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A Survey on the k-Path Vertex Cover Problem

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Abstract: Given an integer $k \geq 2$, a k-path is a path on k vertices. A set of vertices in a graph G is called *a k-path vertex cover* if it includes at least one vertex of every k-path of G. A minimum k-path vertex cover in G is a k-path vertex cover having the smallest possible number of vertices and its cardinality is called the *k-path vertex cover number* of G. In the *k-path vertex cover problem*, the goal is to find a minimum k-path vertex cover in a given graph. In this paper, we present a brief survey of the current state of the art in the study of the k-path vertex cover problem and the k-path vertex cover number.

Keywords: k-path vertex cover problem; k-path vertex cover number; exact algorithm; approximation algorithm; parameterized algorithm

MSC: 05C70, 05C85, 05C90

1. Introduction

We consider only simple undirected graphs. We will use standard graph theory notations, for notations not defined here, we refer the reader to [1]. The number of vertices and edges in a graph G are called the order and size, respectively, of G. In what follows n and m will always be the order and size, respectively, of the given graph.

Given an integer $k \geq 2$, a k-path, denoted by P_k, is a path of order k. A set of vertices in a graph G is called *a k-path vertex cover* (for short, VCP$_k$) if it includes at least one vertex of every k-path of G. A *minimum* VCP$_k$ is a VCP$_k$ having the smallest possible number of vertices and its cardinality, denoted by $\psi_k(G)$, is called the *k-path vertex cover number* of G. In the *k-path vertex cover problem* (for short, MinVCP$_k$), the goal is to find a minimum VCP$_k$ in a given graph. In the literature, a VCP$_k$ is also called a vertex cover P_k [2,3], or a vertex k-path cover [4], or a k-observer [5,6], or a k-path transversal [7], or a P_k-hitting set [8].

Clearly, MinVCP$_2$ is exactly the classic vertex cover problem in which the goal is to find a set of vertices with minimum cardinality in a graph that intersects every edge of the graph. Thus, MinVCP$_k$ is a natural generalization of MinVCP$_2$ that has been intensively studied.

In addition, MinVCP$_k$ also belongs to the vertex deletion problem [9–11]. If a graph property is closed under removal of vertices, it is said to be hereditary. If a graph property holds for some graphs and does not include all graphs, it is said to be non-trival. Given a hereditary and non-trivial graph property Π, the objective of the Π-vertex deletion problem is to find a set of vertices with minimum cardinality whose deletion results in a graph satisfying Π. Many classic optimization problems such as the feedback vertex set problem (Π means "containing no cycle"), the vertex bipartization problem (Π means "containing no odd cycle"), and the d-bounded-degree vertex deletion problem (for short, d-BDD) (Π means "having maximum degree at most d") are examples of vertex deletion problems. MinVCP$_k$ corresponds to the case that Π means "containing no k-path".

Note that a graph that contains no 3-path has vertex degree at most one. Thus, MinVCP$_2$ and MinVCP$_3$ are equivalent to 0-BDD and 1-BDD, respectively. Let G be a

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graph. A dissociation set in G is a set of vertices inducing a subgraph with vertex degree at most one. A maximum dissociation set in G is a dissociation set having the largest possible number of vertices and its cardinality, denoted by $diss(G)$, is called the dissociation number of G. It is worth mentioning that a set of vertices of G is a dissociation set if and only if its complement $V(G) \setminus S$ is a VCP$_3$ of G. It follows that $diss(G) = |V(G)| - \psi_3(G)$. Finding a maximum dissociation set in a given graph G is called *the maximum dissociation set problem* and is the dual problem of MinVCP$_3$. The problem was introduced by Yannakakis [11] in 1981 and generalizes both the maximum independent set problem and the maximum induced matching problem. It has been demonstrated that the problem remains NP-hard even in bipartite graphs [11].

The study of MinVCP$_k$ is motivated by the two real world problems that are related to the design of security protocols of wireless sensor networks (for short, WSNs) [12,13] and to installation of traffic cameras [2], respectively.

Nowadays, WSNs have been widely used in industry and everyday life. A WSN can be described by a graph in which vertices represent sensor devices and edges represent communication channels between pairs of sensor devices. In order to ensure some security properties of WSNs, one needs to design specific security protocols for WSNs. The k-generalized Canvas scheme generalizes the Canvas scheme designed by Vogt [14] and can ensure data integrity if at least one node of every k-path is a protected node. Since a protected node is very costly, the problem of minimizing the number of protected nodes naturally arises and is exactly MinVCP$_k$.

Another motivation for studying MinVCP$_k$ is related to the installation of traffic cameras. The increasing number of cars and buses lead to an increase in road traffic accidents, hence it is necessary to install cameras at traffic intersections. A road network can be described by a graph composed of vertices and edges denoting, respectively, traffic intersections and connections between pairs of traffic intersections. If every traffic intersection is installed with some cameras, it would cost enormous sums of money. For a given integer k, we aim to choose as few traffic intersections as possible to install cameras so that an offending vehicle should be captured at least once when it passes k traffic intersections. The corresponding optimization problem can be formulated as MinVCP$_k$.

MinVCP$_k$ also finds applications in monitoring message flows in WSNs. As requested, for any message that continuously passes k nodes, it should be monitored at least once. The corresponding optimization problem is also MinVCP$_k$.

Based on the real world problems mentioned above, three variants of MinVCP$_k$ have also been raised and studied.

- *The weighted version of MinVCP$_k$* (for short, $\mathcal{W}$-MinVCP$_k$). A graph $G = (V, E)$ with a weight function $w : V \to \mathcal{R}^+$ is given, and our goal is to find a minimum weight VCP$_k$ of G.
- *The connected version of MinVCP$_k$* (for short, $\mathcal{C}$-MinVCP$_k$). A connected graph G is given, and our goal is to find a VCP$_k$ S of G with minimum cardinality so that $G[S]$ is connected.
- *The weighted version of $\mathcal{C}$-MinVCP$_k$* (for short, $\mathcal{WC}$-MinVCP$_k$). A connected graph $G = (V, E)$ with a weight function $w : V \to \mathcal{R}^+$ is given, and our goal is to find a VCP$_k$ S of G with minimum weight so that $G[S]$ is connected.

Due to their importance in theory and application, MinVCP$_k$, $\mathcal{W}$-MinVCP$_k$ and $\mathcal{WC}$-MinVCP$_k$ have been widely studied. In particular, a large number of results on exact algorithms, approximation algorithms, and parameterized algorithms for these problems have been reported. In this paper, we aim to provide a brief survey of the current state of the art in the study of MinVCP$_k$ and its variants, and $\psi_k(G)$. We mainly focus on the cases with $k \geq 3$.

2. Computational Complexity

The decision version of MinVCP$_k$ is stated as follows:

INPUT: A graph G and a positive integer t.

OUTPUT: Is there a k-path vertex cover F in G of size at most t?

We abuse notation and let MinVCP$_k$ refer to the k-path vertex cover problem and its decision version.

Theorem 1 ([5,12])**.** *For any $k \geq 2$, MinVCP$_k$ is NP-complete.*

The 2-subdivision of a graph G is obtained from G by replacing every edge $e = uv$ by a 4-path $uxyv$. Poljak [15] showed that MinVCP$_2$ is NP-complete in 2-subdivision graphs. By Poljak's result and the fact that for any integer $3 \leq t \leq 8$, 2-subdivision graphs are C_t-free, we have

Theorem 2 ([16])**.** *For any $k \geq 2$, MinVCP$_k$ is NP-complete in graphs that contains no cycle of size t for integer $3 \leq t \leq 8$.*

The NP-completeness of MinVCP$_3$ has been studied intensively. Yannakakis [11] proved that MinVCP$_3$ is NP-complete even in bipartite graphs. The author and Yang [17] studied the NP-completeness of MinVCP$_3$ in cubic planar graphs. The girth of a graph is the size of one of its (if any) shortest cycles. The girth of acyclic graphs is thought to be infinite.

Theorem 3 ([17])**.** *MinVCP$_3$ is NP-complete in cubic planar graphs with girth 3.*

If P$\neq$NP is assumed, NP-hard problems cannot be solved efficiently in polynomial time, it follows that approximation algorithms have been developed to solve a lot of NP-hard problems. For an NP-hard problem Π, an *approximation algorithm*, given an instance I of Π, returns a feasible solution of I within a factor α, called the approximation factor, of the optimal one and runs in time polynomial in $|I|$. A *polynomial time approximation scheme* (PTAS) for a minimization problem Π is an algorithm so that, given an instance I of Π and a parameter $\varepsilon > 0$, it is an $(1 + \varepsilon)$-approximation algorithm and runs in time polynomial in $|I|$ when ε is fixed. A problem that allows a constant-factor approximation algorithm is said to be in the class APX. Furthermore, if there is a PTAS reduction from every problem in APX to it, it is said to be *APX-hard*. A problem is said to be *APX-complete* if it is APX-hard and also in APX. If P$\neq$NP is assumed, there does not exist a PTAS for any APX-hard problem.

Theorem 4 ([18])**.** *MinVCP$_3$ is APX-complete in bipartite graphs.*

Theorem 5 ([19])**.** *MinVCP$_4$ is NP-complete in cubic planar graphs, and APX-complete in cubic bipartite graphs as well as $K_{1,4}$-free graphs.*

3. Exact Algorithms

MinVCP$_k$ in trees was firstly investigated by Brešar et al. [12].

Theorem 6 ([12])**.** *MinVCP$_k$ in trees is solvable in linear time.*

We write $G_1 \square G_2$ to denote the Cartesian product of two disjoint graphs G_1 and G_2. The Cartesian product of two disjoint paths is called a grid graph.

Theorem 7 ([5])**.** *MinVCP$_k$ in grid graphs is solvable in linear time.*

A connected graph is called a cactus if every edge of the graph belongs to at most one cycle, that is, any two cycles in such a connected graph have at most one vertex in common. The author [20] studied MinVCP$_k$ in cacti.

Theorem 8 ([20])**.** *Computing $\psi_k(G)$ of a cactus G is solvable in $O(n^2)$ time.*

Brešar et al. [21] studied $\mathcal{W}$-MinVCP$_k$ in three special classes of graphs.

Theorem 9 ([21]). *$\mathcal{W}$-MinVCP$_k$ in complete graphs K_n, cycles C_n and trees of order n are solvable with time complexity $O(n \cdot k)$, $O(n \cdot k^2)$ and $O(n \cdot k)$, respectively.*

Li et al. [22] considered $\mathcal{WC}$-MinVCP$_k$.

Theorem 10 ([22]). *$\mathcal{WC}$-MinVCP$_k$ in a tree is solvable in $O(n)$ time, and $\mathcal{WC}$-MinVCP$_k$ in a unicyclic graph which contains a cycle of size r is solvable in $O(r \cdot n)$ time.*

There have been a lot of works on exact algorithms for MinVCP$_2$ and MinVCP$_3$. Throughout this paper, the $\mathcal{O}^*()$ notation will always suppress all factors that are polynomial in the size of the input size. MinVCP$_2$ is solvable in $\mathcal{O}^*(1.1996^n)$ time [23]. Since the maximum dissociation set problem is the dual problem of MinVCP$_3$, in terms of exact algorithms, there is no need to distinguish these two problems. The exact algorithm for MinVCP$_3$ and the maximum dissociation problem have been improved several times [24–26]. Xiao and Kou [27] reduced the time complexity at the cost of an exponential space complexity.

Theorem 11 ([27]). *MinVCP$_3$ is solvable in $\mathcal{O}^*(1.3659^n)$ time and space.*

On the other hand, MinVCP$_3$ is solvable in polynomial time in many special classes of graphs, such as chordal graphs, AT-free graphs, (chair, bull)-free graphs, (claw, bull)-free graphs, $(P_k, K_{1,n})$-free graphs, ℓK_2-free graphs, P_5-free graphs and so on [28–32]. In particular, for P_4-tidy graphs as well as line graphs of graphs that contain a Hamiltonian path, MinVCP$_3$ is solvable in linear time [16,32].

Let U be a universe set of n elements and let $\mathcal{C}$ be a family of subsets of U each of which contains at most k elements. In the *k-hitting set problem* (for short, k-Hit), the goal is to find a subset $U_0 \subseteq U$ with the smallest possible number of elements so that for every subset of $\mathcal{C}$, at least one element of it is in the subset. It is easy to see that MinVCP$_k$ also belongs to k-Hit. By the results on k-Hit due to Fomin et al. [33] and Fernau [34], one can obtain the following two results.

Theorem 12 ([33]). *MinVCP$_4$ is solvable in $\mathcal{O}^*(1.8704^n)$ time.*

Theorem 13 ([34]). *$\mathcal{W}$-MinVCP$_4$ is solvable in $\mathcal{O}^*(1.97^n)$ time.*

4. Approximation Algorithms

A trivial k-approximation algorithm for MinVCP$_k$ is easily obtained. One way to compute a VCP$_k$ of a given graph G is to repeat the following process: seek a k-path, put its vertices into solution S, and remove every edge incident to any vertex in S from G. As any VCP$_k$ of G must contain at least one vertex of each k-path that was considered in the process, the solution produced, therefore, is within a factor k of the optimal one.

Brešar et al. [12] proved that for any $c > 1$, a c-approximation algorithm for MinVCP$_k$, with polynomial running time, yields directly a c-approximation for MinVCP$_2$. MinVCP$_2$ is APX-complete even in cubic graphs [35]. Furthermore, MinVCP$_2$ does not admit c-approximation for any constant $c < 2$ if the unique games conjecture holds [36]. Thus, we have

Theorem 14. *[12,36] For any $k \geq 3$, MinVCP$_k$ does not admit a c-approximation for any constant $c < 2$ if the unique games conjecture holds.*

Ries et al. [6] proposed a 3-approximation algorithm for MinVCP$_k$ in d-regular graphs for $k \leq \frac{d+2}{2}$. Their algorithm is the first one with a factor less than k for general k. Zhang et al. [37] improved Ries et al.'s result.

Theorem 15 ([37])**.** *When $1 \leq k-2 < d$, MinVCP$_k$ in d-regular graphs admits a $\frac{\lfloor d/2 \rfloor(2d-k+2)}{(\lfloor d/2 \rfloor+1)(d-k+2)}$-approximation that runs in time $O(d^2 \cdot n)$.*

Note that when $k \leq \frac{d+2}{2}$, $\frac{\lfloor d/2 \rfloor(2d-k+2)}{(\lfloor d/2 \rfloor+1)(d-k+2)} < 3$. Lee [7] studied MinVCP$_k$ in general graphs.

Theorem 16 ([7])**.** *MinVCP$_k$ admits an $O(\log k)$-approximation that runs in time $2^{O(k^3 \log k)}$ $n^2 \log n + n^{O(1)}$.*

Note that if k is fixed, the runtime of Lee's algorithm is polynomial. Since finding a k-path is NP-hard, the runtime of any approximation algorithm for MinVCP$_k$ cannot be polynomial in k.

Very recently, Brüstle et al. [8] investigated a wider problem called the H-hitting set problem (for short, H-Hit). Let H be a fixed graph of order k. In H-Hit, the goal is to find a set of vertices of a given graph G with minimum cardinality so that the set includes at least one vertex of every subgraph of G isomorphism to H. Clearly, the P_k-hitting set problem is exactly MinVCP$_k$. Brüstle et al. [8] proved that T-Hit for a k-vertex tree T admits a $(k-\frac{1}{2})$-approximation. Thus, we have

Theorem 17 ([8])**.** *MinVCP$_k$ admits a $(k-\frac{1}{2})$-approximation.*

A d-dimensional ball graph is a graph composed of vertices and edges denoting, respectively, balls in $\mathbb{R}^d$ and nonempty intersections between pairs of balls. Denote by r_{max} the largest radius of those balls, and by r_{min} the smallest radius of those balls. A disk graph is a 2-dimensional ball graph, and if its all disks have the same radii, it is called a unit disk graph. Zhang et al. [38] proposed a PTAS for MinVCP$_k$ in a ball graph with the ratio $r_{max}/r_{min} \leq c$ for a constant c.

Liu et al. [39]] presented a PTAS for $\mathcal{C}$-MinVCP$_k$ in unit disk graphs. A simpler PTAS given by Chen et al. [40] not only simplifies Liu et al.'s algorithm, but also reduces the time-complexity.

Li et al. [22] showed that $\mathcal{C}$-MinVCP$_k$ in graphs of girth of at least k can be approximable within k. Later, Fujito [41] extended Li et al.'s result to *all* graphs.

Theorem 18. *$\mathcal{C}$-MinVCP$_k$ admits a k-approximation.*

Fujito [41] also studied $\mathcal{WC}$-MinVCP$_k$.

Theorem 19 ([41])**.** *For any fixed integer $k \geq 2$, $\mathcal{WC}$-MinVCP$_k$ is at least as hard as the weighted set cover problem.*

In the last decade, there have been a lot of works on the study of approximation algorithms for MinVCP$_3$, MinVCP$_4$ and their variants. Kardoš et al. [26] gave a randomized algorithm for MinVCP$_3$ with an expected approximation ratio of $\frac{23}{11}$. The author and Zhou presented two 2-approximation algorithms for $\mathcal{W}$-MinVCP$_3$ utilizing the primal–dual method [2] and the local ratio method [3], respectively.

Theorem 20 ([2,3])**.** *Utilizing primal–dual method and local ratio method, one can obtain two 2-approximation algorithms for $\mathcal{W}$-MinVCP$_3$ with runtime $O(m \cdot n)$.*

In recent years, researchers also developed approximation algorithms for NP-hard problems that have better approximation factors but run in super-polynomial time. One such algorthm for MinVCP$_3$ was given by Chang et al. [25].

Theorem 21 ([25])**.** *MinVCP$_3$ admits a $\frac{4}{3}$-approximation that runs in time $\mathcal{O}^*(1.4159^n)$.*

Camby et al. [42] proposed a 3-approximation algorithm for MinVCP$_4$. On the other hand, there are a lot of works on approximation algorithms for MinVCP$_3$ and MinVCP$_4$ in some special classes of graphs [6,17,19,37,43], see Table 1.

Table 1. The best approximation ratios for MinVCP$_3$ and MinVCP$_4$.

MinVCP$_k$	General Graphs	Bipartite Graphs	Cubic Graphs	4-Regular Graphs	Bipartite d-Regular Graphs	d-Regular Graphs ($d \geq 5$)	$K_{1,4}$-Free Graphs
$k = 3$	2 [2,3]		1.25 [37]	$\frac{14}{9}$ [37]	$\frac{2d-1}{2d-2}$ [6]	$\frac{\lfloor \frac{d}{2} \rfloor (2d-1)}{(\lfloor \frac{d}{2} \rfloor + 1)(d-1)}$ [37]	
$k = 4$	3 [42]	2 [18]	$\frac{15}{8} + \varepsilon$ for any $\varepsilon > 0$ [37]	1.852 [37]	$\frac{d^2}{d^2-d+1}$ [37]	$\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ (when d is even) [37]	3 [19]

An *efficient PTAS* (for short, EPTAS) for an NP-hard problem Q is a PTAS so that, given an instance I of Q and a parameter $\varepsilon > 0$, its runtime is bounded by $O(f(\varepsilon)|I|^c)$, for an arbitrary function f and a constant c. The author and Shi [44] presented an EPTAS for MinVCP$_3$ in planar graphs.

For $\mathcal{C}$-MinVCP$_3$, we have

Theorem 22 ([22,41])**.** *$\mathcal{C}$-MinVCP$_3$ admits a 3-approximation.*

Liu et al. [45] showed that if MinVCP$_3$ admits an α-approximation, then $\mathcal{C}$-MinVCP$_3$ admits a $(2\alpha + 1/2)$-approximation. For those classes of graphs in which MinVCP$_3$ can be approximable within $\alpha < 5/4$, their algorithm is a kind of improvement.

For $\mathcal{WC}$-MinVCP$_3$, we have

Theorem 23 ([46])**.** *$\mathcal{WC}$-MinVCP$_3$ admits a $(\ln \Delta(G) + 4 + \ln 2)$-approximation.*

Wang et al. [47] proved that WC-MinVCP$_3$ is NP-hard even in unit disk graphs. If it is assumed that the problem is c-local, they also presented a PTAS for WC-MinVCP$_3$ in unit disk graphs with a minimum degree of at least two. If it is assumed that the weight of the vertices is weak c-local and smooth, Wang et al. [48] gave a PTAS for $\mathcal{WC}$-MinVCP$_3$ in unit ball graphs.

5. Parameterized Algorithms

In classical complexity theory, the runtime of algorithms is expressed as a function of the input size only, while in parameterized algorithmics the runtime is analyzed in greater detail by taking into account some parameters of the problem.

In parameterized algorithmics, the parameter may be the size of the solution sought after, or an index describing the structure of the input instance. For a parameterized problem Π with a parameter p, a *fixed-parameter algorithm* (parameterized algorithm, or FPT algorithm) is an exact algorithm which, for an input instance (I, p) of Π, runs in time $f(p) \cdot |I|^c$, where c is a constant independent of both $|I|$ and p. If a parameterized problem allows an FPT algorithm, it is said to be *fixed parameter tractable*.

In what follows, t will always be the size of the solution sought after. There exists a trivial FPT algorithm for MinVCP$_k$ with runtime $\mathcal{O}^*(k^t)$ [49]. With the results on k-Hit from Fomin et al. [33], we have

Theorem 24 ([33])**.** *MinVCP$_k$ admits an $\mathcal{O}^*((k - 0.9245)^t)$-time FPT algorithm.*

For MinVCP$_2$, the best FPT algorithm known to date is an $\mathcal{O}^*(1.2738^t)$-algorithm by Chen et al. [50]. An $\mathcal{O}^*(2^t)$-algorithm for MinVCP$_3$ was given by the author [51]. This result was improved several times [52–55], and Tsur [56] presented an $\mathcal{O}^*(1.713^t)$-algorithm for MinVCP$_3$.

The author and Jin [57] gave an $\mathcal{O}^*(3^t)$-time FPT algorithm for MinVCP$_4$ using the iterative compression method. Tsur [58] improved the result to $\mathcal{O}^*(2.619^t)$. For $k = 5$, 6, and 7, FPT algorithms for MinVCP$_k$ were also given in [59,60]. Very recently, Červený and Suchý [61] gave FPT algorithms outperforming those previously known for MinVCP$_k$ for $3 \le k \le 8$.

Theorem 25. *The current best running times known of FPT algorithms for MinVCP$_k$ for $3 \le k \le 8$ are given in Table 2.*

Table 2. The running times of MinVCP$_k$ for $3 \le k \le 8$.

MinVCP$_k$	$k = 3$	$k = 4$	$k = 5$	$k = 6$	$k = 7$	$k = 8$
	$\mathcal{O}^*(1.712^t)$	$\mathcal{O}^*(2.151^t)$	$\mathcal{O}^*(2.695^t)$	$\mathcal{O}^*(3.45^t)$	$\mathcal{O}^*(4.872^t)$	$\mathcal{O}^*(5.833^t)$

For MinVCP$_3$ in planar graphs, the author et al. [62] showed that there is an FPT algorithm with subexponential time $\mathcal{O}^*(2^{O(\sqrt{t})})$.

When the parameter is the treewidth of the given graphs, the author et al. [62,63] showed that MinVCP$_3$ is fixed parameter tractable. The treewidth is an important graph parameter which has been often used in parameterized algorithmics.

Theorem 26 ([63])**.** *If a tree decomposition of a graph G of width of at most p is given, MinVCP$_3$ in G admits an $\mathcal{O}^*(3^p)$-algorithm.*

Consider $\mathcal{W}$-MinVCP$_k$. Shachnai and Zehavi [64] introduced a multivariate method that can be useful for solving weighted parameterized problems. From their results, one can obtain FPT algorithms for $\mathcal{W}$-MinVCP$_2$ and $\mathcal{W}$-MinVCP$_3$.

Theorem 27 ([64])**.** *Let $G = (V, E)$ be a graph and w be a vertex weight function $V \to [1, +\infty)$. If the paremeter is the total weight W of the solution searched for and s is the smallest possible size of a solution of weight of at most W, then*

(1) there exists an FPT algorithm for $\mathcal{W}$-MinVCP$_2$ in G that runs in $\mathcal{O}^*(1.381^s)$ time and in polynomial space, or in $\mathcal{O}^*(1.363^s)$ time and space.

(2) there exists an FPT algorithm for $\mathcal{W}$-MinVCP$_3$ in G that runs in $\mathcal{O}^*(2.168^s)$ time and in polynomial space.

Fernau [34] studied FPT algorithms for the weighted version of k-Hit. Following his results, one can obtain FPT algorithms for $\mathcal{W}$-MinVCP$_k$ for any $k \ge 4$.

Theorem 28 ([34])**.** *Let $G = (V, E)$ be a graph and w be a vertex weight function $V \to [1, +\infty)$. When the parameter is the total weight W of the solution sought after, $\mathcal{W}$-MinVCP$_4$ in G admits an $\mathcal{O}^*((3.1479)^W)$-algorithm.*

Theorem 29 ([34])**.** *Let $G = (V, E)$ be a graph and w be a vertex weight function $V \to [1, +\infty)$. When the parameter is the total weight W of the solution sought after, $\mathcal{W}$-MinVCP$_k$ in G admits an $\mathcal{O}^*((c_k)^W)$-algorithm, where c_k is the largest positive root of the characteristic polynomial*

$$x^4 - 3x^3 - (k^2 - 5k + 5)x^2 + x + (k^2 - 6k + 9).$$

Some values of c_k are listed in Table 3:

Table 3. Some values of c_k

k	5	6	7	8	9	10	100
c_k ($\leq$)	4.1017	5.0640	6.0439	7.0320	8.0243	9.0191	99.0002

Kernelization is an important method that is often used for dealing with NP-hard problems. Two instances of a decision problem are equivalent if and only if they are either both yes-instances or both no-instances. A *kernelization algorithm* for a given parameterized problem Π with a parameter p is a polynomial-time algorithm which takes an input instance (I, p) of Π and returns an equivalent instance (I', p'), called a *kernel*, so that $t' \leq t$ and $|I'| \leq g(p)$ for a function $g(p)$. If $g(p)$ is linear, the kernel is called linear.

Following a result on k-Hit [65,66], MinVCP$_k$ admits a kernel with $O(t^{k-1})$ vertices and $O(k \cdot t^k)$ edges. On the other hand, it is not possible to achieve a kernel with $O(t^{2-\varepsilon})$ edges for MinVCP$_k$ unless coNP is in NP/poly [67]. Very recently, Červený et al. [68] gave a kernel with $O(t^3)$ edges for MinVCP$_k$ for any $k \geq 6$.

Theorem 30 ([68])**.** *For any $k \geq 6$, MinVCP$_k$ admits a kernel with $O(t^3 k^{O(k)})$ vertices and edges.*

For MinVCP$_2$, the current best kernel known is a kernel with $2t - c \log t$ vertices for any fixed constant c by Lampis [69]. Nemhauser and Trotter [70] showed a well-known theorem (the NT-Theorem) for MinVCP$_2$.

Theorem 31 ([70])**.** *Given a graph G, there exists an $O(\sqrt{mn} + n)$-time algorithm that can partition the vertex set of G into three subsets A, B and C so that*
(1) if F is a VCP_2 of $G[C]$, then $|F| \geq |C|/2$ and $A \cup F$ is a VCP_2 of G;
(2) there must be a minimum VCP_2 F' of G with $A \subseteq F'$.

Fellows et al. [71] extended the NT-Theorem for d-BDD and Xiao [72] improved Fellow et al.'s result. Since MinVCP$_3$ is equivalent to 1-BDD, one can derive a generalization of the NT-theorem for MinVCP$_3$.

Theorem 32 ([72])**.** *For a graph, there exists an $O(n^{5/2}m)$-time algorithm that can partition the vertex set of G into three subsets A, B and C so that*
(1) if F is a VCP_3 of $G[C]$, then $|F| \geq |C|/13$ and $A \cup F$ is a VCP_3 of G;
(2) there must be a minimum VCP_3 F' of G with $A \subseteq F'$.

The generalization of the NT-theorem for MinVCP$_3$ implies a kernel with $13t$ vertices for MinVCP$_3$. The bound of the size of the kernel for MinVCP$_3$ was subsequently improved several times [55,73]. The current best kernel known is by Xiao and Kou [55].

Theorem 33 ([55])**.** *MinVCP$_3$ admits a kernel with $5t$ vertices.*

Červený et al. [68] also gave kernels with $O(t^2)$ edges for MinVCP$_4$ and MinVCP$_5$ that are asymptotically optimal (unless coNP is in NP/poly).

Theorem 34 ([68])**.** *MinVCP$_4$ admits a kernel with at most $176t^2 + 166t$ edges.*

Theorem 35 ([68])**.** *MinVCP$_5$ admits a kernel with at most $608t^2 + 583t$ edges.*

6. The k-Path Vertex Cover Number

Computing $\psi_k(G)$ is NP-hard. Moreover, Computing $\psi_3(G)$ remains NP-hard in bipartite C_4-free graphs of maximum degree three [28]. Some upper and lower bounds on $\psi_k(G)$ in accordance with different graph parameters were given in the literature.

Theorem 36 ([4]). *For $n \geq k \geq 2$, the exact values of $\psi_k(G)$ of P_n, C_n and K_n are $\lfloor \frac{n}{k} \rfloor$, $\lceil \frac{n}{k} \rceil$ and $n-k+1$, respectively.*

Theorem 37 ([74]). *For $n \geq 4$,*
(i) $\psi_k(K_{n,n}) = \lceil \frac{n+1}{2} \rceil$ when $k = n+1$;
(ii) $\psi_k(K_{n,n}) = n+1-\lfloor \frac{k}{2} \rfloor$ when $n+2 \leq k < 2n$;
(iii) $\psi_k(K_{n,n}) \geq \frac{n^2-nk+2n}{n-\frac{k}{2}+1}$ when $4 \leq k \leq n$.

Theorem 38 ([74]). *If $m \geq n \geq 2$, $\psi_2(K_{m,n}) = \psi_3(K_{m,n}) = n$. If $m > n \geq 2$ and $k \geq 3$,*

$$\psi_k(K_{m,n}) = n+1-\lfloor \frac{k}{2} \rfloor.$$

Theorem 39 ([12]). *For a tree T, $\psi_k(T) \leq n/k$.*

Theorem 40 ([4]). *For $d \geq k-1 \geq 1$ and any d-regular graph G,*

$$\psi_k(G) \geq \frac{d-k+2}{2d-k+2}n.$$

Recently, Bujtás et al. [75] generalized the result of Theorem 40 to general graphs in terms of $\Delta(G)$ and $\delta(G)$.

Theorem 41 ([75]). *If G is a graph and $\delta(G) \geq k-1 \geq 2$, then*

$$\psi_k(G) \geq \frac{\delta(G)-k+2}{\delta(G)+\Delta(G)-k+2}n.$$

Theorem 42 ([75]). *If $k \geq 3$ and $\Delta = 2$ or $\Delta \geq 4$, and G is a graph with vertex degree at most Δ, then*
(i) if $\Delta \geq 2$ is even,

$$\psi_k(G) \leq \frac{(k-1)(\Delta-2)+4}{(k-1)\Delta+4}n;$$

(ii) if $\Delta \geq 5$ is odd,

$$\psi_k(G) \leq \frac{(k-1)(\Delta-3)+8}{(k-1)(\Delta-1)+8}n.$$

Theorem 43 ([75]). *If G is a graph without isolated vertices and $k \geq 3$,*

$$\psi_k(G) \leq n - \frac{2k-3}{k-1}\sum_{v \in V(G)} \frac{1}{1+d(v)}.$$

A graph which contains no induced cycle with a size of at least four is called a chordal graph.

Theorem 44 ([75]). *If G is a chordal graph of clique number ω and $k \geq 3$,*

$$\psi_k(G) \leq \frac{\omega}{\omega+k-1}n.$$

The bounds on $\psi_k(G)$ for small k were also investigated.

Theorem 45 ([12,75]). *For a graph G,*
(i) $\psi_3(G) \leq \max\{\frac{2n+m}{6}, \frac{n+m}{4}, \frac{4n+m}{9}\}$,
(ii) $\psi_4(G) \leq \frac{n+3m}{10}$.

Theorem 46 ([12]). *Let G be a graph,*

$$\psi_3(G) \le \frac{\lceil \frac{\Delta(G)-1}{2} \rceil}{\lceil \frac{\Delta(G)+1}{2} \rceil} n.$$

Corollary 1 ([17]). *If G is a cubic graph,* $2n/5 \le \psi_3(G) \le n/2$.

Theorem 47 ([4]). *For a graph G, let* $d(G)$ *be the average degree of G and let* ℓ *be the smallest possible positive integer so that* $\ell \ge (d(G)-2)/2$, *then*

$$\psi_3(G) \le \frac{\ell n}{\ell+2} + \frac{m}{(\ell+1)(\ell+2)}.$$

Theorem 48 ([75]). *Let G be a planar graph,* $\psi_3(G) \le 11n/15$ *and* $\psi_6(G) \le 2n/3$.

Theorem 49 ([12]). *Let G be an outerplanar graph,* $\psi_3(G) \le n/2$.

Theorem 50 ([75]). *Let G be a chordal planar graph,* $\psi_3(G) \le 2n/3$, $\psi_4(G) \le 4n/7$, *and* $\psi_5(G) \le n/2$.

Some bounds and exact values for $\psi_k(G)$ on Cartesian product graphs, rooted product graphs, and Kneser graphs were also investigated [4,74,76,77].

7. Conclusions

MinVCP$_k$ is a natural generalization of the classic MinVCP$_2$ and has received increasing attention in recent years. This work aims to present a survey on the problem that could be used to gain insight into the topic. In this survey, we mainly focus on the cases with $k \ge 3$. It is important to point out that although there are various methods to deal with NP-hard problems, we focus mainly on the results of exact algorithms, approximation algorithms and parameterized algorithms for MinVCP$_k$ and its variants. In fact, some results on online algorithms and heuristic algorithms for MinVCP$_k$ have also been reported in the literature [78,79]. Finally, we present some important problems which could be used to draw future research directions in this area.

Problem 1. *For general k and a fixed constant* $0 < c < 1$, *find an* $(1-c)k$*-approximation algorithm for MinVCP*$_k$.

Problem 2. *Determine whether MinVCP*$_k$ *admits an* $\mathcal{O}^*((k-1-\varepsilon)^t)$*-algorithm for any k and any constant* $\varepsilon > 0$.

Problem 3. *Determine whether MinVCP*$_k$ *admits a kernel with* $O(t)$ *vertices, or a kernel with* $O(t^2)$ *edges for any k.*

Problem 4. *Find a generalization of the NT-theorem for MinVCP*$_k$ *for general k.*

Problem 5. *Prove and disprove the following conjecture: If G is a planar graph,* $\psi_3(G) \le 2n/3$.

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References

1. Bondy, J.A.; Murty, U.S.R. *Graph Theory*; Springer: New York, NY, USA, 2008.
2. Tu, J.; Zhou, W. A primal–dual approximation algorithm for the vertex cover P_3 problem. *Theor. Comput. Sci.* **2011**, *412*, 7044–7048. [CrossRef]
3. Tu, J.; Zhou, W. A factor 2 approximation algorithm for the vertex cover P_3 problem. *Inform. Process. Lett.* **2011**, *111*, 683–686. [CrossRef]
4. Brešar, B.; Jakovac, M.; Katrenič, J.; Semanišin, G.; Taranenko, A. On the vertex k-path cover. *Discret. Appl. Math.* **2013**, *161*, 1943–1949. [CrossRef]
5. Acharya, H.B.; Choi, T.; Bazzi, R.A.; Gouda, M.G. The k-observer problem in computer networks. *Netw. Sci.* **2012**, *1*, 15–22. [CrossRef]
6. Ries, B.; Schamberg, B.; Unger, W. The k-Observer Problem on d-regular Graphs. In Proceedings of the Stabilization, Safety, and Security of Distributed Systems (SSS 2015), Edmonton, AB, Canada, 18–21 August 2015; pp. 81–93.
7. Lee, E. Partitioning a graph into small pieces with applications to path transversal. *Math. Program. Ser. A* **2019**, *177*, 1–19. [CrossRef]
8. Brüstle, N.; Elbaz, T.; Hatami, H.; Kocer, O.; Ma, B. Approximation Algorithms for Hitting Subgraphs. In Proceedings of the 32nd International Workshop on Combinatorial Algorithms (IWOCA 2021), Ottawa, ON, Canada, 5–7 July 2021; Lecture Notes in Computer Science; Springer: Berlin/Heidelberg, Germany, 2021; Volume 12757.
9. Fujito, T. A unified approximation algorithm for node-deletion problems. *Discret. Appl. Math.* **1998**, *86*, 213–231. [CrossRef]
10. Krishnamoorthy, M.S.; Deo, N. Node-deletion NP-complete problems. *SIAM J. Comput.* **1979**, *8*, 619–625. [CrossRef]
11. Yannakakis, M. Node-deletion problems on bipartite graphs. *SIAM J. Comput.* **1981**, *10*, 310–327. [CrossRef]
12. Brešar, B.; Kardoš, F.; Katrenič, J.; Semanišin, G. Minimum k-path vertex cover. *Discret. Appl. Math.* **2011**, *159*, 1189–1195. [CrossRef]
13. Novotný, M. Design and analysis of a generalized canvas protocol. In Proceedings of the 4th Workshop on Information Security Theory and Practice, WISTP 2010, Passau, Germany, 12–14 April 2010; Lecture Notes in Computer Science; Springer: Berlin/Heidelberg, Germany, 2010; Volume 6033, pp. 106–121.
14. Vogt, H. *Integrity Preservation for Communication in Sensor Networks*; Technical Report 434; Institute for Pervasive Computing, ETH Zurich, Zurich, Switzerland, 2004.
15. Poljak, S. A note on the stable sets and coloring of graphs. *Comment. Math. Univ. Carolin.* **1974**, *15*, 307–309.
16. Brešar, B.; Kos, T.; Krivoš-Belluš, R.; Semanišin, G. Hitting subgraphs in P_4-tidy graphs. *Appl. Math. Comput.* **2019**, *352*, 211–219. [CrossRef]
17. Tu, J.; Yang, F. The vertex cover P_3 problem in cubic graphs. *Inform. Process. Lett.* **2013**, *113*, 481–485. [CrossRef]
18. Kumar, M.; Mishra, S.; Devi, N.S.; Saurabh, S. Approximation algorithms for node deletion problems on bipartite graphs with finite forbidden subgraph characterization. *Theor. Comput. Sci.* **2014**, *526*, 90–96. [CrossRef]
19. Devi, N.S.; Mane, A.C.; Mishra, S. Computational complexity of minimum P_4 vertex cover problem for regular and $K_{1,4}$-free graphs. *Discret. Appl. Math.* **2015**, *184*, 114–121. [CrossRef]
20. Tu, J. Efficient algorithm for the vertex cover P_k problem on cacti. *Appl. Math. Comput.* **2017**, *311*, 217–222. [CrossRef]
21. Brešar, B.; Krivoš-Belluš, R.; Semanišin, G.; Šparl, P. On the weighted k-path vertex cover problem. *Discret. Appl. Math.* **2014**, *177*, 14–18. [CrossRef]
22. Li, X.; Zhang, Z.; Huang, X. Approximation algorithms for minimum (weight) connected k-path vertex cover. *Discret. Appl. Math.* **2016**, *205*, 101–108. [CrossRef]
23. Xiao, M.; Nagamochi, H. Exact algorithms for maximum independent set. *Inform. Comput.* **2017**, *255*, 126–146. [CrossRef]
24. Chang, M-S.; Chen, L-H.; Hung, L-J.; Liu, Y-Z.; Rossmanith, P.; Sikdar, S. An $O^*(1.4658^n)$-time exact algorithm for the maximum bounded-degree-1 set problem. In Proceedings of the 31st Workshop on Combinatorial Mathematics and Computation Theory, CMCT 2014, Taipei, Taiwan, 25–26 April 2014; pp. 9–18.
25. Chang, M-S.; Chen, L-H.; Hung, L-J.; Liu, Y-Z.; Rossmanith, P.; Sikdar, S. Moderately exponential time algorithms for the maximum bounded-degree-1 set problem. *Discret. Appl. Math.* **2018**, *251*, 114–125. [CrossRef]
26. Kardoš, F.; Katrenič, J.; Schiermeyer, I. On computing the minimum 3-path vertex cover and dissociation number of graphs. *Theor. Comput. Sci.* **2011**, *412*, 7009–7017. [CrossRef]
27. Xiao, M.; Kou, S. Exact algorithms for the maximum dissociation set and minimum 3-path vertex cover problems. *Theor. Comput. Sci.* **2017**, *657*, 86–97. [CrossRef]
28. Boliac, R.; Cameron, K.; Lozin, V. On computing the dissociation number and the induced matching number of bipartite graphs. *Ars Combin.* **2004**, *72*, 241–253.
29. Brause, C.; Krivoš-Belluš, R. On a relation between k-path partition and k-path vertex cover. *Discret. Appl. Math.* **2017**, *223*, 28–38. [CrossRef]
30. Cameron, K.; Hell, P. Independent packings in structured graphs. *Math. Program., Ser. B* **2006**, *105*, 201–213. [CrossRef]
31. Lozin, V.; Rautenbach, D. Some results on graphs without long induced paths. *Inform. Process. Lett.* **2003**, *88*, 167–171. [CrossRef]
32. Orlovich, Y.; Dolgui, A.; Finke, G.; Gordon, V.; Werner, F. The complexity of dissociation set problems in graphs. *Discret. Appl. Math.* **2011**, *159*, 1352–1366. [CrossRef]

33. Fomin, F.V.; Gaspers, S.; Kratsch, D.; Liedloff, M.; Saurabh, S. Iterative compression and exact algorithms. *Theor. Comput. Sci.* **2010**, *411*, 1045–1053. [CrossRef]
34. Fernau, H. Parameterized algorithmics for d-hitting set. *Int. J. Comput. Math.* **2010**, *87*, 3157–3174. [CrossRef]
35. Alimonti, P.; Kann, V. Some APX-completeness results for cubic graphs. *Theor. Comput. Sci.* **2000**, *237*, 123–134. [CrossRef]
36. Khot, S.; Regev, O. Vertex cover might be hard to approximate to within 2-ε. *J. Comput. Syst. Sci.* **2008**, *74*, 335–349. [CrossRef]
37. Zhang, A.; Chen, Y.; Chen, Z-Z.; Lin, G. Improved approximation algorithms for path vertex covers in regular graphs. *Algorithmica* **2020**, *82*, 3041–3064. [CrossRef]
38. Zhang, Z.; Li, X.; Shi, Y.; Nie, H.; Zhu, Y. PTAS for minimum k-path vertex cover in ball graph. *Inform. Process. Lett.* **2017**, *119*, 9–13. [CrossRef]
39. Liu, X.; Lu, H.; Wang, W.; Wu, W. PTAS for the minimum k-path connected vertex cover problem in unit disk graphs. *J. Glob. Optim.* **2013**, *56*, 449–458. [CrossRef]
40. Chen, L.; Huang, X.; Zhang, Z. A simpler PTAS for connected k-path vertex cover in homogeneous wireless sensor network. *J. Combin. Optim.* **2018**, *36*, 35–43. [CrossRef]
41. Fujito, T. On approximability of connected path vertex cover. In Proceedings of the Workshop on Approximation and Online Algorithms (WAOA 2017), Vienna, Austria, 7–8 September 2017; pp. 17–25.
42. Camby, E.; Cardinal, J.; Chapelle, M.; Fiorini, S.; Joret, G. A primal-dual 3-approximation algorithm for hitting 4-vertex paths. In Proceedings of the 9th International Colloquium on Graph Theory and Combinatorics (ICGT 2014), Grenoble, France, 30 June–4 July 2014; p. 61.
43. Li, Y.; Tu, J. A 2-approximation algorithm for the vertex cover P_4 problem in cubic graphs. *Int. J. Comput. Math.* **2014**, *91*, 2103–2108. [CrossRef]
44. Tu, J.; Shi, Y. An efficient polynomial time approximation scheme for the vertex cover P_3 problem on planar graphs. *Discuss. Math. Graph Theory* **2019**, *39*, 55–65.
45. Liu, P.; Zhang, Z.; Li, X.; Wu, W. Approximation algorithm for minimum connected 3-path vertex cover. *Discret. Appl. Math.* **2020**, **287**, 77–84. [CrossRef]
46. Ran, Y.; Zhang, Z.; Huang, X.; Li, X.; Du, D. Approximation algorithms for minimum weight connected 3-path vertex cover. *Appl. Math. Comput.* **2019**, *347*, 723–733. [CrossRef]
47. Wang, L.; Zhang, X.; Zhang, Z.; Broersma, H. A PTAS for the minimum weight connected vertex cover P_3 problem on unit disk graphs. *Theor. Comput. Sci.* **2015**, *571*, 58–66. [CrossRef]
48. Wang, L.; Zhang, X.; Du, W.; Zhang, Z. A PTAS for minimum weighted connected vertex cover P_3 problem in 3-dimensional wireless sensor networks. *J. Combin. Optim.* **2017**, *33*, 106–122. [CrossRef]
49. Cai, L. Fixed-parameter tractability of graph modification problems for hereditary properties. *Inform. Process. Lett.* **1996**, *58*, 171–176. [CrossRef]
50. Chen, J.; Kanj, I.A.; Xia, G. Improved upper bounds for vertex cover. *Theor. Comput. Sci.* **2010**, *411*, 3736–3756. [CrossRef]
51. Tu, J. A fixed-parameter algorithm for the vertex cover P_3 problem. *Inform. Process. Lett.* **2015**, *115*, 96–99. [CrossRef]
52. Chang, M-S.; Chen, L-H.; Hung, L-J.; Rossmanith, P.; Su, P-C. Fixed-parameter algorithms for vertex cover P_3. *Discret. Optim.* **2016**, *19*, 12–22. [CrossRef]
53. Katrenič, J. A faster FPT algorithm for 3-path vertex cover. *Inform. Process. Lett.* **2016**, *116*, 273–278. [CrossRef]
54. Wu, B. A measure and conquer approach for the parameterized bounded degree-one vertex deletion. In Proceedings of the 21st International Computing and Combinatorics Conference (COCOON 2015), Beijing, China, 4–6 August 2015; pp. 469–480.
55. Xiao, M.; Kou, S. Kernelization and parameterized algorithms for 3-path vertex cover. In Proceedings of the 14th International Conference on Theory and Applications of Models of Computation, TAMC 2017, Bern, Switzerland, 20–22 April 2017; pp. 654–668.
56. Tsur, D. Parameterized algorithm for 3-path vertex cover. *Theor. Comput. Sci.* **2019**, *783*, 1–8. [CrossRef]
57. Tu, J.; Jin, Z. An FPT algorithm for the vertex cover P_4 problem. *Discret. Appl. Math.* **2016**, *200*, 186–190. [CrossRef]
58. Tsur, D. An $O^*(2.619^k)$ algorithm for 4-Path Vertex Cover. *Discret. Appl. Math.* **2021**, *291*, 1–14. [CrossRef]
59. Červený, R.; Suchvý, O. Faster FPT algorithm for 5-path vertex cover. In Proceedings of the 44th Symposium on Mathematical Foundations of Computer Science (MFCS 2019), Aachen, Germany, 26–30 August 2019.
60. Tsur, D. l-path vertex cover is easier than l-hitting set for small l. *arXiv* **2019**, arXiv:1906.10523.
61. Červený, R.; Suchvý, O. Generating faster algorithms for d-Path Vertex Cover. *arXiv* **2021**, arXiv:2111.05896v1.
62. Tu, J.; Wu, L.; Yuan, J.; Cui, L. On the vertex cover P_3 problem parameterized by treewidth. *J. Combin. Optim.* **2017**, *34*, 414–425. [CrossRef]
63. Bai, Z.; Tu, J.; Shi, Y. An improved algorithm for the vertex cover P_3 problem on graphs of bounded treewidth. *Discret. Math. Theor. Comput. Sci.* **2019**, *21*, #17.
64. Shachnai, H.; Zehavi, M. A Multivariate Approach for Weighted FPT Algorithms. In Proceedings of the 23rd Annual European Symposium on Algorithms (ESA 2015), Patras, Greece, 14–16 September 2015; Lecture Notes in Computer Science; Springer: Berlin/Heidelberg, Germany, 2015; Volume 9294.
65. Abu-Khzam, F.N. A kernelization algorithm for d-hitting set. *J. Comput. Syst. Sci.* **2010**, *76*, 524–531. [CrossRef]
66. Fafianie, S.; Kratsch, S. A Shortcut to (Sun)Flowers: Kernels in Logarithmic Space or Linear Time. In Proceedings of the 40th International Symposium on Mathematical Foundations of Computer Science (MFCS 2015), Milano, Italy, 24–28 August 2015; Lecture Notes in Computer Science; Springer: Berlin/Heidelberg, Germany, 2015; Volume 9235.

67. Dell, H.; Melkebeek, D. Satisfiability allows no nontrivial sparsification unless the polynomial-time hierarchy collapses. In Proceedings of the 42nd ACM Symposium on Theory of Computing (STOC 2010), Cambridge, MA, USA, 6–8 June 2010; pp. 251–260.
68. Červený, R.; Choudhary, P.; Suchvý, O. On Kernels for d-Path Vertex Cover. *arXiv* **2021**, arXiv:2107.12245v1.
69. Lampis, M. A kernel of order $2k - c\log k$ for vertex cover. *Inform. Process. Lett.* **2011**, *111*, 1089–1091. [CrossRef]
70. Nemhauser, G.; Trotter, L. Vertex packings: Structural properties and algorithms. *Math. Program.* **1975**, *8*, 232–248. [CrossRef]
71. Fellows, M.R.; Guo, J.; Moser, H.; Niedermeier, R. A generalization of Nemhauser and Trotter's local optimization theorem. *J. Comput. Syst. Sci.* **2011**, *77*, 1141–1158. [CrossRef]
72. Xiao, M. On a Generalization of Nemhauser and Trotter's Local Optimization Theorem. *J. Comput. Syst. Sci.* **2017**, *84*, 97–106. [CrossRef]
73. Brause, C.; Schiermeyer, I. Kernelization of the 3-path vertex cover problem. *Discret. Math.* **2016**, *339*, 1935–1939. [CrossRef]
74. Li, Z.; Zuo, L. The k-path vertex cover in Cartesian product graphs and complete bipartite graphs. *Appl. Math. Comput.* **2018**, *331*, 69–79. [CrossRef]
75. Bujtás, C.; Jakovac, M.; Tuza, Z. The k-path vertex cover: General bounds and chordal graphs. *arXiv* **2021**, arXiv:2105.02018.
76. Brešar, B.; Dravec, T. On the dissociation number of Kneser graphs. *arXiv* **2021**, arXiv:2108.10801.
77. Jakovac, M. The k-path vertex cover of rooted product graphs. *Discret. Appl. Math.* **2015**, *187*, 111–119. [CrossRef]
78. Zhang, W.; Tu, J.; Wu, L. A multi-start iterated greedy algorithm for the minimum weight vertex cover P_3 problem. *Appl. Math. Comput.* **2019**, *349*, 359–366. [CrossRef]
79. Zhang, Y.; Zhang, Z.; Shi, Y.; Li, X. Algorithm for Online 3-Path Vertex Cover. *Theory Comput. Syst.* **2020**, *64*, 327–338. [CrossRef]