

Article

BMO and Asymptotic Homogeneity

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Abstract: First, we prove that the BMO condition by John–Nirenberg leads in the natural way to the asymptotic homogeneity at the origin of regular homeomorphic solutions of the degenerate Beltrami equations. Then, on this basis we establish a series of criteria for the existence of regular homeomorphic solutions of the degenerate Beltrami equations in the whole complex plane with asymptotic homogeneity at infinity. These results can be applied to the fluid mechanics in strongly anisotropic and inhomogeneous media because the Beltrami equation is a complex form of the main equation of hydromechanics.

Keywords: BMO; degenerate Beltrami equations; asymptotic homogeneity at infinity; conformality by Belinskii and by Lavrent'iev; hydromechanics; fluid mechanics

MSC: Primary 30C62; 30C65; 30H35; 35J70; Secondary 35Q35; 76B03



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1. Introduction

A real-valued function u in a domain D in $\mathbb{C}$ is said to be of **bounded mean oscillation** in D , abbr. $u \in \text{BMO}(D)$, if $u \in L^1_{\text{loc}}(D)$ and

$$\|u\|_* := \sup_B \frac{1}{|B|} \int_B |u(z) - u_B| dm(z) < \infty, \quad (1)$$

where the supremum is taken over all discs B in D and

$$u_B = \frac{1}{|B|} \int_B u(z) dm(z).$$

Recall that the class BMO was introduced by John and Nirenberg (1961) in the paper [1] and soon became an important concept in harmonic analysis, partial differential equations and related areas, see, e.g., [2,3].

A function φ in BMO is said to have **vanishing mean oscillation**, abbr. $\varphi \in \text{VMO}$, if the supremum in (1) taken over all balls B in D with $|B| < \varepsilon$ converges to 0 as $\varepsilon \rightarrow 0$. Recall that VMO has been introduced by Sarason in [4]. There are a number of papers devoted to the study of partial differential equations with coefficients of the class VMO, see, e.g., [5–9]. Note, by the way, that $W^{1,2}(D) \subset \text{VMO}(D)$, see [10].

Let D be a domain in the complex plane $\mathbb{C}$, i.e., a connected open subset of $\mathbb{C}$, and let $\mu : D \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in D . A **Beltrami equation** is an equation of the form

$$\bar{\partial}f(z) = \mu(z) \cdot \partial f(z) \quad (2)$$

with the formal complex derivatives $\bar{\partial}f = (f_x + if_y)/2$, $\partial f = (f_x - if_y)/2$, $z = x + iy$, where f_x and f_y are usual partial derivatives of f in x and y , correspondingly. The function μ is said to be the **complex coefficient** and

$$K_\mu(z) := \frac{1 + |\mu(z)|}{1 - |\mu(z)|} \quad (3)$$

the **dilatation quotient** of Equation (2). The Beltrami equation is called **degenerate** if $\text{ess sup } K_\mu(z) = \infty$. Homeomorphic solutions of the Beltrami equations with $K_\mu \leq Q < \infty$ in the Sobolev class $W_{\text{loc}}^{1,1}$ are called **Q-quasiconformal mappings**.

It is known that if K_μ is bounded, then the Beltrami equation has homeomorphic solutions, see, e.g., [11–14]. Recently, a series of effective criteria for the existence of homeomorphic $W_{\text{loc}}^{1,1}$ solutions have been also established for degenerate Beltrami equations, see, e.g., historic comments with relevant references in monographs the [15–17].

These criteria were formulated both in terms of K_μ and the more refined quantity that takes into account not only the modulus of the complex coefficient μ but also its argument

$$K_\mu^T(z, z_0) := \frac{\left| 1 - \frac{\overline{z-z_0}}{z-z_0} \mu(z) \right|^2}{1 - |\mu(z)|^2} \quad (4)$$

that is called the **tangent dilatation quotient** of the Beltrami equation with respect to a point $z_0 \in \mathbb{C}$, see, e.g., [18–23]. Note that

$$K_\mu^{-1}(z) \leq K_\mu^T(z, z_0) \leq K_\mu(z) \quad \forall z \in D, z_0 \in \mathbb{C}. \quad (5)$$

The geometrical sense of K_μ^T can be found, e.g., in the monograph [16].

A function f in the Sobolev class $W_{\text{loc}}^{1,1}$ is called a **regular solution** of the Beltrami Equation (2) if f satisfies it a.e. and its Jacobian $J_f(z) = |\partial f(z)|^2 - |\bar{\partial} f(z)|^2 > 0$ a.e. in $\mathbb{C}$.

By the well-known Gehring–Lehto–Menchoff theorem, see [24,25], or see the monographs [11,13], each homeomorphic $W_{\text{loc}}^{1,1}$ solution f of the Beltrami equation is differentiable a.e. Recall that a function $f : D \rightarrow \mathbb{C}$ is **differentiable by Darboux Stolz at a point** $z_0 \in D$ if

$$f(z) - f(z_0) = \partial f(z_0) \cdot (z - z_0) + \bar{\partial} f(z_0) \cdot \overline{(z - z_0)} + o(|z - z_0|) \quad (6)$$

where $o(|z - z_0|)/|z - z_0| \rightarrow 0$ as $z \rightarrow z_0$. Moreover, f is called **conformal at the point** z_0 if in addition $f_{\bar{z}}(z_0) = 0$ but $f_z(z_0) \neq 0$.

The example $w = z(1 - \ln|z|)$ of B.V. Shabat, see [26], p. 40, shows that, for a continuous complex characteristic $\mu(z)$, the quasiconformal mapping $w = f(z)$ can be non-differentiable by Darboux Stolz at the origin. If the characteristic $\mu(z)$ is continuous at a point $z_0 \in D$, then, as was first established, apparently, by P.P. Belinskij in [26], p. 41, the mapping $w = f(z)$ is differentiable at z_0 in the following meaning:

$$\Delta w = A(\rho) [\Delta z + \mu_0 \overline{\Delta z} + o(\rho)], \quad (7)$$

where $\mu_0 = \mu(z_0)$, $\rho = |\Delta z + \mu_0 \overline{\Delta z}|$, $A(\rho)$ depends only on ρ and $o(\rho)/\rho \rightarrow 0$ as $\rho \rightarrow 0$. As it was clarified later in [27], see also [28], here $A(\rho)$ may not have a limit with $\rho \rightarrow 0$; however,

$$\lim_{\rho \rightarrow 0} \frac{A(t\rho)}{A(\rho)} = 1 \quad \forall t > 0. \quad (8)$$

Following [27], a mapping $f : D \rightarrow \mathbb{C}$ is called **differentiable by Belinskij at a point** $z_0 \in D$ if conditions (7) and (8) hold with some $\mu_0 \in \mathbb{D} := \{\mu \in \mathbb{C} : |\mu| < 1\}$. Note that here, in the case of discontinuous $\mu(z)$, it is not necessary $\mu_0 = \mu(z_0)$. If in addition $\mu_0 = 0$, then f is called **conformal by Belinskij at the point** z_0 .

For quasiconformal mappings $f : D \rightarrow \mathbb{C}$ with $f(0) = 0 \in D$, it was shown in [27], see also [28], that the conformality by Belinskij of f at the origin is equivalent to each of its properties:

$$\lim_{\tau \rightarrow 0} \frac{f(\tau \zeta)}{f(\tau)} = \zeta \quad \text{along the ray } \tau > 0 \quad \forall \zeta \in \mathbb{C}, \quad (9)$$

$$\lim_{z \rightarrow 0} \left\{ \frac{f(z')}{f(z)} - \frac{z'}{z} \right\} = 0 \quad \text{along } z, z' \in \mathbb{C}, |z'| < \delta |z|, \quad \forall \delta > 0, \quad (10)$$

$$\lim_{z \rightarrow 0} \frac{f(z\zeta)}{f(z)} = \zeta \quad \text{along } z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\} \quad \forall \zeta \in \mathbb{C}, \quad (11)$$

and, finally, to the property of the limit in (11) to be locally uniform with respect to $\zeta \in \mathbb{C}$.

Following the article [28], the property (11) of a mapping $f : D \rightarrow \mathbb{C}$ with $f(0) = 0 \in D$ is called its **asymptotic homogeneity** at 0. In the sequel, we sometimes write (11) in the shorter form $f(\zeta z) \sim \zeta f(z)$.

In particular, we obtain from (10) under $|z'| = |z|$ that

$$\lim_{r \rightarrow 0} \frac{\max_{|z|=r} |f(z)|}{\min_{|z|=r} |f(z)|} = 1 \quad (12)$$

i.e., that the Lavrent'iev characteristic is equal 1 at the origin. It is natural to say in the case of (12) that the mapping f is **conformal by Lavrent'iev** at 0. As we see, the usual conformality implies the conformality by Belinskij and the latter implies the conformality by Lavrent'iev at the origin meaning geometrically that the infinitesimal circle centered at zero is transformed into an infinitesimal circle also centered at zero.

However, condition (11) is much stronger than condition (12). We also obtain from (11) the asymptotic preserving angles

$$\lim_{z \rightarrow 0} \arg \left[\frac{f(z\zeta)}{f(z)} \right] = \arg \zeta \quad \forall \zeta \in \mathbb{C}^* \quad (13)$$

and asymptotic preserving moduli of infinitesimal rings

$$\lim_{z \rightarrow 0} \frac{|f(z\zeta)|}{|f(z)|} = |\zeta| \quad \forall \zeta \in \mathbb{C}^*. \quad (14)$$

The latter two geometric properties characterize asymptotic homogeneity and demonstrate that it is close to the usual conformality.

It should be noted that, despite (14), an asymptotically homogeneous map can send radial lines to infinitely winding spirals, as shown by the example $f(z) = ze^{i\sqrt{-\ln|z|}}$, see [26], p. 41. Moreover, the above Shabat example shows that the conformality by Belinskij admits infinitely great tensions and pressures at the corresponding points.

It was shown in [27] that a quasiconformal mapping $f : D \rightarrow \mathbb{C}$, whose complex characteristic $\mu(z)$ is approximately continuous at a point $z_0 \in D$, is differentiable by Belinskij at the point with $\mu_0 = \mu(z_0)$ and, in particular, is asymptotically homogeneous if $\mu(z_0) = 0$. Recall that $\mu(z)$ is called **approximately continuous at the point** z_0 if there is a measurable set E such that $\mu(z) \rightarrow \mu(z_0)$ as $z \rightarrow z_0$ in E and z_0 is a point of density for E , i.e.,

$$\lim_{\varepsilon \rightarrow 0} \frac{|E \cap D(z_0, \varepsilon)|}{|D(z_0, \varepsilon)|} = 1,$$

where $\mathbb{D}(z_0, \varepsilon) = \{z \in \mathbb{C} : |z - z_0| < \varepsilon\}$. Note also that, for functions μ in L^∞ , the points of approximate continuity coincide with the Lebesgue points of μ , i.e., such z_0 for which

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \int_{|z - z_0| < r} |\mu(z) - \mu(z_0)| dm(z) = 0,$$

where $dm(z) := dx dy$, $z = x + iy$, stands to the Lebesgue measure (area) in $\mathbb{C}$.

The above results on the asymptotic homogeneity, i.e., on the conformality by Belinskij, are extended to the degenerate Beltrami equations with its dilatation K_μ in BMO. Just our approximate approach to the study of the degenerate Beltrami equations allowed us significantly to move forward.

As we saw, the asymptotic homogeneity inherits the main geometric properties of conformal mappings. Thus, our research is organically inserted into the stream of numerous works that were devoted to the study of conformality of mappings, see, e.g., [26,29–35].

2. FMO and the Main Lemma with Participation of BMO

Here and later on, we apply the notations

$$\mathbb{D}(z_0, r) := \{z \in \mathbb{C} : |z - z_0| < r\}, \quad \mathbb{D}(r) := \mathbb{D}(0, r), \quad \mathbb{D} := \mathbb{D}(0, 1),$$

and of the mean value of integrable functions φ over the disks $\mathbb{D}(z_0, r)$

$$\oint_{\mathbb{D}(z_0, r)} \varphi(z) dm(z) := \frac{1}{|\mathbb{D}(z_0, r)|} \int_{\mathbb{D}(z_0, r)} \varphi(z) dm(z).$$

Following [36], we say that a function $\varphi : D \rightarrow \mathbb{R}$ has **finite mean oscillation** at a point $z_0 \in D$, abbr. $\varphi \in \text{FMO}(z_0)$, if

$$\overline{\lim}_{\varepsilon \rightarrow 0} \oint_{\mathbb{D}(z_0, \varepsilon)} |\varphi(z) - \tilde{\varphi}_\varepsilon(z_0)| dm(z) < \infty, \quad (15)$$

where

$$\tilde{\varphi}_\varepsilon(z_0) = \oint_{\mathbb{D}(z_0, \varepsilon)} \varphi(z) dm(z). \quad (16)$$

Note that the condition (15) includes the assumption that φ is integrable in some neighborhood of the point z_0 . We say also that a function $\varphi : D \rightarrow \mathbb{R}$ is of **finite mean oscillation in D**, abbr. $\varphi \in \text{FMO}(D)$ or simply $\varphi \in \text{FMO}$, if $\varphi \in \text{FMO}(z_0)$ for all points $z_0 \in D$.

Remark 1. It is evident that $\text{BMO}(D) \subset \text{BMO}(D)_{\text{loc}} \subset \text{FMO}(D)$ and it is well-known by the John–Nirenberg lemma that $\text{BMO}_{\text{loc}} \subset L^p_{\text{loc}}$ for all $p \in [1, \infty)$, see, e.g., [1] or [3]. However, FMO is not a subclass of L^p_{loc} for any $p > 1$ but only of L^1_{loc} , see, e.g., example 2.3.1 in [16]. Thus, the class FMO is much more wider than BMO_{loc} .

The following statement is obvious by the triangle inequality.

Proposition 1. If, for a collection of numbers $\varphi_\varepsilon \in \mathbb{R}$, $\varepsilon \in (0, \varepsilon_0]$,

$$\overline{\lim}_{\varepsilon \rightarrow 0} \oint_{\mathbb{D}(z_0, \varepsilon)} |\varphi(z) - \varphi_\varepsilon| dm(z) < \infty, \quad (17)$$

then φ is of finite mean oscillation at z_0 .

In particular, choosing here $\varphi_\varepsilon \equiv 0$, $\varepsilon \in (0, \varepsilon_0]$ in Proposition 1, we obtain the following.

Corollary 1. *If, for a point $z_0 \in D$,*

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\mathbb{D}(z_0, \varepsilon)} |\varphi(z)| dm(z) < \infty, \quad (18)$$

then φ has finite mean oscillation at z_0 .

Recall that a point $z_0 \in D$ is called a **Lebesgue point** of a function $\varphi : D \rightarrow \mathbb{R}$ if φ is integrable in a neighborhood of z_0 and

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{D}(z_0, \varepsilon)} |\varphi(z) - \varphi(z_0)| dm(z) = 0. \quad (19)$$

It is known that, almost every point in D is a Lebesgue point for every function $\varphi \in L^1(D)$. Thus, we have by Proposition 1 the next corollary.

Corollary 2. *Every locally integrable function $\varphi : D \rightarrow \mathbb{R}$ has a finite mean oscillation at almost every point in D .*

Remark 2. *Note that the function $\varphi(z) = \log(1/|z|)$ belongs to BMO in the unit disk $\mathbb{D}$, see, e.g., [3], p. 5, and hence also to FMO. However, $\tilde{\varphi}_\varepsilon(0) \rightarrow \infty$ as $\varepsilon \rightarrow 0$, showing that condition (18) is only sufficient but not necessary for a function φ to be of finite mean oscillation at z_0 .*

Versions of the next statement has been first proved for the class BMO. For the FMO case, see the paper [36] and the monograph [16]. Here we prefer to use its following version, see Lemma 2.1 in [23], cf. also Lemma 5.3 in the monograph [16]:

Proposition 2. *Let $\varphi : D \rightarrow \mathbb{R}$ be a non-negative function with finite mean oscillation at $0 \in D$ and integrable in the disk $\mathbb{D}(1/2) \subset D$. Then*

$$\int_{A(\varepsilon, 1/2)} \frac{\varphi(z) dm(z)}{\left(|z| \log_2 \frac{1}{|z|}\right)^2} \leq C \cdot \log_2 \log_2 \frac{1}{\varepsilon} \quad \forall \varepsilon \in (0, 1/4), \quad (20)$$

where

$$C = 4\pi (\varphi_0 + 6d_0), \quad (21)$$

φ_0 is the average of φ over the disk $\mathbb{D}(1/2)$ and d_0 is the maximal dispersion of φ in $\mathbb{D}(1/2)$.

Recall that the **maximal dispersion** of the function φ in the disk $\mathbb{D}(z_0, r_0)$ is the quantity

$$\sup_{r \in (0, r_0]} \int_{\mathbb{D}(z_0, r)} |\varphi(z) - \tilde{\varphi}_r(z_0)| dm(z). \quad (22)$$

Here and later on, we also use the following designations for the spherical rings in $\mathbb{C}$:

$$A(z_0, r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}, \quad A(r_1, r_2) := A(0, r_1, r_2). \quad (23)$$

Further, we denote by M the conformal modulus (or 2-modulus) of a family of paths in $\mathbb{C}$, see, e.g., [37]. Moreover, given sets E and F and a domain D in $\mathbb{C}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \rightarrow \mathbb{C}$ joining E and F in D , that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0, 1)$.

Let $Q : \mathbb{C} \rightarrow (0, \infty)$ be a Lebesgue measurable function. A mapping $f : D \rightarrow \mathbb{C}$ is called a **ring Q -mapping at a point $z_0 \in D$** , if

$$M(f(\Gamma(\mathbb{S}(z_0, r_1), \mathbb{S}(z_0, r_2), D))) \leq \int_A Q(z) \cdot \eta^2(|z - z_0|) dm(z) \quad (24)$$

for each spherical ring $A = A(z_0, r_1, r_2)$ with arbitrary $0 < r_1 < r_2 < \delta_0 := \text{dist}(z_0, \partial D)$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \rightarrow [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) dr \geq 1. \quad (25)$$

Here we use also the notations for the circles in $\mathbb{C}$ centered at a point z_0

$$\mathbb{S}(z_0, r_0) = \{z \in \mathbb{C} : |z - z_0| = r_0\}.$$

Remark 3. Recall that regular homeomorphic solutions of the Beltrami Equation (2) are Q_{z_0} -mappings with $Q_{z_0}(z) = K_\mu^T(z, z_0)$ and, in particular, Q -mappings with $Q(z) = K_\mu(z)$ at each point $z_0 \in D$, see [38], see also Theorem 2.2 in [16].

Later on, in the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we use the **spherical (chordal) metric** s defined by the equalities

$$s(z, \zeta) = \frac{|z - \zeta|}{\sqrt{1 + |z|^2} \sqrt{1 + |\zeta|^2}}, \quad z \neq \infty \neq \zeta, \quad s(z, \infty) = \frac{1}{\sqrt{1 + |z|^2}}, \quad (26)$$

see, e.g., [37] (Definition 12.1). For a given set E in $\overline{\mathbb{C}}$, we also use its **spherical diameter**

$$s(E) := \sup_{z, \zeta \in E} s(z, \zeta). \quad (27)$$

Given a domain D in $\mathbb{C}$, a prescribed point $z_0 \in D$ and a measurable $Q : D \rightarrow (0, \infty)$, later on $\mathfrak{R}_Q^\Delta$ denotes the class of all ring Q -homeomorphisms f at z_0 in D with

$$s(\overline{\mathbb{C}} \setminus f(D)) \geq \Delta > 0.$$

The following statement, see Theorem 4.3 in [23], provides us by the effective estimates of the distortion of the spherical distance under the ring Q -homeomorphisms, and it follows just on the basis of Proposition 2 on FMO functions above.

Proposition 3. Let $f \in \mathfrak{R}_Q^\Delta(D)$ with $\Delta > 0$ and $Q : D \rightarrow \mathbb{R}$ be a non-negative function with finite mean oscillation at $\zeta_0 \in D$ and integrable in the disk $\mathbb{D}(\zeta_0, \varepsilon_0) \subset D$, $\varepsilon_0 > 0$. Then

$$s(f(\zeta), f(\zeta_0)) \leq \frac{32}{\Delta} \cdot \left(\log \frac{2\varepsilon_0}{|\zeta - \zeta_0|} \right)^{-\frac{1}{\alpha_0}} \quad \forall \zeta \in \mathbb{D}(\zeta_0, \varepsilon_0/2), \quad (28)$$

where

$$\alpha_0 = 2(q_0 + 6d_0), \quad (29)$$

q_0 is the average of Q over $\mathbb{D}(\zeta_0, \varepsilon_0)$ and d_0 is the maximal dispersion of Q in $\mathbb{D}(\zeta_0, \varepsilon_0)$.

Propositions 2 and 3 are key in establishing equicontinuity of classes of mappings associated with asymptotic homogeneity in the proof of the central lemma involving BMO.

Lemma 1. Let D be a domain in $\mathbb{C}$, $0 \in D$, and let $f : D \rightarrow \mathbb{C}$ be a regular homeomorphic solution of the Beltrami Equation (2) with $f(0) = 0$. Suppose that its dilatation K_μ has a majorant $Q \in \text{BMO}(D)$. Then the family of mappings $f_z(\zeta) := f(\zeta z) / f(z)$ is equicontinuous with respect to the spherical metric at each point $\zeta_0 \in \mathbb{C}$ as $z \rightarrow 0$ along $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$.

Proof. Indeed, for $\zeta_0 \in \mathbb{D}(\delta)$, $\delta > 1$, $0 < \delta_* < \text{dist}(0, \partial D)$, $\tau_* := \delta_* / \delta < \delta_*$, we see that

$$\mathbb{D}(z\zeta_0, \rho_z) \subseteq \mathbb{D}(\delta_*) \subseteq D, \quad \text{where } \rho_z := \delta_* - |z\zeta_0| \geq \delta_*(1 - |\zeta_0|/\delta) > 0, \quad z \in \overline{\mathbb{D}(\tau_*)} \setminus \{0\}.$$

Thus, by the construction the disks

$$\mathbb{D}(\zeta_0, R_z) \subseteq \mathbb{D}(\delta_*/|z|), \quad \text{where } R_z := \delta_*/|z| - |\zeta_0| \geq \delta - |\zeta_0| > 0, z \in \overline{\mathbb{D}(\tau_*)} \setminus \{0\},$$

belong to the domain of definition for the family of the functions $f_z(\zeta)$, $z \in \overline{\mathbb{D}(\tau_*)} \setminus \{0\}$.

It is clear, see, e.g., I.D(8) in [11], that $f_z(\zeta)$ is a regular homeomorphic solution of the Beltrami equation with the complex coefficient μ_z such that $|\mu_z(\zeta)| = |\mu(z\zeta)|$ and

$$K_{\mu_z}(\zeta) \leq Q_z(\zeta) := Q(z\zeta) \quad \forall \zeta_0 \in \mathbb{D}(\delta), \zeta \in \mathbb{D}(\zeta_0, R_z).$$

Note that the BMO norm of Q as well as its averages over disks are invariant under linear transformations of variables in $\mathbb{C}$. Moreover, the averages $\tilde{Q}_z(\zeta_0)$ of the function Q over the disks $\mathbb{D}(z\zeta_0, \rho_z)$ forms a continuous function with respect to the parameter $z \in \overline{\mathbb{D}(\tau_*)} \setminus \{0\}$ in view of absolute continuity of its indefinite integrals and it can be extended by continuity to $z = 0$ as its (finite !) average over the disk $\mathbb{D}(\delta_*)$. Since the closed disk $\overline{\mathbb{D}(\tau_*)}$ is compact,

$$Q_0 := \max_{z \in \overline{\mathbb{D}(\tau_*)}} \tilde{Q}_z(\zeta_0) < \infty.$$

Note also that by Remark 4 f_z , $z \in \overline{\mathbb{D}(\tau_*)}$, belongs to the class $\mathfrak{R}_{Q_\tau}^\Delta$ at ζ_0 in the punctured disk $\mathbb{D}(\zeta_0, \delta - |\zeta_0|) \setminus \{0\}$ with $\Delta = 1 > 0$ if $\zeta_0 \neq 0$, and in $D(\zeta_0, \delta - |\zeta_0|) \setminus \{1\}$ with $\Delta = 1/\sqrt{2} > 1/2$ if $\zeta_0 \neq 1$. Hence by Proposition 3 in any case we obtain the following estimate

$$s(f_\tau(\zeta), f_\tau(\zeta_0)) \leq 64 \left(\log \frac{2(\delta - |\zeta_0|)}{|\zeta - \zeta_0|} \right)^{-\frac{1}{\alpha_0}} \quad (30)$$

for all $z \in \overline{\mathbb{D}(\tau_*)}$ and $\zeta \in \mathbb{D}(\zeta_0, (\delta - |\zeta_0|)/2)$, where $\alpha_0 = 2(Q_0 + 6\|Q\|_*)$, i.e., the family of the mappings $f_z(\zeta)$, $z \in \overline{\mathbb{D}(\tau_*)}$, is equicontinuous at each point $\zeta_0 \in \mathbb{D}(\delta)$. In view of arbitrariness of $\delta > 1$, the latter is true for all $\zeta_0 \in \mathbb{C}$ at all. $\square$

By the Ascoli theorem, see, e.g., 20.4 in [37], and Lemma 1 we obtain the next conclusion.

Corollary 3. *Let a mapping $f : D \rightarrow \mathbb{C}$ satisfy the hypotheses of Lemma 1. Then mappings $f_z(\zeta) := f(\zeta z)/f(z)$ form a normal family, i.e., every sequence $f_{z_n}(\zeta)$, $n = 1, 2, \dots$ with $|z_n| \rightarrow +0$ as $n \rightarrow \infty$ contains a subsequence $f_{z_{n_k}}(\zeta)$, $k = 1, 2, \dots$ that converges with respect to the spherical metric locally uniformly in $\mathbb{C}$ as $k \rightarrow \infty$ to a continuous mapping $f_0 : \mathbb{C} \rightarrow \overline{\mathbb{C}}$ with $f_0(0) = 0$ and $f_0(1) = 1$.*

Furthermore, we are dealing with the so-called approximate solutions of the Beltrami equations. Namely, given a domain D in $\mathbb{C}$, a homeomorphic ACL (absolutely continuous on lines) solution f of the Beltrami Equation (2) in D is called its **approximate solution** if f is a locally uniform limit in D as $n \rightarrow \infty$ of (quasiconformal) homeomorphic ACL solutions f_n of the Beltrami equations with the complex coefficients

$$\mu_n(z) := \begin{cases} \mu(z), & \text{if } \mu(z) \leq 1 - 1/n, \\ 0, & \text{otherwise.} \end{cases}$$

Let us give a proof of the following important fact.

Proposition 4. *Every approximate solution f of Beltrami Equation (2) with $K_\mu \in L^1_{\text{loc}}$ is its regular homeomorphic solution and, moreover, $f^{-1} \in W^{1,2}_{\text{loc}}$.*

Proof. Indeed, let f be an approximate solution of the Beltrami Equation (2) and let f_n be its approximating sequence. Then first of all $f \in W^{1,1}_{\text{loc}}$ by Theorem 2.1 in [16].

Let us now prove that $f^{-1} \in W_{\text{loc}}^{1,2}$. Indeed, by Lemma 2.16 in [16] $g_n := f_n^{-1} \rightarrow g := f^{-1}$ uniformly in $\overline{\mathbb{C}}$ as $n \rightarrow \infty$. Note that f_n and $g_n \in W_{\text{loc}}^{1,2}$, $n = 1, 2, \dots$, because they are quasiconformal mappings. Consequently, these homeomorphisms are locally absolutely continuous, see, e.g., Theorem III.6.1 in [13]. Observe also that $\mu_n := (g_n)_{\bar{w}} / (g_n)_w = -\mu_n \circ g_n$, see, e.g., Section I.C in [11]. Thus, replacing variables in the integrals, see, e.g., Lemma III.2.1 in [13], we obtain that

$$\int_B |\partial g_n(w)|^2 dm(w) = \int_{g_n(B)} \frac{dm(z)}{1 - |\mu_n(z)|^2} \leq \int_{B^*} K_\mu(z) dm(z) < \infty$$

for sufficiently large n , where B and B^* are arbitrary domains in $\mathbb{C}$ with compact closures in $f(D)$ and D , respectively, such that $g(\overline{B}) \subset B^*$. It follows from the latter that the sequence g_n is bounded in the space $W^{1,2}(B)$ in each such domain B . Hence $f^{-1} \in W_{\text{loc}}^{1,2}$, see, e.g., Lemma III.3.5 in [39].

Finally, the latter brings in turn that g has (N) -property, see Theorem III.6.1 in [13]. Hence $J_f(z) \neq 0$ a.e., see Theorem 1 in [40]. Thus, f is really a regular solution of the Beltrami Equation (2). $\square$

Note also that Lemma 2.12 in the monograph in [16] is extended from quasiconformal mappings to approximate solutions of the Beltrami Equation (2) immediately by the definition of such solutions.

Proposition 5. Let $f : \mathbb{D} \rightarrow \overline{\mathbb{C}} \setminus \{a, b\}$, $a, b \in \overline{\mathbb{C}}$, $s(a, b) \geq \delta > 0$, be an approximate solution of the Beltrami Equation (2). Suppose that $s(f(z_1), f(0)) \geq \delta$ for $z_1 \in \mathbb{D} \setminus \{0\}$. Then, for every point z with $|z| < \min(1 - |z_1|, |z_1|/2)$,

$$s(f(z), f(0)) \geq \psi(|z|) \quad (31)$$

where ψ is a nonnegative strictly increasing function depending only on δ and $\|K_\mu\|_1$.

In turn, Propositions 4 and 5 make it possible to prove the following useful statement.

Proposition 6. Let D be a domain in $\mathbb{C}$ and $f_n : D \rightarrow \overline{\mathbb{C}}$ be a sequence of approximate solutions of the Beltrami equations $\bar{\partial} f_n = \mu_n \partial f_n$. Suppose that $f_n \rightarrow f$ as $n \rightarrow \infty$ locally uniformly in D with respect to the spherical metric and the norms $\|K_{\mu_n}\|_1$, $n = 1, 2, \dots$ are locally equipotentially bounded. Then either f is constant or it is a homeomorphism.

Proof. Consider the case when f is not constant in D . Let us first show that then no point in D has a neighborhood of the constancy for f . Indeed, assume that there is at least one point $z_0 \in D$ such that $f(z) \equiv c$ for some $c \in \overline{\mathbb{C}}$ in a neighborhood of z_0 . Note that the set Ω_0 of such points z_0 is open. The set $E_c = \{z \in D : s(f(z), c) > 0\}$ is also open by continuity of f and not empty if f is not constant. Thus, there is a point $z_0 \in \partial\Omega_0 \cap D$ because D is connected. By continuity of f we have that $f(z_0) = c$. However, by the construction there is a point $z_1 \in E_c = D \setminus \overline{\Omega_0}$ such that $|z_0 - z_1| < r_0 = \text{dist}(z_0, \partial D)$ and, thus, by the lower estimate of the distance $s(f(z_0), f(z_1))$ in Proposition 5 we obtain a contradiction for $z \in \Omega_0$. Then again by Proposition 5 we obtain that the mapping f is discrete. Hence f is a homeomorphism by Proposition 2.6 in the monograph [16]. $\square$

Corollary 4. Let a mapping $f : D \rightarrow \mathbb{C}$ satisfy the hypotheses of Lemma 1 and f be an approximate solution of the Beltrami Equation (2) and, moreover,

$$\limsup_{r \rightarrow 0} \frac{1}{r^2} \int_{|z| < r} |K_\mu(z)| dm(z) < \infty. \quad (32)$$

Then each limit mapping f_0 of a sequence $f_{z_n}(\zeta) := f(\zeta z_n)/f(z_n)$, $z_n \in \mathbb{C} \setminus \{0\}$, $n = 1, 2, \dots$ with $z_n \rightarrow 0$ as $n \rightarrow \infty$ is a homeomorphism of $\mathbb{C}$ into $\mathbb{C}$.

Proof. Indeed, f_n are approximate solutions of the Beltrami equations $\bar{\partial} f_n = \mu_n \partial f_n$ with $|\mu_n(\zeta)| = |\mu(z_n \zeta)|$, see, e.g., Section I.C in [11], and by simple calculations, for all $R > 0$,

$$\overline{\lim}_{n \rightarrow \infty} \int_{|\zeta| < R} |K_{\mu_n}(\zeta)| dm(\zeta) = R^2 \cdot \overline{\lim}_{r \rightarrow 0} \frac{1}{(R|z_n|)^2} \int_{|z| < R|z_n|} |K_{\mu}(z)| dm(z) < \infty \quad (33)$$

and, thus, by Proposition 6 the mapping f_0 is a homeomorphism in $\mathbb{C}$.

Now, let us assume that $f_0(\zeta_0) = \infty$ for some $\zeta_0 \in \mathbb{C}$. Since f_n are homeomorphisms, there exist points $\zeta_n \in \mathbb{S}(\zeta_0, 1)$ such that $s(\zeta_n, \infty) > s(\zeta_0, \infty)$ for all large enough n . We may assume in addition, with no loss of generality, that $\zeta_n \rightarrow \zeta_* \in \mathbb{S}(\zeta_0, 1)$ because the circle $\mathbb{S}(\zeta_0, 1)$ is a compact set. Then $f_0(\zeta_*) = \lim_{n \rightarrow \infty} f_n(\zeta_n) = \infty$ because by Lemma 1 the sequence f_n is equicontinuous and, for such sequences, the pointwise convergence $f_n \rightarrow f_0$ is equivalent to its continuous convergence, see, e.g., Theorem 7.1 in [17]. However, the latter leads to a contradiction because $\zeta_* \neq \zeta_0$ and by the first part f_0 is a homeomorphism. The obtained contradiction disproves the above assumption and, thus, really $f_0(\zeta) \neq \infty$ for all $\zeta \in \mathbb{C}$, i.e., f_0 is a homeomorphism of $\mathbb{C}$ into $\mathbb{C}$. $\square$

3. The Main Theorems and Consequences on Asymptotic Homogeneity at the Origin

The following theorem shows, in particular, that the Belinskij conformality still remains to be equivalent to the property of asymptotic homogeneity for regular homeomorphic solutions of the degenerate Beltrami Equations (2) if its dilatation K_{μ} has a majorant Q in BMO.

Theorem 1. Let D be a domain in $\mathbb{C}$, $0 \in D$, and let $f : D \rightarrow \mathbb{C}$ be a regular homeomorphic solution of the Beltrami equation with $f(0) = 0$ and K_{μ} have a majorant $Q \in \text{BMO}(D)$. Then the following assertions are equivalent:

- (1) f is conformal by Belinskij at the origin,
- (2) for all $\zeta \in \mathbb{C}$,

$$\lim_{\substack{\tau \rightarrow 0, \\ \tau > 0}} \frac{f(\tau \zeta)}{f(\tau)} = \zeta, \quad (34)$$

- (3) for all $\delta > 0$, along $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $z' \in \mathbb{C}$ with $|z'| \leq \delta|z|$,

$$\lim_{z \rightarrow 0} \left\{ \frac{f(z')}{f(z)} - \frac{z'}{z} \right\} = 0, \quad (35)$$

- (4) for all $\zeta \in \mathbb{C}$,

$$\lim_{\substack{z \rightarrow 0, \\ z \in \mathbb{C}^*}} \frac{f(z\zeta)}{f(z)} = \zeta, \quad (36)$$

- (5) the limit in (36) is uniform in the parameter ζ on each compact subset of $\mathbb{C}$.

Proof. Let us follow the scheme (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (1) and set

$$f_0(\zeta) = \zeta, \quad f_z(\zeta) = f(\zeta z)/f(z) \quad \forall z \in D \setminus \{0\}, \quad \zeta \in \mathbb{C}: z\zeta \in D.$$

(1) $\Rightarrow$ (2). Immediately the definition of the conformality by Belinskij yields the convergence $f_{\tau}(\zeta) \rightarrow f_0(\zeta)$ as $\tau \rightarrow 0$ along $\tau > 0$ for every fixed $\zeta \in \mathbb{C}$, i.e., just (34).

(2) $\Rightarrow$ (3). In view of Lemma 1, the pointwise convergence in (34) for each $\zeta \in \mathbb{C}$ implies the uniform convergence there on compact sets in $\mathbb{C}$, see, e.g., Theorem 7.1 in [17]. To obtain on this basis the implication (2) $\Rightarrow$ (3), let us note the identities

$$f_z(\zeta) = \frac{f_{|z|}(\zeta z/|z|)}{f_{|z|}(z/|z|)} = \frac{f(z')}{f(z)} \quad \forall \zeta = \frac{z'}{z} \in \mathbb{C}, z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}.$$

Hence to prove (35) it is sufficient to show that $f_z(\zeta) - f_0(\zeta) \rightarrow 0$ as $z \rightarrow 0$, $z \in \mathbb{C}^*$ uniformly with respect to the parameter ζ in the closed disks $\mathbb{D}_\delta := \{\zeta \in \mathbb{C} : |\zeta| \leq \delta\}$, $\delta > 0$.

Indeed, let us assume the inverse. Then there is a number $\varepsilon > 0$ and consequences $\zeta_n \in \mathbb{D}_\delta$, $z_n \rightarrow 0$, $z_n \in \mathbb{C}^*$, such that $|g_n(\zeta_n) - \zeta_n| \geq \varepsilon$, where $g_n(\zeta) = f_{z_n}(\zeta)$, $\zeta \in \mathbb{C}$. Since the closed disk $\mathbb{D}_\delta$ and the unit circle $\partial\mathbb{D}_1$ are compact sets, then with no loss of generality we may in addition to assume that $\zeta_n \rightarrow \zeta_0 \in \mathbb{D}_\delta$ and $\eta_n = z_n/|z_n| \rightarrow \eta_0 \in \partial\mathbb{D}_1$ as $n \rightarrow \infty$.

Let us denote by $\varphi_n(\zeta)$ the mappings $f_{|z_n|}(\zeta)$, $\zeta \in \mathbb{C}$, $n = 1, 2, \dots$. Then $\varphi_n(\zeta) \rightarrow \zeta$ as $n \rightarrow \infty$ uniformly on $\mathbb{D}_\delta \cup \partial\mathbb{D}_1$ and $g_n(\zeta) = \varphi_n(\eta_n \zeta)/\varphi_n(\eta_n)$. Consequently, $g_n(\zeta) \rightarrow \zeta$ as $n \rightarrow \infty$ uniformly on $\mathbb{D}_\delta$. Hence $g_n(\zeta_n) \rightarrow \zeta_0$ as $n \rightarrow \infty$ because the uniform convergence of continuous mappings on compact sets implies the so-called continuous convergence, see, e.g., Remark 7.1 in [17]. Thus, the obtained contradiction disproves the above assumption.

(3) $\Rightarrow$ (4). Setting in (35) $z' = z\zeta$ and $\delta = |\zeta|$, we immediately obtain (36).

(4) $\Rightarrow$ (5). The limit relation (36) means in the other words that $f_z(\zeta) \rightarrow f_0(\zeta)$ as $z \rightarrow 0$ along $z \in \mathbb{C}^*$ pointwise in $\mathbb{C}$. In view of Lemma 1, the latter implies the locally uniform convergence $f_z \rightarrow f_0$ as $z \rightarrow 0$ in $\mathbb{C}$, see again Theorem 7.1 in [17].

(5) $\Rightarrow$ (1). From (36) for $z = \rho > 0$, $\zeta = e^{i\theta}$, $\theta \in \mathbb{R}$, and $w = \zeta z = \rho e^{i\theta}$ we obtain that $f(w) = f(\rho)(\zeta + \alpha(\rho))$, where $\alpha(\rho) \rightarrow 0$ as $\rho \rightarrow 0$. Consequently,

$$f(w) = A(\rho)(w + o(\rho)),$$

where $A(\rho) = f(\rho)/\rho$ and $o(\rho)/\rho \rightarrow 0$ as $\rho \rightarrow 0$. Moreover, by (36) with $z = \rho > 0$ and $\zeta = t > 0$ we have that A satisfies the condition

$$\lim_{\rho \rightarrow 0} \frac{A(t\rho)}{A(\rho)} = 1 \quad \forall t > 0,$$

i.e., f is conformal by Belinskij at the origin. $\square$

The following result is fundamental for further study of asymptotic homogeneity because it facilitates considerably the verification of (36) and at the same time reveals the nature of the notion. Let Z be an arbitrary set in the complex plane $\mathbb{C}$, $0 \notin Z$, with the origin as its accumulation point. Further, we use the following characteristic of its sparseness:

$$\mathbb{S}_Z(\rho) := \frac{\inf_{z \in Z, |z| \geq \rho} |z|}{\sup_{z \in Z, |z| \leq \rho} |z|} \quad \forall \rho > 0. \quad (37)$$

Theorem 2. Let f satisfy the hypotheses of Theorem 1. Suppose that

$$\limsup_{\rho \rightarrow 0} \mathbb{S}_Z(\rho) < \infty \quad (38)$$

and

$$\lim_{\substack{z \rightarrow 0, \\ z \in Z}} \frac{f(z\zeta)}{f(z)} = \zeta \quad \forall \zeta \in \mathbb{C}. \quad (39)$$

Then f is asymptotically homogeneous at the origin.

Remark 4. For Theorem 2 to be true, the condition (38) on the extent of possible sparseness of Z is not only sufficient but also necessary as Proposition 2.1 in [28] in the case $Q \in L^\infty \subset BMO$ shows.

In particular, any continuous path to the origin or a discrete set, say $1/n, n = 1, 2, \dots$, can be taken as the set Z in Theorem 2. For instance, the conclusion of Theorem 2 is also true if Z has at least one point on each circle $|z| = \rho$ for all small enough $\rho > 0$.

Proof. Indeed, by (39) we have that, for functions $f_z(\zeta) := f(z\zeta)/f(z)$, pointwise

$$\lim_{\substack{z \rightarrow 0, \\ z \in Z}} f_z(\zeta) = \zeta \quad \forall \zeta \in \mathbb{C} \quad (40)$$

and, by Theorem 7.1 in [17] and Lemma 1, the limit in (40) is locally uniform in $\zeta \in \mathbb{C}$.

Let us assume that (36) does not hold for f , in other words, there exist $\zeta \in \mathbb{C}, \varepsilon > 0$ and a sequence $z_n \in \mathbb{C}^*, n = 1, 2, \dots$ such that $z_n \rightarrow 0$ as $n \rightarrow \infty$ and

$$|f_{z_n}(\zeta) - \zeta| > \varepsilon. \quad (41)$$

On the other hand, by (38) there is a sequence $z_n^* \in Z$ such that

$$0 < \delta \leq |\tau_n| \leq 1 < \infty$$

for all large enough $n = 1, 2, \dots$, where

$$\tau_n = \frac{z_n}{z_n^*}, \quad \delta = 1/2 \limsup_{\rho \rightarrow 0} \mathbb{S}_Z(\rho).$$

With no loss of generality, we may assume in addition that $\tau_n \rightarrow \tau_0$ with $\delta \leq |\tau_0| \leq 1$ as $n \rightarrow \infty$ because the closed ring $R := \{z \in \mathbb{C} : \delta \leq |z| \leq 1\}$ is a compact set. Note also that

$$f_{z_n}(\zeta) = \frac{f_{z_n^*}(\zeta \tau_n)}{f_{z_n^*}(\tau_n)}.$$

Thus, $f_{z_n^*}(\zeta \tau_n) \sim \zeta \tau_0$ and $f_{z_n^*}(\tau_n) \sim \tau_0$ as $n \rightarrow \infty$ because the uniform convergence in (40) with respect to ζ over any compact set implies the so-called continuous convergence, see, e.g., Remark 7.1 in [17]. Consequently, $f_{z_n}(\zeta) \sim \zeta$ as $n \rightarrow \infty$ because $\tau_0 \neq 0$. However, the latter contradicts (41). The obtained contradiction disproves the above assumption and the conclusion of the theorem is true. $\square$

Now, recall that the abstract spaces $\mathfrak{F}$ in which convergence is a primary notion were first considered by Frechet in his thesis in 1906. Later on, Uryson introduced the third axiom in these spaces: if a compact sequence $f_n \in \mathfrak{F}$ has its unique accumulation point $f \in \mathfrak{F}$, then $\lim_{n \rightarrow \infty} f_n = f$, see, e.g., [41], Chapter 2, 20, I-II. Recall that $f_n \in \mathfrak{F}, n = 1, 2, \dots$ is called a **compact sequence** if each its subsequence contains a converging subsequence and, moreover, $f \in \mathfrak{F}$ is said to be an **accumulation point** of the sequence $f_n \in \mathfrak{F}$ if f is a limit of some its subsequence. It is customary to call such spaces $\mathfrak{L}^*$ -spaces.

Remark 5. In particular, any convergence generated by a metric satisfies Uryson's axiom, see, e.g., [41], Chapter 2, 21, II. However, the well-known convergence almost everywhere of measurable functions yields a counter-example to Uryson's axiom: any sequence converging in measure is compact with respect to convergence almost everywhere, but not every such sequence converges almost everywhere. Later on, we apply the convergence generated by the uniform convergence of continuous functions, generated as known by the uniform norm.

To prove the corresponding sufficient criteria for the asymptotic homogeneity at the origin for solutions of degenerate Beltrami equations, we need also the following general lemma.

Lemma 2. Let D be a bounded domain in $\mathbb{C}$ and $f_n : D \rightarrow \mathbb{C}, n = 1, 2, \dots$ be a sequence of $W^{1,1}$ solutions of the Beltrami equations $\bar{\partial} f_n = \mu_n \partial f_n$. Suppose that $f_n \rightarrow f$ as $n \rightarrow \infty$ in L^1 and the

norms $\|\bar{\partial}f_n\|_1$ and $\|\partial f_n\|_1$ are equipotentially bounded. Then $f \in W^{1,1}$ and ∂f_n and $\bar{\partial}f_n$ converge weakly in L^1 to ∂f and $\bar{\partial}f$, respectively. Moreover, if $\mu_n \rightarrow \mu$ a.e. or in measure as $n \rightarrow \infty$, then $\bar{\partial}f = \mu \partial f$ a.e.

Proof. The first part of conclusions follow from Lemma III.3.5 in [39]. Let us prove the latter of these conclusions. Namely, assuming that $\mu_n(z) \rightarrow \mu(z)$ a.e. as $n \rightarrow \infty$ and, setting

$$\zeta(z) = \bar{\partial}f(z) - \mu(z) \cdot \partial f(z),$$

let us show that $\zeta(z) = 0$. Indeed, since $\bar{\partial}f_n(z) - \mu_n(z)\partial f_n(z) = 0$, by the triangle inequality

$$\int_D |\zeta(z)| dm(z) \leq I_1(n) + I_2(n) + I_3(n),$$

where

$$I_1(n) := \int_D |\bar{\partial}f(z) - \bar{\partial}f_n(z)| dm(z),$$

$$I_2(n) := \int_D |\mu(z)| \cdot |\partial f(z) - \partial f_n(z)| dm(z),$$

$$I_3(n) := \int_D |\mu(z) - \mu_n(z)| \cdot |\partial f_n(z)| dm(z).$$

By the first part of conclusions, with no loss of generality, assume that $|\bar{\partial}f(z) - \bar{\partial}f_n(z)| \rightarrow 0$ and $|\partial f(z) - \partial f_n(z)| \rightarrow 0$ as $n \rightarrow \infty$ weakly in L^1 , see Corollary IV.8.10 in [42]. Thus, $I_1(n) \rightarrow 0$ and $I_2(n) \rightarrow 0$ as $n \rightarrow \infty$ because the dual space of L^1 is naturally isometric to L^∞ , see, e.g., Theorem IV.8.5 in [42].

Moreover, by Corollary IV.8.11 in [42], for each $\varepsilon > 0$, there is $\delta > 0$ such that over every measurable set E in D with $|E| < \delta$

$$\int_E |\partial f_n(z)| dm(z) < \varepsilon, \quad n = 1, 2, \dots \quad (42)$$

Further, by the Egoroff theorem, see, e.g., III.6.12 in [42], $\mu_n(z) \rightarrow \mu(z)$ as $n \rightarrow \infty$ uniformly on some set S in D with $|E| < \delta$ where $E = D \setminus S$. Hence $|\mu_n(z) - \mu(z)| < \varepsilon$ on S and

$$I_3(n) \leq \varepsilon \int_S |\partial f_n(z)| dm(z) + 2 \int_E |\partial f_n(z)| dm(z) \leq \varepsilon (\|\partial f_n(z)\|_1 + 2)$$

for large enough n , i.e., $I_3(n) \rightarrow 0$ because $\varepsilon > 0$ is arbitrary. Thus, really $\zeta = 0$ a.e. $\square$

Theorem 3. Let D be a domain in $\mathbb{C}$, $0 \in D$, $f : D \rightarrow \mathbb{C}$, $f(0) = 0$, be an approximate solution of the Beltrami Equation (2) and K_μ have a majorant $Q \in \text{BMO}(D)$. Suppose that

$$\limsup_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z| < r} K_\mu(z) dm(z) < \infty, \quad (43)$$

and

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z| < r} |\mu(z)| dm(z) = 0. \quad (44)$$

Then f is asymptotically homogeneous at the origin.

Proof. By Theorem 2 with $Z := \{2^{-n}\}_{n=N}^{\infty}$, where $2^{-N} < \text{dist}(0, \partial D)$, it is sufficient to show that

$$\lim_{n \rightarrow \infty} f_{2^{-n}}(\zeta) = \zeta \quad \forall \zeta \in \mathbb{C}, \quad f_{2^{-n}}(\zeta) := \frac{f(2^{-n}\zeta)}{f(2^{-n})}.$$

By Corollary 3 the sequence $f_{2^{-n}}(\zeta)$ is compact with respect to locally uniform convergence in $\mathbb{C}$ and by Remark 5 it remains to prove that each its converging subsequence $f_k^* = f_{n_k}$ with $n_k \rightarrow \infty$ as $k \rightarrow \infty$ has the identity mapping of the complex plane $\mathbb{C}$ as its limit f_0 .

Indeed, the mappings f_k^* are approximate solutions of Beltrami equations $\bar{\partial} f_k^* = \mu_k^* \cdot \partial f_k^*$ with $|\mu_k^*(\zeta)| = |\mu(2^{-n_k}\zeta)|$, see, e.g., calculations of Section I.C in [11]. Since such solutions are regular by Proposition 4, we have by the calculations that

$$|\bar{\partial} f_k^*| \leq |\partial f_k^*| \leq |\partial f_k^*| + |\bar{\partial} f_k^*| \leq K_{\mu_k^*}^{1/2} J_{f_k^*}^{1/2} \quad \text{a.e.}, \quad k = 1, 2, \dots$$

where

$$K_{\mu_k^*}(\zeta) = K_{\mu}(2^{-n_k}\zeta), \quad J_{f_k^*}(\zeta) = |\partial f_k^*(\zeta)|^2 - |\bar{\partial} f_k^*(\zeta)|^2 = J_{f_{n_k}}(\zeta) = J_f(2^{-n_k}\zeta)/|f(2^{-n_k})|^2.$$

Consequently, by the Hölder inequality for integrals, see, e.g., Theorem 189 in [43], and Lemma III.3.3 in [13], we obtain that

$$\|\partial f_k^*\|_1(\mathbb{D}_l) \leq \|K_{\mu_k^*}\|_1^{1/2}(\mathbb{D}_l) \cdot |f_k^*(\mathbb{D}_l)|^{1/2} \quad \forall l = 1, 2, \dots, \quad \mathbb{D}_l := \mathbb{D}(2^l).$$

Now, by the condition (43) and simple calculations, for each fixed $l = 1, 2, \dots$,

$$\overline{\lim}_{k \rightarrow \infty} \|K_{\mu_k^*}\|_1(\mathbb{D}_l) = 2^{2l} \cdot \overline{\lim}_{k \rightarrow \infty} \frac{1}{(2^l 2^{-n_k})^2} \int_{|z| < 2^l 2^{-n_k}} |K_{\mu}(z)| dm(z) < \infty.$$

Next, choosing ζ_k in $\mathbb{S}_l := \{\zeta \in \mathbb{C} : |\zeta| = 2^l\}$ with $|f(2^{-n_k}\zeta_k)| = \max_{\zeta \in \mathbb{S}_l} |f(2^{-n_k}\zeta)|$, we see that

$$|f_k^*(\mathbb{D}_l)| = |f_{n_k}(\mathbb{D}_l)| = \frac{|f(\mathbb{D}(2^{l-n_k}))|}{|f(2^{-n_k}\zeta_k)|^2} \cdot \frac{|f(2^{-n_k}\zeta_k)|^2}{|f(2^{-n_k})|^2} \leq \pi |f_{2^{-n_k}}(\zeta_k)|^2.$$

With no loss of generality, we may assume that $\zeta_k \rightarrow \zeta_0 \in \mathbb{S}_l$ as $k \rightarrow \infty$ because the circle $\mathbb{S}_l$ is a compact set. Then $f_{2^{-n_k}}(\zeta_k) \rightarrow f_0(\zeta_0)$ because the uniform convergence implies the so-called continuous convergence, see, e.g., Remark 7.1 in [17]. However, $f_0(\zeta_0) \neq \infty$, see Corollary 4.

Thus, the norms of ∂f_k^* and $\bar{\partial} f_k^*$ are locally equipotentially bounded in L^1 . Then f_0 is $W_{\text{loc}}^{1,1}$ solution of the Beltrami equation with $\mu \equiv 0$ in $\mathbb{C}$ by Lemma 2 in view of (44). Moreover, f_0 is a homeomorphism of $\mathbb{C}$ into $\mathbb{C}$ by Corollary 4. Hence f_0 is a conformal mapping of $\mathbb{C}$ into $\mathbb{C}$, see, e.g., Corollary II.B.1 in [11]. Hence $f_0(\zeta)$ is a linear function $a + b\zeta$, see, e.g., Theorem 2.31.1 in [44]. In addition, by the construction $f_0(0) = 0$ and $f_0(1) = 1$. Thus, $f_0(\zeta) \equiv \zeta$ in the whole complex plane $\mathbb{C}$ and the proof is thereby complete. $\square$

Remark 6. Note that, in particular, both conditions (43) and (44) follow from the only one stronger condition

$$\lim_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z| < r} K_{\mu}(z) dm(z) = 1 \quad (45)$$

because

$$|\mu(z)| \leq \frac{|\mu(z)|}{1 - |\mu(z)|} = \frac{K_{\mu}(z) - 1}{2}. \quad (46)$$

Combining Theorems 1 and 3, see also Proposition 4, we obtain the following conclusions.

Corollary 5. Under hypotheses of Theorem 3, f is conformal by Lavrent'iev at the origin, i.e., f preserves infinitesimal circles centered at the origin:

$$\lim_{r \rightarrow 0} \frac{\max_{|z|=r} |f(z)|}{\min_{|z|=r} |f(z)|} = 1, \quad (47)$$

asymptotically preserves angles, i.e.,

$$\lim_{z \rightarrow 0} \arg \left[\frac{f(z\zeta)}{f(z)} \right] = \arg \zeta \quad \forall \zeta \in \mathbb{C}, |\zeta| = 1, \quad (48)$$

and asymptotically preserves the moduli of infinitesimal rings, i.e.,

$$\lim_{z \rightarrow 0} \frac{|f(z\zeta)|}{|f(z)|} = |\zeta| \quad \forall \zeta \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}. \quad (49)$$

Corollary 6. Under hypotheses of Theorem 3, for all $\delta > 0$, along $z \in \mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $z' \in \mathbb{C}$ with $|z'| \leq \delta|z|$,

$$\lim_{z \rightarrow 0} \left\{ \frac{|f(z')|}{|f(z)|} - \frac{|z'|}{|z|} \right\} = 0. \quad (50)$$

Moreover, by the theorem of Stolz (1885) and Cesaro (1888), see, e.g., Problem 70 in [45], we derive from Corollary 6 the next assertion on logarithms.

Corollary 7. Under hypotheses of Theorem 3,

$$\lim_{\substack{z \rightarrow 0 \\ z \in \mathbb{C}^*}} \frac{\ln |f(z)|}{\ln |z|} = 1. \quad (51)$$

Proof. For brevity, let us introduce designations $t_n = -\ln |z_n|$, $\tau_n = -\ln |f(z_n)|$ and assume that (51) does not hold, i.e., there exist $\varepsilon > 0$ and a sequence $z_n \rightarrow 0$ such that

$$\left| \frac{\tau_n}{t_n} - 1 \right| \geq \varepsilon \quad \forall n = 1, 2, \dots \quad (52)$$

Passing, if necessary, to a subsequence, we can consider that $t_n - t_{n-1} \geq 1$ for all $n = 1, 2, \dots$. Then, we can achieve that $t_n - t_{n-1} < 2$, by inserting, if necessary, the mean arithmetic values between neighboring terms of the subsequence t_n , $n = 1, 2, \dots$. In this case, inequality (52) holds for the infinite number of terms of the subsequence.

Thus, the sequence $\rho_n = |z_n| = e^{-t_n}$ satisfies the inequalities $e^{-2} < \rho_n / \rho_{n-1} \leq e^{-1}$. Relations (50) implies that $\exp(\tau_{n-1} - \tau_n) = \exp(t_{n-1} - t_n) + \alpha_n$, where $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$, or, in the other form, $\exp(\tau_{n-1} - \tau_n) = (1 + \beta_n) \exp(t_{n-1} - t_n)$ with $\beta_n \rightarrow 0$ as $n \rightarrow \infty$. The latter gives that $(\tau_n - \tau_{n-1}) = (t_n - t_{n-1}) + \gamma_n$ with $\gamma_n \rightarrow 0$ as $n \rightarrow \infty$ and, since $t_n - t_{n-1} \geq 1$, we have that $(\tau_n - \tau_{n-1}) / (t_n - t_{n-1}) = 1 + \delta_n$, where $\delta_n \rightarrow 0$ as $n \rightarrow \infty$. By the Stolz theorem, then we conclude that $\tau_n / t_n \rightarrow 1$ in contradiction with (52). This contradiction disproves the above assumption, i.e., (51) is true. $\square$

Theorem 4. Let D be a domain in $\mathbb{C}$ and let $f : D \rightarrow \mathbb{C}$ be an approximate solution of the Beltrami Equation (2), K_μ have a majorant $Q \in \text{BMO}(D)$ and at a point $z_0 \in D$

$$\limsup_{r \rightarrow 0} \frac{1}{\pi r^2} \int_{|z-z_0| < r} K_\mu(z) \, dm(z) < \infty. \quad (53)$$

Suppose that $\mu(z)$ is approximately continuous at z_0 . Then the mapping f is differentiable by Belinskij at this point with $\mu_0 = \mu(z_0)$.

Proof. First of all, $|\mu(z_0)| < 1$ because by the hypotheses $K_\mu \in L^1_{\text{loc}}$ and $\mu(z)$ is approximately continuous at z_0 . Note also that f is differentiable by Belinskij with $\mu_0 = \mu(z_0)$ at z_0 if and only if $g := h \circ \varphi^{-1}$ is conformal by Belinskij at zero, where $h(z) = f(z_0 + z) - f(z_0)$ and $\varphi(z) = z + \mu_0 \bar{z}$. It is evident that $\mu_h(z) = \mu(z + z_0)$ and $K_{\mu_h} = K_\mu(z + z_0)$ and by elementary calculations, see, e.g., Section I.C(6) in [11], $\mu_g \circ \varphi = (\mu_h - \mu_0)/(1 - \overline{\mu_0}\mu_h)$ and $K_{\mu_g} \leq K_0 \cdot K_{\mu_h} \circ \varphi^{-1} \leq K_0 Q_0$, where $K_0 = (1 + |\mu_0|)/(1 - |\mu_0|)$ and $Q_0(w) = Q(z_0 + \varphi^{-1}(w))$ belongs to BMO in $D_0 := \varphi(D)$ because φ and φ^{-1} are K_0 -quasiconformal mappings, see the paper [46] and the monograph [3]. Thus, Theorem 4 follows from Theorem 3. $\square$

4. On Homeomorphic Solutions in Extended Complex Plane

Here we start from establishing a series of criteria for existence of approximate solutions $f : \mathbb{C} \rightarrow \mathbb{C}$ to the degenerate Beltrami equations in the whole complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

It is easy to give examples of locally quasiconformal mappings of $\mathbb{C}$ onto the unit disk $\mathbb{D}$, consequently, there exist locally uniform elliptic Beltrami equations with no such solutions. Hence, compared with our previous articles, the main goal here is to find the corresponding additional conditions on dilatation quotients of the Beltrami equations at infinity.

Lemma 3. Let a function $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\text{loc}}(\mathbb{C})$. Suppose that, for every $z_0 \in \overline{\mathbb{C}}$, there exist $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0, \varepsilon} : (0, \infty) \rightarrow (0, \infty)$ such that

$$I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0) \quad (54)$$

and

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \mathbb{C} \quad (55)$$

and, moreover,

$$\int_{\varepsilon < |\zeta| < \varepsilon_\infty} K_\mu^T(\zeta, \infty) \cdot \psi_{\infty, \varepsilon}^2(|\zeta|) \frac{dm(\zeta)}{|\zeta|^4} = o(I_\infty^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0, \quad (56)$$

where $K_\mu^T(\zeta, \infty) := K_\mu^T(1/\zeta, 0)$.

Then the Beltrami Equation (2) has an approximate homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Remark 7. After the replacements of variables $\zeta \mapsto z := 1/\zeta$, $\varepsilon \mapsto R := 1/\varepsilon$, $\varepsilon_\infty \mapsto R_0 := 1/\varepsilon_\infty$ and functions $\psi_{\infty, \varepsilon}(t) \mapsto \psi_R(t) := \psi_{\infty, 1/R}(1/t)$, the condition (56) can be rewritten in the more convenient form:

$$\int_{R_0 < |z| < R} K_\mu^T(z, 0) \psi_R^2(|z|) \frac{dm(z)}{|z|^4} = o(I^2(R)) \quad \text{as } R \rightarrow \infty, \quad (57)$$

with the family of measurable functions $\psi_R : (0, \infty) \rightarrow (0, \infty)$ such that

$$I(R) := \int_{R_0}^R \psi_R(t) \frac{dt}{t^2} < \infty \quad \forall R \in (R_0, \infty). \quad (58)$$

Before arriving at the proof of Lemma 3, let us recall that a **condenser** in $\mathbb{C}$ is a domain $\mathcal{R}$ in $\mathbb{C}$ whose complement in $\overline{\mathbb{C}}$ is the union of two distinguished disjoint compact sets C_1 and C_2 . For convenience, it is written $\mathcal{R} = \mathcal{R}(C_1, C_2)$. A **ring** in $\mathbb{C}$ is a condenser $\mathcal{R} = \mathcal{R}(C_1, C_2)$ with connected C_1 and C_2 that are called the **complementary components** of $\mathcal{R}$. It is known that the (conformal) capacity of a ring $\mathcal{R} = \mathcal{R}(C_1, C_2)$ in $\mathbb{C}$ is equal to the (conformal) modulus of all paths in $\mathcal{R}$ connecting C_1 and C_2 , see, e.g., Theorem A.8 in [17].

Proof. By the first item of the proof of Lemma 3 in [21] the Beltrami Equation (2) has under the conditions (55) an approximate homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$ and $f(1) = 1$. Moreover, by Lemma 3 in [21] we may also assume that f is a ring Q -homeomorphism with $Q(z) = K_\mu^T(z, 0)$ at the origin, i.e., for every ring $A = A(r_1, r_2) := \{z \in \mathbb{C} : r_1 < |z| < r_2\}$, we have the estimate of the capacity $C_f(r_1, r_2)$ of its image under the mapping f :

$$C_f(r_1, r_2) \leq \int_{A(r_1, r_2)} K_\mu^T(z, 0) dm(z) \quad \forall r_1, r_2 : 0 < r_1 < r_2 < \infty.$$

Let us consider the mapping $F(z) := 1/f(1/z)$ in $\mathbb{C}_* := \overline{\mathbb{C}} \setminus \{0\}$. Note that $F(\infty) = \infty$ because $f(0) = 0$. Since the capacity is invariant under conformal mappings, we have by the change of variables $z \mapsto \zeta := 1/z$ as well as $r_1 \mapsto \varepsilon_2 := 1/r_1$ and $r_2 \mapsto \varepsilon_1 := 1/r_2$ that

$$C_F(\varepsilon_1, \varepsilon_2) \leq \int_{A(\varepsilon_1, \varepsilon_2)} K_\mu^T(1/\zeta, 0) \frac{dm(\zeta)}{|\zeta|^4} \quad \forall \varepsilon_1, \varepsilon_2 : 0 < \varepsilon_1 < \varepsilon_2 < \infty,$$

i.e., F is a ring $\tilde{Q}$ -homeomorphism at the origin with $\tilde{Q}(\zeta) := K_\mu^T(1/\zeta, 0)/|\zeta|^4$. Thus, in view of the condition (56), we obtain by Lemma 6.5 in [16] that F has a continuous extension to the origin. Let us assume that $c := \lim_{\zeta \rightarrow 0} F(\zeta) \neq 0$.

However, $\overline{\mathbb{C}}$ is homeomorphic to the sphere $\mathbb{S}^2$ by stereographic projection and hence by the Brouwer theorem in $\mathbb{S}^2$ on the invariance of domain the set $C_* := F(\mathbb{C}_*)$ is open in $\overline{\mathbb{C}}$, see, e.g., Theorem 4.8.16 in [47]. Consequently, $c \notin C_*$ because F is a homeomorphism. Then the extended mapping $\tilde{F}$ is a homeomorphism of $\overline{\mathbb{C}}$ into $\mathbb{C}_*$ because $f \neq \infty$ in $\mathbb{C}$. Thus, again by the Brouwer theorem, the set $C := \tilde{F}(\overline{\mathbb{C}})$ is open in $\overline{\mathbb{C}}$ and $0 \in \overline{\mathbb{C}} \setminus C \neq \emptyset$. On the other hand, the set C is compact as a continuous image of the compact space $\overline{\mathbb{C}}$. Hence the set $\overline{\mathbb{C}} \setminus C \neq \emptyset$ is also open in $\overline{\mathbb{C}}$. The latter contradicts the connectivity of $\overline{\mathbb{C}}$, see, e.g., Proposition I.1.1 in [48].

The obtained contradiction disproves the assumption that $c \neq 0$. Thus, we have proved that f is extended to a homeomorphism of $\overline{\mathbb{C}}$ onto itself with $f(\infty) = \infty$. $\square$

Choosing $\psi_{z_0, \varepsilon}(t) \equiv 1/(t \log(1/t))$ in Lemma 3, we obtain by Proposition 2 the following.

Theorem 5. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L_{\text{loc}}^1(\mathbb{C})$ and

$$\int_{R_0 < |z| < R} K_\mu(z) \psi^2(|z|) \frac{dm(z)}{|z|^4} = o(I^2(R)) \quad \text{as } R \rightarrow \infty \quad (59)$$

for some $R_0 > 0$ and a measurable function $\psi : (0, \infty) \rightarrow (0, \infty)$ such that

$$I(R) := \int_{R_0}^R \psi(t) \frac{dt}{t^2} < \infty \quad \forall R \in (R_0, \infty). \quad (60)$$

Suppose also that $K_\mu^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in \mathbb{C}$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \rightarrow [0, \infty]$ in the class $\text{FMO}(z_0)$.

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

In particular, by Proposition 1 the conclusion of Theorem 5 holds if every point $z_0 \in \mathbb{C}$ is the Lebesgue point of the function Q_{z_0} .

By Corollary 1 we obtain the next nice consequence of Theorem 5, too.

Corollary 8. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\text{loc}}(\mathbb{C})$, (59) and

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\mathbb{D}(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \mathbb{C}. \quad (61)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

By (5), we also obtain the following consequences of Theorem 5.

Corollary 9. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., (59) and K_μ have a dominant $Q : \mathbb{C} \rightarrow [1, \infty)$ in the class BMO_{loc} . Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Remark 8. In particular, the conclusion of Corollary 7 holds if $Q \in W^{1,2}_{\text{loc}}$ because $W^{1,2}_{\text{loc}} \subset \text{VMO}_{\text{loc}}$, see, e.g., [10].

Corollary 10. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., (59) and $K_\mu(z) \leq Q(z)$ a.e. in $\mathbb{C}$ with a function Q in the class $\text{FMO}(\mathbb{C})$. Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Similarly, choosing $\psi_{z_0, \varepsilon}(t) \equiv 1/t$ in Lemma 3, we come to the next statement.

Theorem 6. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\text{loc}}(\mathbb{C})$, (59) and

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \mathbb{C} \quad (62)$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Remark 9. Choosing $\psi_{z_0, \varepsilon}(t) \equiv 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ in Lemma 2, we are able to replace (62) by

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right) \quad (63)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

Now, choosing in Lemma 3 the functional parameter $\psi_{z_0, \varepsilon}(t) \equiv \psi_{z_0}(t) := 1/[tk_\mu^T(z_0, t)]$, where $k_\mu^T(z_0, r)$ is the integral mean value of $K_\mu^T(z, z_0)$ over the circle $S(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}$, we obtain one more important conclusion.

Theorem 7. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\text{loc}}(\mathbb{C})$, (59) and

$$\int_0^{\varepsilon_0} \frac{dr}{rk_\mu^T(z_0, r)} = \infty \quad \forall z_0 \in \mathbb{C} \quad (64)$$

for some $\varepsilon_0 = \varepsilon(z_0) > 0$. Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Corollary 11. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\text{loc}}(\mathbb{C})$, (59) and

$$k_\mu^T(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \mathbb{C}. \quad (65)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Remark 10. In particular, the conclusion of Corollary 10 holds if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \overline{D}. \quad (66)$$

Moreover, the condition (65) can be replaced by the whole series of more weak conditions

$$k_\mu^T(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \overline{D}. \quad (67)$$

For further consequences, the following statement is useful, see e.g., Theorem 3.2 in [22].

Proposition 7. Let $Q : \mathbb{D} \rightarrow [0, \infty]$ be a measurable function such that

$$\int_{\mathbb{D}} \Phi(Q(z)) \, dm(z) < \infty \quad (68)$$

where $\Phi : [0, \infty] \rightarrow [0, \infty]$ is a non-decreasing convex function such that

$$\int_{\delta}^{\infty} \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty \quad (69)$$

for some $\delta > \Phi(+0)$. Then

$$\int_0^1 \frac{dr}{rq(r)} = \infty \quad (70)$$

where $q(r)$ is the average of the function $Q(z)$ over the circle $|z| = r$.

Here we use the following notions of the inverse function for monotone functions. Namely, for every non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$ the inverse function $\Phi^{-1} : [0, \infty] \rightarrow [0, \infty]$ can be well-defined by setting

$$\Phi^{-1}(\tau) := \inf_{\Phi(t) \geq \tau} t. \quad (71)$$

Here $\inf$ is equal to ∞ if the set of $t \in [0, \infty]$ such that $\Phi(t) \geq \tau$ is empty. Note that the function Φ^{-1} is non-decreasing, too. It is evident immediately by the definition that $\Phi^{-1}(\Phi(t)) \leq t$ for all $t \in [0, \infty]$ with the equality except intervals of constancy of the function $\Phi(t)$.

Let us recall the connection of condition (69) with other integral conditions, see, e.g., Theorem 2.5 in [22].

Remark 11. Let $\Phi : [0, \infty] \rightarrow [0, \infty]$ be a non-decreasing function and set

$$H(t) = \log \Phi(t). \quad (72)$$

Then the equality

$$\int_{\Delta}^{\infty} H'(t) \frac{dt}{t} = \infty, \quad (73)$$

implies the equality

$$\int_{\Delta}^{\infty} \frac{dH(t)}{t} = \infty, \quad (74)$$

and (74) is equivalent to

$$\int_{\Delta}^{\infty} H(t) \frac{dt}{t^2} = \infty \quad (75)$$

for some $\Delta > 0$, and (75) is equivalent to each of the equalities

$$\int_0^{\delta_*} H\left(\frac{1}{t}\right) dt = \infty \quad (76)$$

for some $\delta_* > 0$,

$$\int_{\Delta_*}^{\infty} \frac{d\eta}{H^{-1}(\eta)} = \infty \quad (77)$$

for some $\Delta_* > H(+0)$ and to (69) for some $\delta > \Phi(+0)$.

Moreover, (73) is equivalent to (74) and hence to (75)–(77) as well as to (69) are equivalent to each other if Φ is in addition absolutely continuous. In particular, all the given conditions are equivalent if Φ is convex and non-decreasing.

Note that the integral in (74) is understood as the Lebesgue–Stieltjes integral and the integrals in (73) and (75)–(77) as the ordinary Lebesgue integrals. It is necessary to give one more explanation. From the right hand sides in the conditions (73)–(77) we have in mind $+\infty$. If $\Phi(t) = 0$ for $t \in [0, t_*]$, then $H(t) = -\infty$ for $t \in [0, t_*]$ and we complete the definition $H'(t) = 0$ for $t \in [0, t_*]$. Note, the conditions (74) and (75) exclude that t_* belongs to the interval of integrability because in the contrary case the left hand sides in (74) and (75) are either equal to $-\infty$ or indeterminate. Hence we may assume in (73)–(76) that $\delta > t_0$, correspondingly, $\Delta < 1/t_0$ where $t_0 := \sup_{\Phi(t)=0} t$, and set $t_0 = 0$ if $\Phi(0) > 0$.

The most interesting of the above conditions is (75) that can be rewritten in the form:

$$\int_{\Delta}^{\infty} \log \Phi(t) \frac{dt}{t^2} = +\infty \quad \text{for some } \Delta > 0. \quad (78)$$

Combining Theorems 7, Proposition 7 and Remark 11, we obtain the following result.

Theorem 8. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_{\mu} \in L^1_{\text{loc}}(\mathbb{C})$, (59) and

$$\int_{U_{z_0}} \Phi_{z_0} \left(K_{\mu}^T(z, z_0) \right) dm(z) < \infty \quad \forall z_0 \in \mathbb{C} \quad (79)$$

for a neighborhood U_{z_0} of z_0 and a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ with

$$\int_{\Delta(z_0)}^{\infty} \log \Phi_{z_0}(t) \frac{dt}{t^2} = +\infty \quad (80)$$

for some $\Delta(z_0) > 0$. Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Corollary 12. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., $K_\mu \in L^1_{\text{loc}}(\mathbb{C})$, (59) and

$$\int_{U_{z_0}} e^{\alpha(z_0)K_\mu^T(z, z_0)} dm(z) < \infty \quad \forall z_0 \in \mathbb{C} \quad (81)$$

for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 . Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Since $K_\mu^T(z, z_0) \leq K_\mu(z)$ for z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 8.

Corollary 13. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., (59) and

$$\int_C \Phi(K_\mu(z)) dm(z) < \infty \quad (82)$$

over each compact C in $\mathbb{C}$ for a convex non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$ with

$$\int_\delta^\infty \log \Phi(t) \frac{dt}{t^2} = +\infty \quad (83)$$

for some $\delta > 0$. Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

Corollary 14. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., (59) and, for some $\alpha > 0$, over each compact C in $\mathbb{C}$,

$$\int_C e^{\alpha K_\mu(z)} dm(z) < \infty. \quad (84)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$.

5. On Existence of Solutions with Asymptotics at Infinity

In the extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, we will use the so-called **spherical area** whose element can be given through the element $dm(z)$ of the Lebesgue measure (usual area)

$$dS(z) := \frac{4 dm(z)}{(1 + |z|^2)^2} = \frac{4 dx dy}{(1 + |z|^2)^2}, \quad z = x + iy. \quad (85)$$

Let us start from the following general lemma on the existence of regular homeomorphic solutions for the Beltrami equations in $\mathbb{C}$ with asymptotic homogeneity at infinity.

Lemma 4. Let a function $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a connected open (punctured at ∞) neighborhood U of infinity,

$$\int_{|z|>R} |\mu(z)| dS(z) = o\left(\frac{1}{R^2}\right) \quad (86)$$

and, moreover,

$$\int_{|z|>R} K_\mu(z) dS(z) = O\left(\frac{1}{R^2}\right). \quad (87)$$

Suppose also that, for every $z_0 \in \mathbb{C} \setminus U$, there exist $\varepsilon_0 = \varepsilon(z_0) > 0$ and a family of measurable functions $\psi_{z_0, \varepsilon} : (0, \infty) \rightarrow (0, \infty)$ such that

$$I_{z_0}(\varepsilon) := \int_{\varepsilon}^{\varepsilon_0} \psi_{z_0, \varepsilon}(t) dt < \infty \quad \forall \varepsilon \in (0, \varepsilon_0) \quad (88)$$

and

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_{\mu}^T(z, z_0) \cdot \psi_{z_0, \varepsilon}^2(|z - z_0|) dm(z) = o(I_{z_0}^2(\varepsilon)) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \mathbb{C}. \quad (89)$$

Then the Beltrami Equation (2) has an approximate homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity, $f(\zeta z) \sim \zeta f(z)$ as $z \rightarrow \infty$ for all $\zeta \in \mathbb{C}$, i.e.,

$$\lim_{\substack{z \rightarrow \infty \\ z \in \mathbb{C}}} \frac{f(z\zeta)}{f(z)} = \zeta \quad \forall \zeta \in \mathbb{C} \quad (90)$$

and the limit (90) is locally uniform with respect to the parameter ζ in $\mathbb{C}$.

Remark 12. (86) and (87) can be replaced by only one (stronger) condition

$$\lim_{r \rightarrow \infty} \frac{R^2}{\pi} \int_{|z| > R} K_{\mu}(z) dS(z) = 1. \quad (91)$$

Note also that, arguing similarly to the proofs of Theorem 1 and Corollary 7, we see that the locally uniform property of the asymptotic homogeneity of f at infinity (90) implies its **conformality by Belinskij at infinity**, i.e.,

$$f(z) = A(\rho) \cdot [z + o(\rho)] \quad \text{as } z \rightarrow \infty, \quad (92)$$

where $A(\rho)$ depends only on $\rho = |z|$, $o(\rho)/\rho \rightarrow 0$ as $\rho \rightarrow \infty$ and, moreover,

$$\lim_{\rho \rightarrow \infty} \frac{A(t\rho)}{A(\rho)} = 1 \quad \forall t > 0, \quad (93)$$

its **conformality by Lavrent'iev at infinity**, i.e.,

$$\lim_{R \rightarrow \infty} \frac{\max_{|z|=R} |f(z)|}{\min_{|z|=R} |f(z)|} = 1, \quad (94)$$

the logarithmic property at infinity

$$\lim_{z \rightarrow \infty} \frac{\ln |f(z)|}{\ln |z|} = 1, \quad (95)$$

asymptotic preserving angles at infinity, i.e.,

$$\lim_{z \rightarrow \infty} \arg \left[\frac{f(z\zeta)}{f(z)} \right] = \arg \zeta \quad \forall \zeta \in \mathbb{C}^* \quad (96)$$

and asymptotic preserving moduli of rings at infinity, i.e.,

$$\lim_{z \rightarrow \infty} \frac{|f(z\zeta)|}{|f(z)|} = |\zeta| \quad \forall \zeta \in \mathbb{C}^*. \quad (97)$$

The latter two geometric properties characterize asymptotic homogeneity at infinity and demonstrate that it is very close to the usual conformality at infinity.

Proof. The extended complex plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ is a metric space with a measure with respect to the spherical (chordal) metric s , see (26), and the spherical area S , see (85). This space is regular by Ahlfors that is evident from the geometric interpretation of $\overline{\mathbb{C}}$ as the so-called stereographic projection of a sphere in $\mathbb{R}^3$, see details, e.g., in Section 13 and Supplement B in the monograph [17].

Let us recall only here that, if the function Q belongs to the class BMO in U with respect to the Euclidean distance and the usual area in $\mathbb{C}$, then Q is in BMO with respect to the spherical distance and the spherical area not only in U but also in $U \cup \{\infty\}$, see Lemma B.3 and Proposition B.1 in [17]. Moreover, we have an analog of Proposition 2 in terms of spherical metric and area, see Lemma 13.2 and Remark 13.3 in [17], that in turn can be rewritten in terms of the Euclidean distance and area at infinity in the following form:

$$\int_{R_0 < |z| < R} \frac{Q(z)}{\log^2 |z|} \frac{dm(z)}{|z|^2} = O(\log \log R) \quad \text{as } R \rightarrow \infty \quad (98)$$

for large enough R_0 with $\{z \in \mathbb{C} : |z| > R_0\} \subseteq U$. Consequently, we have the condition (57) with $\psi_R(t) \equiv \psi(t) := t^{-1} \log t$ and by Lemma 3, see also Remark 7, the Beltrami Equation (2) has an approximate solution f in $\mathbb{C}$ with the normalization $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$. Recall that f is its regular homeomorphic solution by Proposition 4.

Setting $f^*(\zeta) := 1/f(1/\zeta)$ in $\overline{\mathbb{C}}$, we see that $f^*(0) = 0$, $f^*(1) = 1$, $f^*(\infty) = \infty$ and that f^* is an approximate solution in $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ of the Beltrami equation with

$$\mu^*(\zeta) := \mu\left(\frac{1}{\zeta}\right) \cdot \frac{\zeta^2}{\bar{\zeta}^2}, \quad K_{\mu^*}(\zeta) = K_{\mu}\left(\frac{1}{\zeta}\right), \quad (99)$$

because

$$f_{\bar{\zeta}}^*(\zeta) = \frac{1}{\bar{\zeta}^2} \cdot \frac{f_{\bar{z}}(\frac{1}{\zeta})}{f^2(\frac{1}{\zeta})}, \quad f_{\zeta}^*(\zeta) = \frac{1}{\zeta^2} \cdot \frac{f_z(\frac{1}{\zeta})}{f^2(\frac{1}{\zeta})} \quad \text{a.e. in } \mathbb{C}, \quad (100)$$

see, e.g., Section I.C and the proof of Theorem 3 of Section V.B in [11].

Note that f^* belongs to the class $W_{\text{loc}}^{1,1}(\mathbb{C}^*)$ and, consequently, f^* is ACL (absolutely continuous on lines) in $\mathbb{C}$, see, e.g., Theorems 1 and 2 of Section 1.1.3 and Theorem of Section 1.1.7 in [49]. However, it is not clear directly from (100) whether the derivatives $f_{\bar{\zeta}}^*$ and f_{ζ}^* are integrable in a neighborhood of the origin, because of the first factors in (100). Thus, to prove that f^* is a regular homeomorphic solution of the Beltrami equation in $\mathbb{C}$, it remains to establish the latter fact in another way.

Namely, after the replacements of variables $z \mapsto \zeta := 1/z$ and $R \mapsto r := 1/R$, in view of (99), the condition (87) can be rewritten in the form

$$\limsup_{r \rightarrow 0} \frac{1}{r^2} \int_{|\zeta| < r} K_{\mu^*}(\zeta) dm(\zeta) < \infty, \quad (101)$$

and the latter implies, in particular, that, for some $r_0 \in (0, 1]$,

$$\frac{1}{r_0^2} \int_{|\zeta| < r_0} K_{\mu^*}(\zeta) dm(\zeta) < \infty, \quad (102)$$

i.e., the dilatation quotient K_{μ^*} of the given Beltrami equation is integrable in the disk $\mathbb{D}(r_0)$.

Now, since f^* is a regular homeomorphism in $\mathbb{C}^*$, in particular, its Jacobian $J(\xi) = |f_{\xi}^*|^2 - |f_{\bar{\xi}}^*|^2 \neq 0$ a.e. and hence $|f_{\xi}^*| - |f_{\bar{\xi}}^*| \neq 0$ a.e. as well as $f_{\xi}^* \neq 0$ a.e., the following identities are also correct a.e.

$$|f_{\xi}^*(\xi)| + |f_{\bar{\xi}}^*(\xi)| = \left[\frac{|f_{\xi}^*(\xi)| + |f_{\bar{\xi}}^*(\xi)|}{|f_{\xi}^*(\xi)| - |f_{\bar{\xi}}^*(\xi)|} \right]^{\frac{1}{2}} \cdot J^{\frac{1}{2}}(\xi) = K_{\mu^*}^{\frac{1}{2}}(\xi) \cdot J^{\frac{1}{2}}(\xi). \quad (103)$$

Hence by the Hölder inequality for integrals, see, e.g., Theorem 189 in [43], we have that

$$\int_{|\xi| < r_0} (|f_{\xi}^*(\xi)| + |f_{\bar{\xi}}^*(\xi)|) dm(\xi) \leq \left(\int_{|\xi| < r_0} K_{\mu^*}(\xi) dm(\xi) \right)^{\frac{1}{2}} \cdot \left(\int_{|\xi| < r_0} J(\xi) dm(\xi) \right)^{\frac{1}{2}} \quad (104)$$

and, since the latter factor in (104) is estimated by the area of $f^*(\mathbb{D}(r_0))$, see, e.g., the Lebesgue theorem in Section III.2.3 of the monograph [13], we conclude that both partial derivatives f_{ξ}^* and $f_{\bar{\xi}}^*$ are integrable in the disk $\mathbb{D}(r_0)$.

Next, note that the function $Q_*(\xi) := Q(1/\xi)$ is of the class BMO in a neighborhood of the origin with respect to the spherical area as well as with respect to the usual area, see, e.g., again Lemma B.3 in [17], because also the spherical area is invariant under rotations of the sphere $\mathbb{S}^2$ in the stereographic projection. Moreover, by (86) and (99), we obtain that

$$\lim_{r \rightarrow 0} \frac{1}{r^2} \int_{|\xi| < r} |\mu^*(\xi)| dm(\xi) = 0. \quad (105)$$

Thus, by Theorems 3 we conclude that f^* is asymptotically homogeneous at the origin, i.e.,

$$\lim_{\substack{\xi \rightarrow 0, \\ \xi \in \mathbb{C}^*}} \frac{f^*(\xi \zeta)}{f^*(\xi)} = \zeta \quad \forall \zeta \in \mathbb{C} \quad (106)$$

and, furthermore, the limit in (106) is locally uniform in the parameter ζ .

After the inverse replacements of the variables $\xi \mapsto w := 1/\xi$ and the functions $f^*(\xi) \mapsto f(w) = 1/f^*(1/w)$ the relation (106) can be rewritten in the form

$$\lim_{\substack{w \rightarrow \infty, \\ w \in \mathbb{C}}} \frac{f(w)}{f(w\zeta^{-1})} = \zeta \quad \forall \zeta \in \mathbb{C}. \quad (107)$$

Finally, after one more change of variables $w \mapsto z := w\zeta^{-1}$, the latter is transformed into (90), where the limit is locally uniform with respect to the parameter $\zeta \in \mathbb{C}$. $\square$

Choosing $\psi_{z_0, \varepsilon}(t) \equiv 1/(t \log(1/t))$ in Lemma 4, we obtain by Proposition 2 the following.

Theorem 9. Let a function $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., K_{μ} have a majorant Q of the class BMO in a neighborhood U of ∞ and satisfy (91). Suppose also that $K_{\mu}^T(z, z_0) \leq Q_{z_0}(z)$ a.e. in U_{z_0} for every point $z_0 \in \mathbb{C} \setminus U$, a neighborhood U_{z_0} of z_0 and a function $Q_{z_0} : U_{z_0} \rightarrow [0, \infty]$ in the class FMO(z_0). Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

As a particular case of Theorem 9, we obtain the following central theorem in terms of BMO.

Theorem 10. Let a function $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be measurable with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class $BMO(\mathbb{C})$ and satisfy (91). Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Note also that, in particular, by Proposition 1 the conclusion of Theorem 9 holds if every point $z_0 \in \mathbb{C} \setminus U$ is the Lebesgue point of the function Q_{z_0} .

By Corollary 1 we obtain the next fine consequence of Theorem 9, too.

Corollary 15. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and

$$\overline{\lim}_{\varepsilon \rightarrow 0} \int_{\mathbb{D}(z_0, \varepsilon)} K_\mu^T(z, z_0) dm(z) < \infty \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (108)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

By (5), we also obtain the following consequences of Theorem 9.

Corollary 16. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in $\mathbb{C}$ and satisfy (91). Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Corollary 17. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and $K_\mu(z) \leq Q_*(z)$ a.e. in $\mathbb{C} \setminus U$ with a function $Q : \mathbb{C} \rightarrow \mathbb{R}^+$ of the class $FMO(\mathbb{C} \setminus U)$. Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Remark 13. In particular, the conclusion of Corollary 17 holds if $Q_* \in W_{loc}^{1,2}$ because $W_{loc}^{1,2} \subset VMO_{loc}$, see, e.g., [10].

Similarly, choosing $\psi_{z_0, \varepsilon}(t) \equiv 1/t$ in Lemma 4, we come also to the next statement.

Theorem 11. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and, for some $\varepsilon_0 = \varepsilon(z_0) > 0$,

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} K_\mu^T(z, z_0) \frac{dm(z)}{|z - z_0|^2} = o\left(\left[\log \frac{1}{\varepsilon}\right]^2\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (109)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Remark 14. Choosing $\psi_{z_0, \varepsilon}(t) \equiv 1/(t \log 1/t)$ instead of $\psi(t) = 1/t$ in Lemma 4, we are able to replace (109) by

$$\int_{\varepsilon < |z - z_0| < \varepsilon_0} \frac{K_\mu^T(z, z_0) dm(z)}{\left(|z - z_0| \log \frac{1}{|z - z_0|}\right)^2} = o\left(\left[\log \log \frac{1}{\varepsilon}\right]^2\right) \quad (110)$$

In general, we are able to give here the whole scale of the corresponding conditions in log using functions $\psi(t)$ of the form $1/(t \log 1/t \cdot \log \log 1/t \cdot \dots \cdot \log \dots \log 1/t)$.

Now, choosing in Lemma 4 the functional parameter $\psi_{z_0, \varepsilon}(t) \equiv \psi_{z_0}(t) := 1/[tk_\mu^T(z_0, t)]$, where $k_\mu^T(z_0, r)$ is the average of $K_\mu^T(z, z_0)$ over the circle $S(z_0, r) := \{z \in \mathbb{C} : |z - z_0| = r\}$, we obtain one more important conclusion.

Theorem 12. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and, for some $\varepsilon_0 = \varepsilon(z_0) > 0$,

$$\int_0^{\varepsilon_0} \frac{dr}{rk_\mu^T(z_0, r)} = \infty \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (111)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Corollary 18. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and

$$k_\mu^T(z_0, \varepsilon) = O\left(\log \frac{1}{\varepsilon}\right) \quad \text{as } \varepsilon \rightarrow 0 \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (112)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Remark 15. In particular, the conclusion of Corollary 18 holds if

$$K_\mu^T(z, z_0) = O\left(\log \frac{1}{|z - z_0|}\right) \quad \text{as } z \rightarrow z_0 \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (113)$$

Moreover, the condition (112) can be replaced by the whole series of more weak conditions

$$k_\mu^T(z_0, \varepsilon) = O\left(\left[\log \frac{1}{\varepsilon} \cdot \log \log \frac{1}{\varepsilon} \cdot \dots \cdot \log \dots \log \frac{1}{\varepsilon}\right]\right) \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (114)$$

Combining Theorems 12, Proposition 4 and Remark 1, we obtain the following result.

Theorem 13. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and

$$\int_{U_{z_0}} \Phi_{z_0}(K_\mu^T(z, z_0)) dm(z) < \infty \quad \forall z_0 \in \mathbb{C} \setminus U \quad (115)$$

for a neighborhood U_{z_0} of z_0 and a convex non-decreasing function $\Phi_{z_0} : [0, \infty] \rightarrow [0, \infty]$ with

$$\int_{\Delta(z_0)}^{\infty} \log \Phi_{z_0}(t) \frac{dt}{t^2} = +\infty \quad \text{for some } \Delta(z_0) > 0. \quad (116)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Corollary 19. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and, for some $\alpha(z_0) > 0$ and a neighborhood U_{z_0} of the point z_0 ,

$$\int_{U_{z_0}} e^{\alpha(z_0)K_\mu^T(z, z_0)} dm(z) < \infty \quad \forall z_0 \in \mathbb{C} \setminus U. \quad (117)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Since $K_\mu^T(z, z_0) \leq K_\mu(z)$ for z and $z_0 \in \mathbb{C}$, we also obtain the following consequences of Theorem 13.

Corollary 20. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and

$$\int_{\mathbb{C} \setminus U} \Phi(K_\mu(z)) \, dm(z) < \infty \quad (118)$$

for a convex non-decreasing function $\Phi : [0, \infty] \rightarrow [0, \infty]$ such that, for some $\delta > 0$,

$$\int_{\delta}^{\infty} \log \Phi(t) \frac{dt}{t^2} = +\infty. \quad (119)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Corollary 21. Let $\mu : \mathbb{C} \rightarrow \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e., K_μ have a majorant Q of the class BMO in a neighborhood U of ∞ , satisfy (91) and, for some $\alpha > 0$,

$$\int_{\mathbb{C} \setminus U} e^{\alpha K_\mu(z)} \, dm(z) < \infty. \quad (120)$$

Then the Beltrami Equation (2) has a regular homeomorphic solution f in $\mathbb{C}$ with $f(0) = 0$, $f(1) = 1$ and $f(\infty) = \infty$ that is asymptotically homogeneous at infinity.

Corollary 22. Recall that by Theorem 5.1 in [22] the condition (119) is not only sufficient but also necessary for the existence of regular homeomorphic solutions for all Beltrami Equation (2) with the integral constraints (118), see also Remark 11.

6. Conclusions

Thus, this paper contains a number of effective criteria for the existence of regular homeomorphic solutions for the Beltrami equations with asymptotic homogeneity at infinity where the BMO condition in a neighborhood of infinity plays a key role.

Finally, these results can be applied to the fluid mechanics in strongly anisotropic and inhomogeneous media because the Beltrami equation is a complex form of the main equation of hydromechanics, see, e.g., Theorem 16.1.6 in [15]; these results will be published elsewhere.

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References

1. John, F.; Nirenberg, L. On functions of bounded mean oscillation. *Comm. Pure Appl. Math.* **1961**, *14*, 415–426. [CrossRef]
2. Heinonen, J.; Kilpelainen, T.; Martio, O. *Nonlinear Potential Theory of Degenerate Elliptic Equations*; Oxford Mathematical Monographs, Clarendon Press: Oxford, UK, 1993.

3. Reimann, H.M.; Rychener, T. *Funktionen Beschränkter Mittlerer Oscillation*; Lecture Notes in Math.; Springer: Berlin/Heidelberg, Germany, 1975; Volume 487.
4. Sarason, D. Functions of vanishing mean oscillation. *Trans. Am. Math. Soc.* **1975**, *207*, 391–405. [\[CrossRef\]](#)
5. Chiarenza, F.; Frasca, M.; Longo, P. $W^{2,p}$ -solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients. *Trans. Am. Math. Soc.* **1993**, *336*, 841–853.
6. Iwaniec, T.; Sbordone, C. Riesz transforms and elliptic PDEs with VMO coefficients. *J. d'Anal. Math.* **1998**, *74*, 183–212. [\[CrossRef\]](#)
7. Palagachev, D.K. Quasilinear elliptic equations with VMO coefficients. *Trans. Am. Math. Soc.* **1995**, *347*, 2481–2493. [\[CrossRef\]](#)
8. Ragusa, M.A. Elliptic boundary value problem in vanishing mean oscillation hypothesis. *Comment. Math. Univ. Carol.* **1999**, *40*, 651–663.
9. Ragusa, M.A.; Tachikawa, A. Partial regularity of the minimizers of quadratic functionals with VMO coefficients. *J. Lond. Math. Soc. II. Ser.* **2005**, *72*, 609–620. [\[CrossRef\]](#)
10. Brezis, H.; Nirenberg, L. Degree theory and BMO. I. Compact manifolds without boundaries. *Selecta Math.* **1995**, *1*, 197–263. [\[CrossRef\]](#)
11. Ahlfors, L. *Lectures on Quasiconformal Mappings*; Van Nostrand: New York, NY, USA, 1966.
12. Bojarski, B. Generalized solutions of a system of differential equations of the first order of the elliptic type with discontinuous coefficients. *Mat. Sb. N. Ser.* **1958**, *43*, 451–503.
13. Lehto, O.; Virtanen, K.I. *Quasiconformal Mappings in the Plane*; Die Grundlehren der Mathematischen Wissenschaften 126; Springer: Berlin/Heidelberg, Germany, 1973.
14. Vekua, I.N. *Generalized Analytic Functions*; Pergamon Press: London, UK, 1962.
15. Astala, K.; Iwaniec, T.; Martin, G. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*; Princeton Mathematical Series 48; Princeton University Press: Princeton, NJ, USA, 2009.
16. Gutlyanskii, V.; Ryazanov, V.; Srebro, U.; Yakubov, E. *The Beltrami Equation: A Geometric Approach*; Developments in Mathematics 26; Springer: Berlin/Heidelberg, Germany, 2012.
17. Martio, O.; Ryazanov, V.; Srebro, U.; Yakubov, E. *Moduli in Modern Mapping Theory*; Springer Monographs in Mathematics; Springer: New York, NY, USA, 2009.
18. Andreian Cazacu, C. On the length-area dilatation. *Complex Var. Theory Appl. Int. J.* **2005**, *50*, 765–776. [\[CrossRef\]](#)
19. Gutlyanskii, V.; Martio, O.; Sugawa, T.; Vuorinen, M. On the degenerate Beltrami equation. *Trans. Am. Math. Soc.* **2005**, *357*, 875–900. [\[CrossRef\]](#)
20. Lehto, O. Homeomorphisms with a prescribed dilatation. *Lect. Notes Math.* **1968**, *118*, 58–73.
21. Ryazanov, V.; Srebro, U.; Yakubov, E. On the theory of the Beltrami equation. *Ukr. Math. J.* **2006**, *58*, 1786–1798. [\[CrossRef\]](#)
22. Ryazanov, V.; Srebro, U.; Yakubov, E. Integral conditions in the theory of the Beltrami equations. *Complex Var. Elliptic Equ.* **2012**, *57*, 1247–1270. [\[CrossRef\]](#)
23. Ryazanov, V.; Srebro, U.; Yakubov, E. The Beltrami equation and ring homeomorphisms. *Ukrainian Math. Bull.* **2007**, *4*, 79–115.
24. Gehring, F.W.; Lehto, O. On the total differentiability of functions of a complex variable. *Ann. Acad. Sci. Fenn. A I Math.* **1959**, *272*, 1–9.
25. Menchoff, D. Sur les différentielles totales des fonctions univalentes. *Math. Ann.* **1931**, *105*, 75–85. [\[CrossRef\]](#)
26. Belinskii, P.P. *General Properties of Quasiconformal Mappings*; Nauka: Novosibirsk, Russia, 1974. (In Russian)
27. Ryazanov, V.I. A criterion for differentiability in the sense of Belinskii and its consequences. *Ukr. Math. J.* **1992**, *44*, 254–258. [\[CrossRef\]](#)
28. Gutlyanskii, V.Ya.; Ryazanov, V.I. On the local behaviour of quasi-conformal mappings. *Izv. Math.* **1995**, *59*, 471–498. [\[CrossRef\]](#)
29. Brakalova, M. Sufficient and necessary conditions for conformality at a point. I. Geometric viewpoint. *Complex Var. Elliptic Equ.* **2010**, *55*, 137–155. [\[CrossRef\]](#)
30. Brakalova, M. Sufficient and necessary conditions for conformality. II. Analytic viewpoint. *Ann. Acad. Sci. Fenn. Math.* **2010**, *35*, 235–254. [\[CrossRef\]](#)
31. Brakalova, M.; Jenkins, J.A. On the local behavior of certain homeomorphisms. *Kodai Math. J.* **1994**, *17*, 201–213. [\[CrossRef\]](#)
32. Brakalova, M.; Jenkins, J.A. On the local behavior of certain homeomorphisms. II. *J. Math. Sci.* **1999**, *95*, 2178–2184; *Zap. Nauchn. Semin. POMI.* **1997**, *236*, 11–20. [\[CrossRef\]](#)
33. Gutlyanskii, V.; Martio, O. Conformality of a quasiconformal mapping at a point. *J. Anal. Math.* **2003**, *91*, 179–192. [\[CrossRef\]](#)
34. Lehto, O. On the differentiability of quasi-conformal mappings with prescribed complex dilatation. *Ann. Acad. Sci. Fenn. Ser. A I* **1960**, *275*, 28.
35. Teichmüller, O. Untersuchungen über konforme und quasikonforme Abbildung. *Dtsch. Math.* **1938**, *3*, 621–678.
36. Ignat'ev, A.A.; Ryazanov, V.I. Finite mean oscillation in the mapping theory. *Ukrainian Math. Bull.* **2005**, *2*, 403–424.
37. Väisälä, J. *Lectures on n-Dimensional Quasiconformal Mappings*; Lecture Notes in Mathematics 229; Springer: Berlin/Heidelberg, Germany, 1971.
38. Salimov, R. On regular homeomorphisms in the plane. *Ann. Acad. Sci. Fenn. Math.* **2010**, *35*, 285–289. [\[CrossRef\]](#)
39. Reshetnyak, Y.G. *Space Mappings with Bounded Distortion*; Translations of Mathematical Monographs 73; American Mathematical Society (AMS): Providence, RI, USA, 1989.
40. Ponomarev, S.P. The N^{-1} –property of mappings, and Lusin's (N) –condition. *Mat. Zametki.* **1995**, *58*, 411–418; *Math. Notes.* **1995**, *58*, 960–965.

41. Kuratowski, K. *Topology*; New Edition, Revised and Augmented. (English); Academic Press: London, UK; PWN-Polish Scientific Publishers: Warszawa, Poland, 1966; Volume I.
42. Dunford, N.; Schwartz Jacob, T. *Linear Operators. I. General Theory*; Pure and Applied Mathematics; Interscience Publishers: New York, NY, USA; London, UK, 1958; Volume 7.
43. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Cambridge University Press: Cambridge, UK, 1988.
44. Lawrentjew, M.A.; Schabat, B.W. *Methoden der Komplexen Funktionentheorie. (German) Mathematik für Naturwissenschaft und Technik 13*; VEB Deutscher Verlag der Wissenschaften: Berlin, Germany, 1967.
45. Polya, G.; Szego, G. *Aufgaben und Lehrsätze aus der Analysis. Bd. I: Reihen, Integralrechnung, Funktionentheorie*; (German)-Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen 19; Springer: Berlin/Heidelberg, Germany, 1925.
46. Reimann, H.M. Functions of bounded mean oscillation and quasiconformal mappings. *Comment. Math. Helv.* **1974**, *49*, 260–276. [[CrossRef](#)]
47. Spanier, E.H. *Algebraic Topology*; Springer: Berlin/Heidelberg, Germany, 1995.
48. Fischer, W.; Lieb, I. *A Course in Complex Analysis. From Basic Results to Advanced Topics*; Vieweg + Teubner: Wiesbaden, Germany, 2012.
49. Mazya, V.G. *Sobolev Spaces*; Springer: Berlin/Heidelberg, Germany, 1985.