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*Article*

# Existence and Uniqueness Theorem for a Solution to a Class of a Third-Order Nonlinear Differential Equation in the Domain of Analyticity

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**Abstract:** The paper considers the specifics of nonlinear differential equations that have applications in different areas. Earlier, the authors proved the existence and uniqueness theorem for a solution to a class of non-linear differential equations in a neighborhood of a moving singular point. In this paper, we consider the first problem of studying a third-order nonlinear differential equation in the domain of analyticity. An analytical approximate solution is built, taking into account the solution search area. A priori estimates of the analytical approximate solution are obtained, and the technology of their optimization using a posteriori ones is illustrated. The result of a numerical experiment is presented. The presented results allow to expand the class of nonlinear differential equations for describing various phenomena and processes.

**Keywords:** nonlinear differential equations; wave processes; analytical approximate solution; Cauchy problem; a priori estimate

**MSC:** 34G20; 35A05



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## 1. Introduction

Recently, much attention has been paid to nonlinear differential equations for applications in various fields. In particular, a mathematical model based on a nonlinear differential equation [1,2] with moving singular points is used to study cantilever structures. To study wave processes in elastic beams, in [3], a third-order differential equation is considered in an implicit form. The paper [4] considers wave processes in beams based on the generalized Korteweg–de Vries–Burgers equation. During the transition to a stationary process, the equation is reduced to an ordinary differential equation. By varying the parameters of the equation, it is possible to ensure that this equation passes to the class of differential equations we are considering. One of the main points to unite these papers consists on the authors not taking into account the features of nonlinear differential equations. In [5], this study was continued, a solution was proposed to a class of nonlinear differential equations, where the presence of moving singular points was proved. At the same time, the authors demonstrate the practical application of series with fractional negative powers that do not currently have a generally accepted terminology. If in the paper [5] a study was carried out with regard to a neighborhood of a moving singular point, then in this paper the study is continued in the domain of analyticity. An analytical approximate solution is constructed, with a guarantee that there are no moving singular points in the area under consideration. Taking into account the specifics of the equations, we can conclude that the solution search area is divided into two parts: the analyticity area and the neighborhood of the moving singular point. A technique for optimizing a priori estimates of an analytical approximate solution using a posteriori estimates is shown. We pay attention to the results of publications [6–12], which present the development of the theory of nonlinear differential equations for other classes of equations.

## 2. Research Methods

Let us consider the following differential equation:

$$y''' = a_7(x)y^7 + a_6(x)y^6 + a_5(x)y^5 + a_4(x)y^4 + a_3(x)y^3 + a_2(x)y^2 + a_1(x)y + a_0(x). \quad (1)$$

Based on the replacement proposed in paper [5]

$$y(x) = u(x) \cdot z(x) + v(x), \quad (2)$$

we bring Equation (1) to the normal form

$$z''' = z^7(x) + r(x). \quad (3)$$

Let us consider the Cauchy problem

$$y''' = y^7(x) + r(x), \quad (4)$$

$$\begin{cases} y(x_0) = y_0, \\ y'(x_0) = y_1, \\ y''(x_0) = y_2. \end{cases} \quad (5)$$

**Theorem 1.** *We require the fulfillment of two conditions:*

1. *$r(x) \in C^\infty$ in the domain $|x - x_0| < \rho_1$ where $0 < \rho_1 = const$;*
2. *$\exists M_n : \frac{|r^{(n)}(x_0)|}{n!} \leq M_n, M_n = const$, then there is a unique solution to the Cauchy problem (4)–(5) that can be represented as a regular series*

$$y(x) = \sum_0^\infty C_n (x - x_0)^n, \quad (6)$$

*in the domain $|x - x_0| < \rho_2$, where*

$$\rho_2 = \min\left\{\rho_1, \frac{1}{(M+1)^2}\right\}, \; M = \max\left\{|y_0|, |y_1|, |y_2|, \sup_n\left\{\frac{\left|r^{(n)}(x_0)\right|}{n!}\right\}\right\}, \; n = 0, 1, 2, \ldots$$

**Proof.** We build a solution to the problem (4)–(5) for the domain of analyticity in the form of a regular series (6), where $C_0 \neq 0$. By the condition of the theorem, the function can also be represented by a regular series:

$$r(x) = \sum_0^\infty A_n (x - x_0)^n. \quad (7)$$

Let us place (6) and (7) into the Equation (4) and we obtain

$$\sum_0^\infty C_n (x - x_0)^{n-3} n(n-1)(n-2) = \sum_0^\infty C^{****}{}_n (x - x_0)^n + \sum_0^\infty A_n (x - x_0)^n,$$

The equality of the series in the latter implies the equality of the coefficients at the corresponding powers of the left and right sides. This procedure leads to the following recurrence relation:

$$n(n-1)(n-2)C_n = C^{***}_{n-3} + A_{n-3},$$

where $C^{****}_n = \sum_{i=0}^n C_i C^{***}_j, \quad C^{***}_n = \sum_{i=0}^n C^*_i C^{**}_j, \quad C^{**}_n = \sum_{i=0}^n C^*_i C^*_j, \quad C^*_n = \sum_{i=0}^n C_i C_{n-i}.$

The uniqueness of the coefficients implies the uniqueness of solution (6).

Let us prove the validity of the following estimates for the coefficients of the desired series (6):

$$\begin{aligned} |C_{3k}| &\le \tfrac{(M+1)^{6k+1}}{3k(3k-1)(3k-2)} = E_{3k}, \\ |C_{3k+1}| &\le \tfrac{(M+1)^{6k+1}}{3k(3k-1)(3k+1)} = E_{3k+1}, \\ |C_{3k+2}| &\le \tfrac{(M+1)^{6k+1}}{3k(3k+1)(3k+2)} = E_{3k+2}, \end{aligned} \tag{8}$$

where

$$M = \max\left\{|y_0|, |y_1|, |y_2|, \sup_n \left\{\frac{\left|r^{(n)}(x_0)\right|}{n!}\right\}\right\}, n = 0, 1, 2, \ldots$$

We limit ourselves to the variant of estimating the coefficient $C_{3k+3}$. In this case, taking into account estimates (8) and decomposition $C_n^{****}$, we have:

$$|C_{3k+3}| = \left|\frac{1}{(3k+2)3k(3k+1)}\left(C^{****}_{3k-2} + A_{3k-2}\right)\right| =$$

$$= \left|\frac{1}{(3k+2)3k(3k+1)}\sum_{i=0}^{3k} C_i\left(\sum_{j=0}^{3k-i} C_j\left(\sum_{l=0}^{3k-i-j} C_l\left(\sum_{m=0}^{3k-i-j-l} C_m C_{3k-i-j-l-m}\right)\right)\right) + A_{3k-2}\right| \le$$

$$\le \frac{1}{(3k+2)3k(3k+1)}\left|\sum_{i-0}^{k} \frac{(M+1)^{6i+1}}{(3i+2)3i^*(3i+1)}\left(\sum_{j-0}^{k-i} \frac{(M+1)^{6j+1}}{(3j+2)3j^*(3j+1)}\times\right.\right.$$

$$\times\left(\sum_{l-0}^{k-i-j} \frac{(M+1)^{6l+1}}{(3l+2)3l^*(3l+1)}\left(\sum_{m-0}^{k-i-j-l} \frac{(M+1)^{6m+1}}{(3m+2)3m^*(3m+1)}\times\right.\right.$$

$$\left.\left.\left.\left.\times\frac{(M+1)^{6(k-i-j-l-m)+1}}{(3(k-i-j-l-m)+2)3(k-i-j-l-m)^*(3(k-i-j-l-m)+1)}\right)\right)\right) + M\right| =$$

$$\frac{(M+1)^{6k+5}}{(3k+2)3k(3k+1)}\sum_{i-0}^{k}\frac{1}{(3i+2)3i^*(3i+1)}\left(\sum_{j-0}^{k-i}\frac{1}{(3j+2)3j^*(3j+1)}\times\right.$$

$$\times\left(\sum_{l-0}^{k-i-j}\frac{1}{(3l+2)3l^*(3l+1)}\left(\sum_{m-0}^{k-i-j-l}\frac{1}{(3m+2)3m^*(3m+1)}\times\right.\right.$$

$$\left.\left.\left.\times\frac{1}{(3(k-i-j-l-m)+2)3(k-i-j-l-m)^*(3(k-i-j-l-m)+1)}\right)\right)\right) + M \le$$

$$\le \frac{(M+1)^{6k+5}}{(6k+5)(6k+3)(6k+1)} + M \le \frac{(M+1)^{6k+7}}{(6k+5)(6k+3)(6k+1)},$$

where

$$i^* = \begin{cases} 1, \; if \; i = 0 \\ i, \; if \; i \ne 0 \end{cases}, \; j^* = \begin{cases} 1, \; if \; j = 0 \\ j, \; if \; j \ne 0 \end{cases}, \; l^* = \begin{cases} 1, \; if \; l = 0 \\ l, \; if \; l \ne 0 \end{cases}, \; m^* = \begin{cases} 1, \; if \; m = 0 \\ m, \; if \; m \ne 0 \end{cases},$$

$$(k-i-j-l-m)^* = \begin{cases} 1, \; if \; m = k-i-j-l, \\ (k-i-j-l-m), \; if \; m \ne k-i-j-l. \end{cases}.$$

Thus, we are convinced of the estimate of the coefficient $C_{3k+3}$. The subsequent estimates in (6) are proved by analogy.

We introduce the next series

$$\sum_0^\infty E_n(x-x_0)^n, \tag{9}$$

which is major for the formal series

$$\sum_{0}^{\infty} C_n(x-x_0)^n. \tag{10}$$

Based on the regularity of estimates (8), we represent the series (9) in the following form:

$$\sum_{0}^{\infty} E_n(x-x_0)^n = \sum_{0}^{\infty} E_{3k}(x-x_0)^{3k} + \sum_{0}^{\infty} E_{3k+1}(x-x_0)^{3k+1} + \sum_{0}^{\infty} E_{3k+2}(x-x_0)^{3k+2}.$$

Further, for each of the three series on the right-hand side, taking into account estimates (8), we obtain the convergence domain according to d'Alembert:

$$|x-x_0| < \left(\frac{1}{(M+1)^6}\right)^{\frac{1}{3}} = \frac{1}{(M+1)^2}.$$

Assuming that $\rho_2 = \min\left\{\rho_1, \frac{1}{(M+1)^2}\right\}$ , we are convinced of the convergence of series (10) in the region under consideration.

The proved theorem allows constructing an analytical approximate solution in the way as follows:

$$y_N(x) = \sum_{0}^{N} C_n(x-x_0)^n. \tag{11}$$

□

**Theorem 2.** *We require conditions 1 and 2 of theorem 1 to be satisfied. Then, for an analytical approximate solution (11) of the Cauchy problem (4)–(5) in the domain*

$$|x-x_0| < \rho_2 \tag{12}$$

*the error estimate is fair*

$$\Delta y_N(x) = |y(x) - y_N(x)| \le \Delta,$$

*where*

$$\Delta = \frac{(M+1)^{2N+3}}{1-(M+1)^6|x-x_0|^3}|x-x_0|^{N-1}\times$$

$$\times\left(\frac{1}{N(N-1)(N-2)} + \frac{|x-x_0|}{N(N-1)(N+1)} + \frac{|x-x_0|^2}{N(N+1)(N+2)}\right)$$

*in case $N+1=3k$,*

$$\Delta = \frac{(M+1)^{2N+1}}{1-(M+1)^6|x-x_0|^3}|x-x_0|^{N}\times$$

$$\times\left(\frac{1}{N(N-1)(N-2)} + \frac{|x-x_0|}{N(N-1)(N+1)} + \frac{|x-x_0|^2}{N(N+1)(N+2)}\right)$$

*two variants $N+1=3k+1$, and*

$$\Delta = \frac{(M+1)^{2N-1}}{1-(M+1)^6|x-x_0|^3}|x-x_0|^{N+1}\times$$

$$\times\left(\frac{1}{N(N-1)(N-2)} + \frac{|x-x_0|}{N(N-1)(N+1)} + \frac{|x-x_0|^2}{N(N+1)(N+2)}\right)$$

*with* $N+1 = 3k+2$, *where* $\rho_2 = \min\left\{\rho_1, \frac{1}{(M+1)^4}\right\}$,
$M = \max\left\{|y_0|, |y_1|, |y_2|, \sup_n\left\{\frac{|r^{(n)}(x_0)|}{n!}\right\}\right\}$, $n = 0, 1, 2, \ldots. N \geq 3$.

**Proof.** We limit ourselves to prove the case for $N+1 = 3k$. Let us express$\Delta y_N(x)$ as:

$$\Delta y_N(x) = |y(x) - y_N(x)| = \left|\sum_0^\infty C_n(x-x_0)^n - \sum_0^N C_n(x-x_0)^n\right| = \left|\sum_{N+1}^\infty C_n(x-x_0)^n\right|$$

Taking into account the regularity of the coefficients $C_n$ from Theorem 1, we obtain:

$$\Delta y_N(x) = \left|\sum_{N+1}^\infty C_n(x-x_0)^n\right| \leq \sum_{N+1}^\infty |C_n| \times |x-x_0|^n \leq \sum_{N+1}^\infty E_{3k}|x-x_0|^{3k} +$$

$$+ \sum_{N+1}^\infty E_{3k+1}|x-x_0|^{3k+1} + \sum_{N+1}^\infty E_{3k+2}|x-x_0|^{3k+2} = \sum_{N+1}^\infty \frac{(M+1)^{6k+1}}{3k(3k-1)(3k-2)}|x-x_0|^{3k} +$$

$$\sum_{N+1}^\infty \frac{(M+1)^{6k+1}}{3k(3k-1)(3k+1)}|x-x_0|^{3k+1} + \sum_{N+1}^\infty \frac{(M+1)^{6k+1}}{3k(3k+1)(3k+2)}|x-x_0|^{3k+2} \leq$$

$$\leq \frac{(M+1)^{6k+1}}{1-(M+1)^6|x-x_0|^3}|x-x_0|^{3k}\left(\frac{1}{3k(3k-1)(3k-2)} + \frac{|x-x_0|}{3k(3k-1)(3k+1)} +\right.$$

$$\left.+\frac{|x-x_0|^2}{3k(3k+1)(3k+2)}\right) \leq \frac{(M+1)^{2N+3}}{1-(M+1)^6|x-x_0|^3}|x-x_0|^{N-1}\left(\frac{1}{N(N-1)(N-2)} +\right.$$

$$\left.+\frac{|x-x_0|}{N(N-1)(N+1)} + \frac{|x-x_0|^2}{N(N+1)(N+2)}\right).$$

Similarly, we obtain estimates for the approximate solution (11) in case $N+1 = 3k+1$ and $N+1 = 3k+2$, which is also valid in the domain $|x-x_0| < \rho_2$. □

## 3. The Discussion of the Results Numerical Experiment

Let us consider the Cauchy problem (5) and (6), where $r(x) = 0$, $y(0) = \frac{1}{4}$, $y'(0) = \frac{3}{10}$, $y''(0) = 1$. The Cauchy problem (4) and (5) under given conditions is not solvable in quadratures. The calculation results are presented in Table 1.

**Table 1.** Numerical characteristics of an analytically approximate solution.

| $x_1$ | $y_9(x_1)$ | $\Delta_1$ | $\Delta_2$ |
|---|---|---|---|
| 0.12 | 0,2932 | 0,007 | 0,0005 |

where $x_1$ is the value of the argument, falls within the scope of Theorem 2, (12); $y_9(x)$ is an analytically approximate solution (11); $\Delta_1$—a priori estimate; $\Delta_2$—a posteriori estimate. For $\Delta_2 = 0,0005$ by Theorem 2, we determine $N = 14$. The summands from 10 to 14 in the total sum do not exceed the required accuracy of $\varepsilon = 0,0005$, therefore, for $N = 9$ we obtain a value $y_9(x_1)$ with an accuracy of $\varepsilon = 0,0005$.

## 4. Conclusions

In this paper, we present a study of the considered class of nonlinear equations in the domain of analyticity, and prove the theorem of the existence and uniqueness of the solution. A formula is obtained for determining the area of analyticity of the solution. A priori estimates of the error of the analytical approximate solution are obtained, and a numerical experiment is carried out, confirming the adequacy of the obtained theoretical positions with the experimental calculations. A technique for optimizing a priori estimates by using a posteriori ones is given.

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## References

1. Orlov, V.N.; Kovalchuk, O.A. An analytical solution with a given accuracy for a nonlinear mathematical model of a console-type construction. *J. Phys. Conf. Ser.* **2020**, *1425*, 012127. [CrossRef]
2. Orlov, V.N.; Kovalchuk, O.A. Mathematical modeling of complex structures and nonlinear differential equations with movable points. In Proceedings of the IOP Conference Series. Materials Science and Engineering, Novosibirsk, Russia, 1–8 July 2018.
3. Feng, Y. Existence and uniqueness results for a third-order implicit differential equation. *Comput. Math. Appl.* **2008**, *56*, 2507–2514. [CrossRef]
4. Chugainova, A.P. Nonstationary solutions of the generalized Korteweg–de Vries–Burgers equation. *Proc. Steklov Inst. Math.* **2013**, *281*, 204–212. [CrossRef]
5. Orlov, V.N.; Gasanov, M.V. Existence theorem for a solution of a class of non-linear differential equations of the third order with a polynomial right-hand side of the seventh degree in the neighborhood of a moving singular point. *Bull. Yakovlev Chuvash State Pedagog. Univ. Ser. Mech. Limit State* **2020**, *1*, 92–99. [CrossRef]
6. Astashova, I.V. On the asymptotic classification of solutions to non-linear equations of the third and fourth orders with power non-linearity. ICDDEA 2015: Differential and Difference Equations with Applications. In *Part of the Springer Proceedings in Mathematics & Statistics Book Series*; Springer: Cham, Switzerland, 2016; Volume 164, pp 191–203. [CrossRef]
7. Chichurin, A.; Filipuk, G. The properties of certain linear and nonlinear differential equations of the fourth order arising in beam models. *J. Phys. Conf. Ser.* **2020**, *1425*, 012107. [CrossRef]
8. Leonteva, T.Y. The exact criteria for the existence movable singular points of a class of nonlinear ordinary differential equations of second order in the complex region. *Belgorod State Univ. Sci. Bull. Math. Phys.* **2017**, *13*, 51–57.
9. Leonteva, T.J. The effect of perturbation movable singular point on the approximate solution of a nonlinear differential equation of second order in a complex region. *Bull. Yakovlev Chuvash State Pedagog. Univ. Ser. Mech. Limit State* **2015**, *2*, 109–118.
10. Pchelova, A.Z. Boundaries of the application area of an approximate solution in the vicinity of a perturbed moving singular point of a differential equation in a complex domain. *Vestnik VGU Ser. Phys. Math.* **2014**, *4*, 170–179.
11. Chichurin, A. Computer algorithm for solving of the Chazy equation of the third order with six singular points. *Miskolc Math. Notes* **2017**, *18*, 701–715. [CrossRef]
12. Filipuk, G.; Chichurin, A. The Properties of Certain Linear and Nonlinear Differential Equations. In *Advances in Mathematical Methods and High Performance Computing*; Springer: Cham, Switzerland, 2019; pp. 193–200. [CrossRef]