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Strong Chromatic Index of Outerplanar Graphs

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Abstract: The strong chromatic index $\chi'_s(G)$ of a graph G is the minimum number of colors needed in a proper edge-coloring so that every color class induces a matching in G. It was proved In 2013, that every outerplanar graph G with $\Delta \geq 3$ has $\chi'_s(G) \leq 3\Delta - 3$. In this paper, we give a characterization for an outerplanar graph G to have $\chi'_s(G) = 3\Delta - 3$. We also show that if G is a bipartite outerplanar graph, then $\chi'_s(G) \leq 2\Delta$; and $\chi'_s(G) = 2\Delta$ if and only if G contains a particular subgraph.

Keywords: strong edge-coloring; strong chromatic index; outerplanar graph; bipartite graph

MSC: Graph Theory with Applications

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1. Introduction

Only simple graphs are considered in this paper. For a graph G, we use $V(G)$, $E(G)$, and $\Delta(G)$ to denote its vertex set, edge set and maximum degree, respectively. A vertex v is called a *k-vertex* (or *k^+-vertex*) if the degree $d_G(v)$ of v is k (or at least k). Let $N_G(v)$ denote the set of vertices adjacent to v in G. If no ambiguity arises in the context, $\Delta(G)$, $d_G(v)$, and $N_G(v)$ are simply written as Δ, $d(v)$, and $N(v)$, respectively. A subgraph of G is called a *clique* if any two of its vertices are adjacent in G. A subset $I \subset V(G)$ of a connected graph G is called a *clique-cut* if $G[I]$ is a clique and $G - I$ is disconnected.

A *proper edge-k-coloring* of a graph G is a mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ for any two adjacent edges e and e'. The *chromatic index* $\chi'(G)$ of G is the smallest k such that G has a proper edge k-coloring. An edge coloring of the graph G is called *strong* if every color class induces a matching in G. The *strong chromatic index* of G, denoted $\chi'_s(G)$, is the smallest k such that G has a strong edge-k-coloring.

The strong edge-coloring of graphs was introduced by Fouquet and Jolivet [1]. In 1985, Erdős and Nešetřil raised the following conjecture and showed that the upper bounds are tight:

Conjecture 1. *For a graph G,*

$$\chi'_s(G) \leq \begin{cases} 1.25\Delta^2, & \text{if } \Delta \text{ is even;} \\ 1.25\Delta^2 - 0.5\Delta + 0.25, & \text{if } \Delta \text{ is odd.} \end{cases}$$

Using probabilistic method, Molloy and Reed [2] showed that $\chi'_s(G) \leq 1.998\Delta^2$ when Δ is sufficiently large. This result was further improved in [3] to that $\chi'_s(G) \leq 1.93\Delta^2$ for any graph G. Using Four-Colour Theorem and Vizing Theorem, Faudree et al. [4] showed that every planar graph G has $\chi'_s(G) \le 4\Delta + 4$; and constructed a planar graph G such that $\chi'_s(G) = 4\Delta - 4$.

A planar graph is called *outerplanar* if it has a plane embedding such that all the vertices lie on the boundary of the unbounded face. It was shown in [5] that a graph G is outerplanar if and only if G is K_4-minor-free and $K_{2,3}$-minor-free. Hence outerplanar graphs are special K_4-minor-free graphs. Wang et al. [6] showed that every K_4-minor-free graph G with $\Delta \ge 3$ has $\chi'_s(G) \le 3\Delta - 2$ and the upper bound is tight. Hocquard et al. [7] proved that every outerplanar graph G with $\Delta \ge 3$ has $\chi'_s(G) \le 3\Delta - 3$ and the upper bound is tight.

In this paper we will give a characterization for an outerplanar graph G with $\Delta \ge 3$ to have $\chi'_s(G) = 3\Delta - 3$.

2. Sun-Graphs

Suppose that G is an outerplanar graph. We embed G in the plane so that all the vertices occur in the boundary of unbounded face. Let $F(G)$ denote the set of faces in G. The unbounded face, denoted by $f_0(G)$, of G is called *outer face*, and other faces *inner faces*. For a face $f \in F(G)$, the boundary of f is denoted by $b(f)$. A 3-face with x, y, z as boundary vertices is written as $[xyz]$. The edges lying in the outer face are called *outer edges* and other edges *inner edges*. An inner face f is called an *end-face* if $b(f)$ contains at most one inner edge. A *leaf* of G is a vertex of degree 1, and a *pendant edge* is an edge incident to a leaf. For a vertex $v \in V(G)$, let $L(v)$ denote the set of pendant edges at vertex v. For a cycle C, an edge $xy \in E(G) \setminus E(C)$ is called a *chord* of C if $x, y \in V(C)$.

Let F_1 denote a subgraph of G, which consists of a 3-cycle $C_3 = x_0x_1x_2x_0$ with $d_G(x_i) = \Delta \ge 3$ for $i = 0, 1, 2$.

Let F_2 denote a subgraph of G, which consists of a 4-cycle $C_4 = x_0x_1x_2x_3x_0$ with $d_G(x_0) = d_G(x_1) = \Delta \ge 3$.

Let F_3 denote a subgraph of G, which consists of a 7-cycle $C_7 = x_0x_1 \cdots x_6x_0$ with $d_G(x_i) = 3$ for $i = 0, 1, \ldots, 6$.

We assume that C_4 in F_2 and C_7 in F_3 have no chord.

The configurations F_1, F_2, F_3 are depicted in Figure 1. By the outerplanarity of G, for F_j, $j \in \{1, 2, 3\}$, some vertex $y_i \in N(x_i) \setminus \{x_{i-1}, x_{i+1}\}$ may identify with some vertex $y_{i+1} \in N(x_{i+1}) \setminus \{x_i, x_{i+2}\}$, but there is at most one such pair $\{y_i, y_{i+1}\}$ satisfying $y_i = y_{i+1}$, where indices i are taken as modulo n.

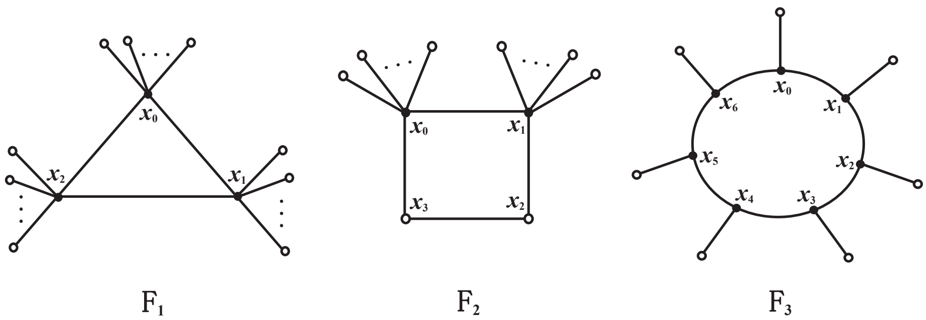

Figure 1. Configurations F_1, F_2, and F_3.

Lemma 1 ([7])**.** *If G is an outerplanar graph with $\Delta \ge 3$, then $\chi'_s(G) \le 3\Delta - 3$.*

Lemma 2. *Let F_1, F_2, F_3 are defined as above. Then*

(1) $\chi'_s(F_1) = 3\Delta - 3$.
(2) $\chi'_s(F_2) = 3\Delta - 3$.
(3) $\chi'_s(F_3) = 6$.

Proof. (1) Since $|E(F_1)| = 3\Delta - 3$ and it is easy to check that any two edges of F_1 have distance at most two, so it follows that $\chi'_s(F_1) = 3\Delta - 3$.

(2) Applying the similar analysis as in (1), we can derive that $\chi'_s(F_2) = 3\Delta - 3$.
(3) It is evident that $\chi'_s(F_3) \le 6$ by Lemma 1. Conversely, assume that F_3 admits a strong edge-5-coloring ϕ using the color set $C = \{1, 2, \ldots, 5\}$. Let E_i denote the set of edges colored with the color i under the coloring ϕ. Set $E^* = E(F_3) - E(C_7)$. First, it is easy to inspect that $|E_i| \le 3$ for each $i \in C$. Next, because $|E(F_3)| = 14$ and $|C| = 5$, we can assume that $|E_i| = 3$ for $i = 1, 2, 3, 4$ and $|E_5| = 2$. Since $|E^*| = 7$, some E_i for $i \in \{1, 2, 3, 4\}$, say $i = 1$, satisfies $|E_1 \cap E^*| \le 1$. It implies that $|E_1 \cap E(C_7)| \ge 2$. On the other hand, it is easy to inspect that $|E_1 \cap E(C_7)| \le 2$. So $|E_1 \cap E(C_7)| = 2$ and $|E_1 \cap E^*| = 1$, however such coloring is impossible, a contradiction. This shows that $\chi'_s(F_3) \ge 6$. Consequently, $\chi'_s(F_3) = 6$.
□

Let $C_n = x_0 x_1 \cdots x_{n-1} x_0$ be a cycle with $n \ge 3$. Let $k \ge 3$ be an integer. At each vertex x_i, we glue $k - 2$ leaves and write the resultant graph as S_n^k. Then S_n^k is an outerplanar graph with maximum degree k and order $n(k - 1)$. We call S_n^k a *sun-graph* with parameters n and k. If $k = 3$, then we use y_i to denote a leaf adjacent to x_i for $i = 0, 1, \ldots, n - 1$.

As an easy observation, we have the following:

Lemma 3. *Let C_n be a cycle with $n \ge 3$. Then*

$$\chi'_s(C_n) = \begin{cases} 5, & \text{if } n = 5; \\ 3, & \text{if } n \equiv 0 (\text{mod } 3); \\ 4, & \text{otherwise.} \end{cases}$$

Lemma 4. *Let S_n^3 be a sun-graph with $n \ge 3$. Then*

$$\chi'_s(S_n^3) = \begin{cases} 6, & \text{if } n = 3, 4, 7; \\ 5, & \text{otherwise.} \end{cases}$$

Proof. If $n = 3, 4, 7$, the conclusion follows immediately from Lemma 2. So suppose that $n \ne 3, 4, 7$. It holds trivially that $\chi'_s(S_n^3) \ge 5$ since S_n^3 contains two adjacent 3-vertices. To show that $\chi'_s(S_n^3) \le 5$, we make use of induction on n. It remains to construct a strong edge-5-coloring ϕ of S_n^3 using the color set $C = \{1, 2, \ldots, 5\}$.

- If $n = 5$, then we color the edges in $\{x_i y_i, x_{i+2} x_{i+3}\}$ with $i + 1$ for $i = 0, 1, 2, 3, 4$, where indices are taken as modulo 5.
- If $n \equiv 0$ (mod 6), then we alternatively color the edges in $E(C_n)$ with $1, 2, 3$, and color alternatively pendant edges with $4, 5$.
- If $n = 8$, then we color $\{x_1 y_1, x_3 x_4, x_6 x_7\}$ with 1, $\{x_3 y_3, x_0 x_1, x_5 x_6,\}$ with 2, $\{x_5 y_5, x_0 x_7,\}$ $\{x_2 x_3\}$ with 3, $\{x_7 y_7, x_1 x_2, x_4 x_5\}$ with 4, and $\{x_0 y_0, x_2 y_2, x_4 y_4, x_6 y_6\}$ with 5.
- If $n = 9$, then we color $\{x_1 y_1, x_3 y_3, x_8 y_8, x_5 x_6\}$ with 1, $\{x_0 y_0, x_5 y_5, x_7 y_7, x_2 x_3\}$ with 2, $\{x_2 y_2, x_4 y_4, x_6 y_6, x_0 x_8\}$ with 3, $\{x_0 x_1, x_3 x_4, x_6 x_7\}$ with 4, and $\{x_1 x_2, x_4 x_5, x_7 x_8\}$ with 5.

Now assume that $n \ge 10$ and $n \not\equiv 0$ (mod 6). Consider the graph S_{n-5}^3. Note that $n - 5 \ge 5$, and $n - 5 \ne 7$. By the induction hypothesis, $\chi'_s(S_{n-5}^3) = 5$. Let ϕ be a strong edge-5-coloring of S_{n-5}^3, so that $\phi(x_0 x_1) = 1$, $\phi(x_0 y_0) = 2$, $\phi(x_{n-7} x_{n-6}) = 3$, $\phi(x_{n-6} y_{n-6}) = 4$, and $\phi(x_0 x_{n-6}) = 5$. Clearly, S_n^3 can be obtained from S_{n-5}^3 by inserting five vertices x_{n-5}, x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1} to the edge $x_0 x_{n-6}$ and adding a leaf y_j at x_j for $j = n - 5, n - 4, \ldots, n - 1$. We extend ϕ to S_n^3 by coloring $\{x_{n-3} x_{n-2}, x_{n-5} y_{n-5}\}$ with 1, $\{x_{n-5} x_{n-4}, x_{n-2} y_{n-2}\}$ with 2, $\{x_{n-2} x_{n-1}, x_{n-4} y_{n-4}\}$ with 3, $\{x_{n-4} x_{n-3}, x_{n-1} y_{n-1}\}$ with 4, and $\{x_0 x_{n-1}, x_{n-6} x_{n-5}, x_{n-3} y_{n-3}\}$ with 5. It is easy to testify that the extended coloring is a strong edge-5-coloring of S_n^3. □

For a sun-graph S_n^k with $C_n = x_0 x_1 \cdots x_{n-1} x_0$, we set $L(x_i) = \{e_i^1, e_i^2, \ldots, e_i^{k-2}\}$ for $i = 0, 1, \ldots, n - 1$. Recall that $L(x_i)$ stands for the set of pendant edges incident to x_i.

Lemma 5. *Let S_n^k be a sun-graph with $k, n \geq 4$ and n being even. Then*

$$\chi_s'(S_n^k) = \begin{cases} 2k, & \text{if } n = 4; \\ 2k-1, & \text{if } n \geq 6. \end{cases}$$

Proof. Since $k \geq 4$, it follows that $k-2 \geq 2$. The proof is split into the following two cases.

- Assume that $n = 4$. Color $x_0x_1, x_1x_2, x_2x_3, x_3x_0$ with $1,2,3,4$, respectively; For i = 0,2, $\ldots, n-2$, we color $k-2$ pendant edges in $L(x_i)$ with colors $5, 6, \ldots, k+2$; For $i = 1, 3, \ldots, n-1$, we color $k-2$ pendant edges in $L(x_i)$ with colors $k+3, k+4, \ldots, 2k$. It is easy to see that the defining coloring is a strong edge-$2k$-coloring of S_4^k. Hence $\chi_s'(S_4^k) \leq 2k$. Conversely, we note that every pendant edge of S_4^k has distance at most two to any edge in $E(C_4)$. This implies that, for any strong edge coloring of S_4^k, the color of any pendant edge is distinct from that of edges in C_4. Moreover, at least $2(k-2)$ colors are needed when we color the $4(k-2)$ pendant edges of S_4^k. It follows therefore that $\chi_s'(S_4^k) \geq 4 + 2(k-2) = 2k$. This yields that $\chi_s'(S_4^k) = 2k$.
- Assume that $n \geq 6$. It is straightforward to conclude that $\chi_s'(S_n^k) \geq 2k-1$ since S_n^k contains two adjacent k-vertices. Conversely, we notice that S_n^3 is a spanning subgraph of S_n^k. By Lemma 4, S_n^3 has a strong edge-5-coloring ϕ using colors $1,2,3,4,5$. Based on ϕ, we can color the remaining $k-3$ pendant edges in $L(x_i)$ with colors $6, 7, \ldots, k+2$ for each $i = 0, 2, \ldots, n-2$; and color the remaining $k-3$ pendant edges in $L(x_i)$ with colors $k+3, k+4, \ldots, 2k-1$ for each $i = 1, 3, \ldots, n-1$. The extended coloring is a strong edge-$(2k-1)$-coloring of S_n^k. It therefore turns out that $\chi_s'(S_n^k) \leq 2k-1$. Consequently, $\chi_s'(S_n^k) = 2k-1$.

□

Lemma 6. *Let $n \geq 4$ be an odd number. Then*
(1) $\chi_s'(S_n^4) \leq 8$.
(2) $\chi_s'(S_n^5) \leq 11$.

Proof. We first prove (1), by discussing two cases below.

- Assume that $n = 7$. Give a strong edge-7-coloring ϕ of S_7^4 as follows: $\phi(x_ix_{i+1}) = i+1$ for $i = 0, 1, \ldots, 6$, where indices are taken as modulo 7; then we color $L(x_0)$ with $3, 5$, $L(x_1)$ with $4, 6$, $L(x_2)$ with $5, 7$, $L(x_3)$ with $1, 6$, $L(x_4)$ with $2, 7$, $L(x_5)$ with $1, 3$, and $L(x_6)$ with $2, 4$.
- Assume that $n \neq 7$. By Lemma 4, S_n^3 admits a strong edge-5-coloring ϕ using the colors $1, 2, \ldots, 5$ so that $e_0^1, e_1^1, \ldots, e_{n-1}^1$ have been colored. Afterward, we extend ϕ to the remaining edges of S_n^4 by coloring e_0^2 with 6, $\{e_1^2, e_3^2, \ldots, e_{n-2}^2\}$ with 7, and $\{e_2^2, e_4^2, \ldots, e_{n-1}^2\}$ with 8. It is easily seen that the resultant coloring is a strong edge-5-coloring of S_n^4.

Next we prove (2). By the result of (1), S_n^4 has a strong edge-8-coloring ϕ using the colors $1, 2, \ldots, 8$. Based on ϕ, we can color e_0^3 with 9, $\{e_1^3, e_3^3, \ldots, e_{n-2}^3\}$ with 10, and $\{e_2^3, e_4^3, \ldots, e_{n-1}^3\}$ with 11. This leads to a strong edge-11-coloring of S_n^5. □

We first establish a useful claim:

Claim 1. *Let $A_i = \{e_i', e_i''\} \subseteq L(x_i)$ for $i = 0, 1, \ldots, n-1$. Let $A = A_0 \cup A_1 \cup \cdots \cup A_{n-1}$. Then A can be strongly edge-5-colored on the graph S_n^k.*

Proof. Since $n \geq 5$ is odd, we can give an edge 5-coloring π of A as follows: coloring A_1 with 2, 4; A_2 with 3, 5; A_3 with 1, 4; each of $A_0, A_5, A_7, \ldots, A_{n-2}$ with 1, 3; and each of $A_4, A_6, A_8, \ldots, A_{n-1}$ with 2, 5. It is easy to confirm that π is a strong edge-5-coloring of A restricted in the graph S_n^k. □

Lemma 7. *Let $k \geq 6$, and let $n \geq 5$ be odd. Then $\chi_s'(S_n^k) \leq \lceil 2.5k - 2 \rceil$.*

Proof. If k is even, then by Lemma 6(1) and repeatedly applying Claim 1, we get that $\chi'_s(S_n^k) \leq 8 + 5 \cdot \frac{k-4}{2} = 2.5k - 2 = \lceil 2.5k - 2 \rceil$.

If k is odd, then by Lemma 6(2) and repeatedly applying Claim 1, we get that $\chi'_s(S_n^k) \leq 11 + 5 \cdot \frac{k-5}{2} = 2.5k - 1.5 = \lceil 2.5k - 2 \rceil$. □

3. Outerplanar Graphs

Suppose that G is a connected outerplanar graph. Let $I \subseteq V(G)$ be a clique-cut of G with $1 \leq |I| \leq 2$; that is, $G[I]$ is K_1 or K_2 such that $G - I$ is disconnected. If $G - I$ has at least two components each containing at least one edge, then I is said to be a *separable* clique-cut.

For a separable clique-cut I of G, let $H_1, H_2, \ldots, H_s$ ($s \geq 2$) denote the components of $G - I$ with $|E(H_1)| \geq 1$ and $|E(H_2)| \geq 1$. We set $G_1 = G[E(H_1) \cup E(I)]$ and $G_2 = G[E(H_2) \cup \cdots \cup E(H_s) \cup E(I)]$, where $E(I)$ denotes the set of edges in G which are incident to at least one vertex in I.

The following lemma plays a crucial role in the proof of our main results.

Lemma 8. *Let G be a connected outerplanar graph with a separable clique-cut I. Suppose that G_1 and G_2 are defined as above. Then*

$$\chi'_s(G) = \max\{\chi'_s(G_1), \chi'_s(G_2)\}$$

Proof. Let $l_1 = \chi'_s(G_1)$, $l_2 = \chi'_s(G_2)$, and $l = \max\{l_1, l_2\}$. For $i = 1, 2$, let ϕ_i be a strong edge-l-coloring of G_i using the color set $C = \{1, 2, \ldots, l\}$. By the definition of G_i, we deduce that $E(I) \subset E(G_i)$ for $i = 1, 2$, and $E(G_1) \cap E(G_2) = E(I)$. Observe that any edge in $E(G_1) \setminus E(I)$ and any edge in $E(G_2) \setminus E(I)$ have distance at least three in G. Moreover, since the distance between any two edges in $E(I)$ is less than three, no two edges in $E(I)$ are assigned same color in both ϕ_1 and ϕ_2. So we may assume that $\phi_1(e) = \phi_2(e)$ for each $e \in E(I)$. Combining ϕ_1 and ϕ_2, we get a strong edge-l-coloring of G. This shows that $\chi'_s(G) \leq l$. On the other hand, since G_i is a subgraph of G, we have naturally that $\chi'_s(G) \geq \max\{\chi'_s(G_1), \chi'_s(G_2)\} = l$. Consequently, $\chi'_s(G) = l$. □

Theorem 1. *Let G be an outerplanar graph with $\Delta \geq 4$. If G does not contain F_1 as a subgraph, then $\chi'_s(G) \leq 3\Delta - 4$.*

Proof. Assume the contrary, let G be a counterexample with $|E(G)|$ being as small as possible. Then G is connected, $|E(G)| \geq 3\Delta - 3$, and possesses the following properties:

(P1) No F_1 is contained in G or its subgraphs.

(P2) G is not strongly edge-$(3\Delta - 4)$-colorable, but any subgraph H of G with $|E(H)| < |E(G)|$ is strongly edge-$(3\Delta - 4)$-colorable.

In fact, if $\Delta(H) < \Delta$, then by Lemma 1, $\chi'_s(H) \leq 3\Delta(H) - 3 \leq 3(\Delta - 1) - 3 < 3\Delta - 4$ since $\Delta \geq 4$. If $\Delta(H) = \Delta$, then by the minimality of G, $\chi'_s(H) \leq 3\Delta(H) - 4 = 3\Delta - 4$.

(P3) G is not a tree; otherwise $\chi'_s(G) \leq 2\Delta - 1 < 3\Delta - 4$, contradicting (P2).

By Lemma 8, the following claim holds:

Claim 2. G does not contain a separable clique-cut $I \subseteq V(G)$ with $1 \leq |I| \leq 2$.

Embed G to the plane so that all the vertices lie in the boundary of $f_0(G)$. Let H denote the graph obtained from G by removing all leaves. By (P3) and Claim 2, we can easily deduce Claims 3 and 4 below.

Claim 3. H is 2-connected, and $b(f_0(H))$ forms a Hamiltonian cycle. This furthermore implies that all vertices in $V(G) \setminus V(H)$ are leaves.

Claim 4. Every inner edge uv of H is incident to an end-3-face $[uvw]$ such that $d_G(w) = d_H(w) = 2$.

Claim 4 implies that $2 \leq \Delta(H) \leq 4$; for otherwise H will contain an inner edge xy with $d_H(x) \geq 5$ and $\{x, y\}$ is a separable clique-cut of G.

Let G^* denote the graph obtained from G by carrying out repeatedly the following operation:

$(*)$ If x is a 2-vertex of H incident to an end-3-face $[xyz]$, then we split x into two new vertices y_1 and z_1 so that y_1 joins with y, and z_1 joins with z.

Intuitively speaking, every 2-vertex of H which is incident to an end-3-face is replaced by two leaves in G. It is easy to see that $\Delta(G^*) = \Delta(G)$, and ϕ is a strong edge-k-coloring of G^* if and only if ϕ is a strong edge-k-coloring of G.

It is easily observed that G^* is a spanning subgraph of some sun-graph S_n^k, where $k = \Delta(G)$ and n is the total number of 3^+-vertices in G and the number of 2-vertices in G which are not on any 3-face. As an example, we observe the graphs G and G^* depicted in Figure 2.

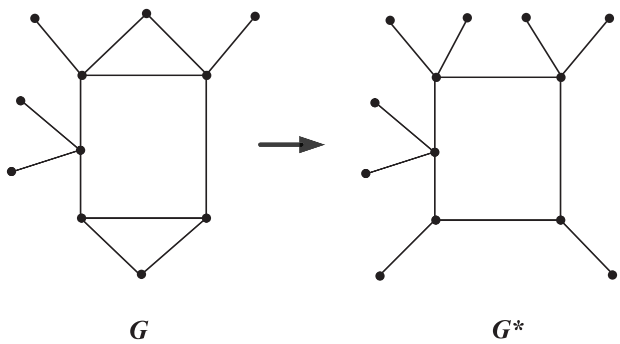

Figure 2. G^* is obtained from G by carrying out $(*)$, and G^* is a subgraph of S_5^4.

Noting that $3k-4 \geq \max\{2k, \lceil 2.5k-2 \rceil\}$, we deduce by Lemmas 5 and 7 that $\chi'_s(G) = \chi'_s(G^*) \leq \chi'_s(S_n^k) \leq 3k-4 = 3\Delta - 4$. This completes the proof of the theorem. □

Combining Theorem 1 and Lemmas 1 and 2(1), the following theorem holds:

Theorem 2. *Let G be an outerplanar graph with $\Delta \geq 4$. Then $\chi'_s(G) \leq 3\Delta - 3$; and $\chi'_s(G) = 3\Delta - 3$ if and only if G contains F_1 as a subgraph.*

Theorem 3. *Let G be an outerplanar graph with maximum degree $\Delta = 3$. If G does not contain F_1, F_2 or F_3 as a subgraph, then $\chi'_s(G) \leq 5$.*

Proof. Assume the contrary, let G be a counterexample with $|E(G)|$ being as small as possible. Then G is connected, $|E(G)| \geq 6$, and possesses the following properties:

(Q1) None of F_1, F_2, F_3 is contained in G or its subgraphs.

(Q2) G is not strongly edge-5-colorable, but any subgraph H of G with $|E(H)| < |E(G)|$ is strongly edge-5- colorable. Actually, if $\Delta(H) \leq 2$, then by Lemma 3, $\chi'_s(H) \leq 5$. If $\Delta(H) = 3$, then by the minimality of G, we obtain that $\chi'_s(H) \leq 5$.

(Q3) G is not a tree; otherwise $\chi'_s(G) \leq 5$, contradicting (Q2).

By Lemma 8, G does not contain a separable clique-cut $I \subseteq V(G)$ with $1 \leq |I| \leq 2$.

Embed G to the plane so that all the vertices lie in $b(f_0(G))$. Removing all the leaves of G, we get a subgraph H of G. Similarly to the proof of Theorem 1, we conclude the following:

- H is 2-connected, and all vertices in $V(G) \setminus V(H)$ are leaves.
- Every inner edge uv of H is incident to an end-3-face $[uvw]$ such that $d_G(w) = d_H(w) = 2$.

Let G^* be the graph obtained from G by doing repeatedly the following operation:

$(*)$ If x is a 2-vertex of H incident to an end-3-face $[xyz]$ in H, then we split x into two new vertices y_1 and z_1 so that y_1 joins with y, and z_1 joins with z.

Then $\Delta(G^*) = \Delta(G)$, and $\chi'_s(G) = \chi'_s(G^*)$. Note that G^* is a spanning subgraph of some sun-graph S_n^3, where n is the total number of 3-vertices in G and the number of 2-vertices in G which are not on any 3-face. By Lemma 4, we derive immediately that $\chi'_s(G) = \chi'_s(G^*) \le \chi'_s(S_n^3) \le 5$. $\square$

Combining Theorem 3 and Lemmas 1 and 2, we have the following:

Theorem 4. *Let G be an outerplanar graph with $\Delta = 3$. Then $\chi'_s(G) \le 6$; and $\chi'_s(G) = 6$ if and only if G contains at least one of F_1, F_2, F_3 as a subgraph.*

When restricted to the family of bipartite outerplanar graphs G, smaller and tight upper bounds for $\chi'_s(G)$ can be obtained.

Theorem 5. *Let G be a bipartite and outerplanar graph with maximum degree $\Delta \ge 3$. Then $\chi'_s(G) \le 2\Delta$; moreover, $\chi'_s(G) = 2\Delta$ if and only if G contains F_2 as a subgraph.*

Proof. We first show that $\chi'_s(G) \le 2\Delta$. Assume the contrary, let G be a counterexample with $|E(G)|$ being as small as possible. Then G is connected, other than a tree, and is not strongly edge 2Δ-colorable, but any subgraph H of G with $|E(H)| < |E(G)|$ is strongly edge-2Δ-colorable. Moreover, by Lemma 8, there is no separable clique-cut $I \subseteq V(G)$ with $1 \le |I| \le 2$.

Embed G to the plane so that all the vertices occur in $b(f_0(G))$. Removing all the leaves of G, we obtain a subgraph H of G. Then H is a Hamiltonian cycle without chords, and $V(G) \setminus V(H)$ are all leaves. So G is a subgraph of some S_n^k where $n = |V(H)|$ is even and $k = \Delta(G)$. By Lemma 5, $\chi'_s(G) \le \chi'_s(S_n^k) \le 2k = 2\Delta$.

If G contains F_2 as a subgraph, then $\chi'_s(G) \ge \chi'_s(F_2) = |E(F_2)| = 2\Delta$. Using the foregoing proof, we get that $\chi'_s(G) = 2\Delta$. Conversely, if G does not contain F_2 as a subgraph, then similarly to the above discussion we can show that $\chi'_s(G) \le 2\Delta - 1$. $\square$

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References

1. Fouquet, J.L.; Jolivet, J.L. Strong edge-colorings of graphs and applications to multi-k-gons. *Ars Combin.* **1983**, *16*, 141–150.
2. Molloy, M.; Reed, B. A bound on the strong chromatic index of a graph. *J. Combin. Theory Ser. B* **1997**, *69*, 103–109. [CrossRef]
3. Bruhn, H.; Joos, F. A stronger bound for the strong chromatic index. *Combin. Probab. Comput.* **2018**, *27*, 21–43. [CrossRef]
4. Faudree, R.J.; Gyárfás, A.; Schelp, R.H.; Tuza, D. The strong chromatic index of graphs. *Ars Combin.* **1990**, *29*, 205–211.
5. Chartrand, G.; Harary, F. Planar permutation graphs. *Ann. Inst. Henri Poincaré Sect. B (N.S.)* **1967**, *3*, 433–438.
6. Wang, Y.; Wang, P.; Wang, W. Strong chromatic index of K_4-minor free graphs. *Inform. Process. Lett.* **2018**, *129*, 53–56. [CrossRef]
7. Hocquard, H.; Ochem, P.; Valicov, P. Strong edge-colouring and induced matching. *Inform. Process. Lett.* **2013**, *113*, 836–843. [CrossRef]