation (22) with $\sigma = 1$



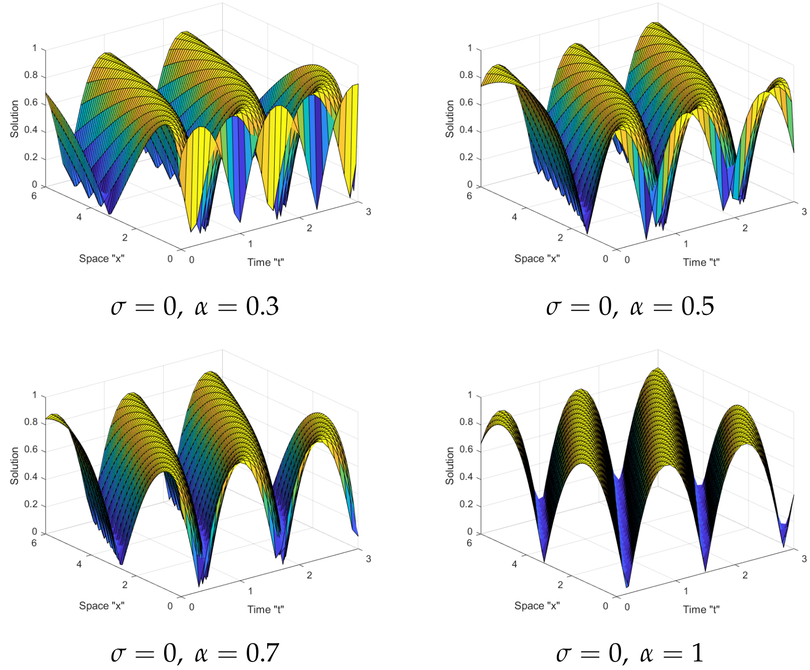
Equation (21) with $\sigma = 2$



Equation (22) with $\sigma = 2$

Figure 2. 3D profile of Equations (21) and (22) with $\sigma = 1, 2$.

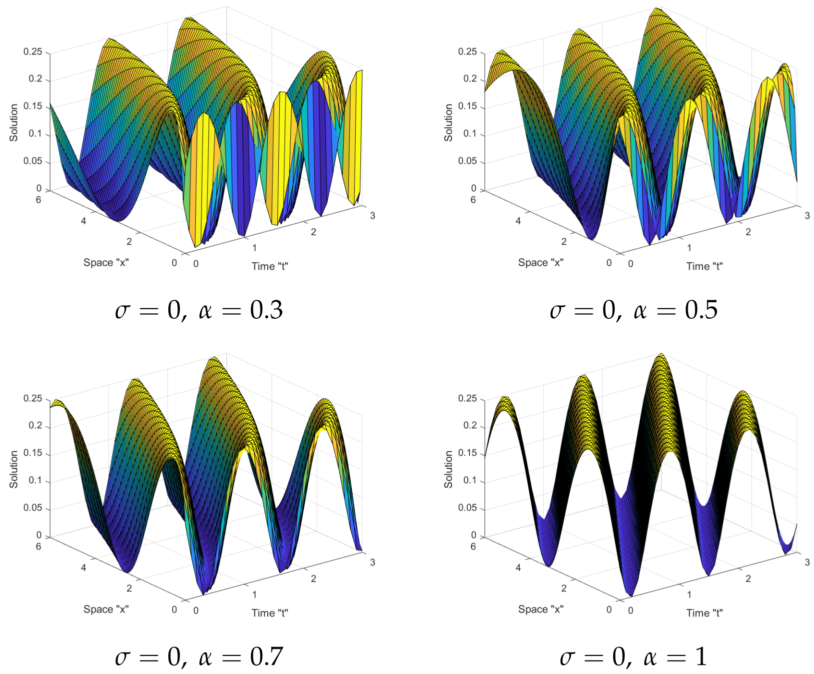
Secondly the effect of fractional order: In Figures 3 and 4, if $\sigma = 0$, we can observe that as α increases, the surface extends:



$\sigma = 0, \ \alpha = 0.3$ $\sigma = 0, \ \alpha = 0.5$

$\sigma = 0, \ \alpha = 0.7$ $\sigma = 0, \ \alpha = 1$

Figure 3. 3D profile of Equation (21) with $\sigma = 0$ and various α.



$\sigma = 0, \ \alpha = 0.3$ $\sigma = 0, \ \alpha = 0.5$

$\sigma = 0, \ \alpha = 0.7$ $\sigma = 0, \ \alpha = 1$

Figure 4. 3D profile of Equation (22) with $\sigma = 0$ and various α.

5. Conclusions

The stochastic fractional-space Hirota–Maccari system (1-2) were taken into consideration in this work. To obtain stochastic trigonometric, elliptic, rational solutions, we used the Jacobi elliptic functions approach. The obtained solutions will be very helpful for further research in disciplines such as optical fibers and others. Finally, an illustration is provided of how multiplicative Brownian motion affects the exact solutions of the SFSHMs (1-2). In future studies, we can consider SDSEs with additive noise.

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References

1. Yuste, S.B.; Acedo, L.; Lindenberg, K. Reaction front in an $A + B \to C$ reaction–subdiffusion process. *Phys. Rev. E* **2004**, *69*, 036126. [CrossRef] [PubMed]
2. Mohammed, W.W.; Iqbal, N.; Botmart, T. Additive noise effects on the stabilization of fractional-space diffusion equation solutions. *Mathematics* **2022**, *10*, 130. [CrossRef]
3. Benson, D.A.; Wheatcraft, S.W.; Meerschaert, M.M. The fractional-order governing equation of Lévy motion. *Water Resour.* **2000**, *36*, 1413–1423. [CrossRef]
4. Mohammed, W.W.; Aly, E.S.; Matouk, A.E.; Albosaily, S.; EM Elabbasy, E.M. An analytical study of the dynamic behavior of Lotka-Volterra based models of COVID-19. *Phys. Rev. Lett.* **2001**, *87*, 118301. [CrossRef] [PubMed]
5. Mohammed, W.W.; Bazighifan, O.; Al-Sawalha M.M.; Almatroud, A.O.; Aly, E.S. The influence of noise on the exact solutions of the stochastic fractional-space chiral nonlinear schrdinger equation. *Fractal Fract.* **2021**, *5*, 262. [CrossRef]
6. Barkai, E.; Metzler, R.; Klafter, J. From continuous time random walks to the fractional Fokker–Planck equation. *Phys. Rev.* **2000**, *61*, 132–138. [CrossRef]
7. Mohammed, W.W.; Blömker, D. Fast-diffusion limit for reaction-diffusion equations with multiplicative noise. *J. Math. Anal. Appl.* **2021**, *496*, 124808. [CrossRef]
8. Weinan, E.; Li, X.; Vanden-Eijnden, E. Some recent progress in multiscale modeling. *Multiscale Model. Simul.* **2004**, *39*, 3–21.
9. Imkeller, P.; Monahan, A.H. Conceptual stochastic climate models. *Stoch. Dynam.* **2002**, *2*, 311–326. [CrossRef]
10. Mohammed, W.W. Modulation equation for the stochastic Swift–Hohenberg equation with cubic and quintic nonlinearities on the Real Line. *Mathematics* **2020**, *6*, 1217. [CrossRef]
11. Rezazadeh, H.; Mirzazadeh, M.; Mirhosseini-Alizamini, S.M.; Neirameh, A.; Eslami, M.; Zhou, Q. Optical solitons of Lakshmanan–Porsezian–Daniel model with a couple of nonlinearities. *Optik* **2018**, *164*, 414–423. [CrossRef]
12. Arshed, S.; Raza, N.; Alansari, M. Soliton solutions of the generalized Davey-Stewartson equation with full nonlinearities via three integrating schemes. *Ain Shams Eng. J.* **2021**, *12*, 3091–3098. [CrossRef]
13. Wang, M.L.; Li, X.Z.; Zhang, J.L. The $(\frac{G'}{G})$-expansion method and travelling wave solutions of nonlinear evolution equations in mathematical physics. *Phys. Lett. A* **2008**, *372*, 417–423. [CrossRef]
14. Mirzazadeh, M.; Eslami, M.; Milovic, D.; Biswas, A. Topological solitons of resonant nonlinear Schödinger's equation with dual-power law nonlinearity by $(\frac{G'}{G})$-expansion technique. *Optik* **2014**, *125*, 5480–5489. [CrossRef]
15. Biswas, A.; Zhou, Q.; Moshokoa, S.P.; Triki, H.; Belic, M.; Alqahtani, R. Resonant 1-soliton solution in anti-cubic nonlinear medium with perturbations. *Optik* **2017**, *145*, 14–17. [CrossRef]
16. Savescu, M.; Bhrawy, A.H.; Hilal, E.M.; Alshaery, A.A.; Biswas, A. Optical Solitons in Birefringent Fibers with Four-Wave Mixing for Kerr Law Nonlinearity. *Rom. J. Phys.* **2014**, *59*, 582–589.
17. Blömker, D.; Mohammed, W.W. Amplitude equations for SPDEs with cubic nonlinearities. *Stochastics Int. J. Probability Stoch. Process.* **2013**, *85*, 181–215. [CrossRef]
18. Mohammed, W.W. Amplitude equation with quintic nonlinearities for the generalized Swift-Hohenberg equation with additive degenerate noise. *Adv. Differ. Equ.* **2016**, *1*, 1–18. [CrossRef]
19. Wen-Xiu, M.; Sumayah, B. A binary darboux transformation for multicomponent NLS equations and their reductions. *Anal. Math. Phys.* **2021**, *11* 44.
20. Al-Askar, F.M.; Mohammed, W.W.; Albalahi, A.M.; El-Morshedy, M. The Impact of the Wiener process on the analytical solutions of the stochastic (2+ 1)-dimensional breaking soliton equation by using tanh–coth method. *Mathematics* **2022**, *10*, 817. [CrossRef]
21. Malfliet, W.; Hereman, W. The tanh method. I. Exact solutions of nonlinear evolution and wave equations. *Phys. Scr.* **1996**, *54*, 563–568. [CrossRef]
22. Khan, K.; Akbar, M.A. The $exp(-\Phi(\varsigma))$ -expansion method for finding travelling wave solutions of Vakhnenko-Parkes equation. *Int. J. Dyn. Syst. Differ. Equ.* **2014**, *5*, 72–83.

23. Yan, Z.L. Abunbant families of Jacobi elliptic function solutions of the-dimensional integrable Davey-Stewartson-type equation via a new method. *Chaos Solitons Fractals* **2003**, *18*, 299–309. [CrossRef]
24. Mohammed, W.W.; FM Al-Askar, F.M.; Cesarano, C.; El-Morshedy, M. The Optical Solutions of the Stochastic Fractional Kundu-Mukherjee-Naskar Model by Two Different Methods. *Mathematics* **2020**, *10*, 1465. [CrossRef]
25. Khalil, R.; Horani, M.A.; Yousef, A.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [CrossRef]
26. Kloeden, P.E.; Platen, E. *Numerical Solution of Stochastic Differential Equations*; Springer: New York, NY, USA, 1995.
27. Maccari, A. A generalized Hirota equation in (2+1) dimensions. *J. Math. Phys.* **1998**, *39*, 6547–6551. [CrossRef]
28. Demiray, S.T.; Pandir, Y.; Bulut, H. All exact travelling wave solutions of Hirota equation and Hirota-Maccari system. *Opt. Int. J. Light Electron Opt.* **2016**, *127*, 1848–1859. [CrossRef]
29. Wazwaz, A.M. Abundant soliton and periodic wave solutions for the coupled Higgs eld equation, the Maccari system and the Hirota-Maccari system. *Phys. Scr.* **2012**, *85*, 1–10. [CrossRef]
30. Malik, A.; Chand, F.; Khatri, H. Exact solutions of some physical models using the (G'/G)-expansion method. *Pramana* **2012**, *78*, 513. [CrossRef]
31. Yu, X.; Gao, Y.T.; Sun, Z.Y.; Meng, X.H.; Liu, Y.; Feng, Q.; Wang, M.Z. N-soliton solutions for the (2+1)-dimensional Hirota-Maccari equation in fluids, plasmas and optical bers. *J. Math. Anal. Appl.* **2011**, *378*, 519–527. [CrossRef]
32. Chen, Y.; Yan, Z. The Weierstrass elliptic function expansion method and its applications in nonlinear wave equations. *Chaos Solitons Fractals* **2006**, *29*, 948–964. [CrossRef]
33. Liang, Z.F.; Tang, X.Y. Modulational instability and variable separation solution for a generalized (2+1)-dimensional Hirota equation. *Chin. Phys. Lett.* **2010**, *27*, 1–4.
34. Xu, G.Q.; Li, Z.B. The Painleve test of nonlinear partial differential equations and its implementation using Maple. *Comput. Algebr. Geom. Algebra Appl.* **2005**, *3519*, 179–190.
35. Raza, N.; Jhangeery, A.; Rezazadehz, H.; Bekir, A. Explicit solutions of the (2 + 1)-dimensional Hirota-Maccarisystem arising in nonlinear optics. *Int. J. Mod. Phys. B* **2019**, *33*, 1950360. [CrossRef]
36. Bai, C.L.; Zhao, H. Complex hyperbolic-function method and its applications to nonlinear equations. *Phys. Lett. A* **2006**, *355*, 32–38. [CrossRef]
37. Mohammed, W.W.; Ahmad, H.; Boulares, H.; Khelifi, F.; El-Morshedy, M. Exact solutions of Hirota–Maccari system forced by multiplicative noise in the Itô sense. Journal of Low Frequency Noise. *Vib. Act. Control* **2022**, *41*, 74–84.
38. Peng, Y.Z. Exact solutions for some nonlinear partial differential equations. *Phys. Lett. A* **2013**, *314*, 401–408. [CrossRef]