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Approximation Properties of the Generalized Abel-Poisson Integrals on the Weyl-Nagy Classes

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Abstract: Asymptotic equalities are obtained for the least upper bounds of approximations of functions from the classes $W^r_{\beta,\infty}$ by the generalized Abel-Poisson integrals $P_\gamma(\delta)$, $0 < \gamma \leq 2$, for the case $r > \gamma$ in the uniform metric, which provide the solution to the Kolmogorov–Nikol'skii problem for the given method of approximation on the Weyl-Nagy classes.

Keywords: Weyl-Nagy classes; generalized Abel-Poisson integral; asymptotic equality; Kolmogorov–Nikol'skii problem; uniform metric

MSC: 42A05; 41A60

1. Introduction

Let L be a space of 2π-periodic summable functions and

$$S[f] = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx)$$

be the Fourier series of $f \in L$.

Further, let C be a subset of the continuous functions from L with the uniform norm $\|f\|_C = \max_t |f(t)|$; L_∞ be a subset of the functions $f \in L$ with the finite norm $\|f\|_\infty = \operatorname{ess\,sup}_t |f(t)|$.

Let $\Lambda = \{\lambda_\delta(k)\}$ be the set of functions depending on $k \in \mathbb{N} \cup 0$ and on the parameter $\delta \in E_\Lambda \subset \mathbb{R}$, the set E_Λ has at least one limit point and $\lambda_\delta(0) = 1$. Using the set Λ to each function $f \in L$ we can associate the series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \lambda_\delta(k)(a_k \cos kx + b_k \sin kx), \quad \delta \in E_\Lambda,$$

which converges for every $\delta \in E_\Lambda$ and all x to the continuous function $U_\delta(f; x; \Lambda)$.

If the series

$$\frac{1}{2} + \sum_{k=1}^{\infty} \lambda_\delta(k) \cos kt$$

is the Fourier series of some summable function, then (similarly to ([1], p. 52)) for almost all $x \in \mathbb{R}$ we have the equality

$$U_\delta(f; x; \Lambda) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t)\Big(\frac{1}{2} + \sum_{k=1}^{\infty} \lambda_\delta(k) \cos kt\Big) dt. \tag{1}$$

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Putting in the equality (1) $\lambda_\delta(k) = e^{-\frac{k^\gamma}{\delta}}$, $0 < \gamma \leq 2$, we obtain the quantity

$$U_\delta(f;x;\Lambda) := P_\gamma(\delta;f;x) = \frac{1}{\pi}\int\limits_{-\pi}^{\pi} f(x+t)\left\{\frac{1}{2} + \sum_{k=1}^{\infty} e^{-\frac{k^\gamma}{\delta}} \cos kt\right\}dt, \ \ \delta > 0, \ 0 < \gamma \leq 2, \quad (2)$$

which is usually called the generalized Abel-Poisson integral of the function f (see, e.g., [2,3]). For $\gamma = 1$ the integral (2) is the Poisson integral (see, e.g., [4]), for $\gamma = 2$ the integral (2) is the Weierstrass integral (see, e.g., [5]).

Let us define the classes of functions that we consider further. Let $f \in L$, $r > 0$ and β be a real number. If the series

$$\sum_{k=1}^{\infty} k^r \left(a_k \cos\left(kx + \frac{\beta\pi}{2}\right) + b_k \sin\left(kx + \frac{\beta\pi}{2}\right)\right)$$

is the Fourier series of a summable function, then it is denoted by f_β^r and is called the (r,β)-derivative of the function f in the Weyl-Nagy sense (see, e.g., [6]). Let $W_{\beta,\infty}^r$ be the classes of the functions f for which $\|f_\beta^r(\cdot)\|_\infty \leq 1$.

In this paper, we consider the problem of asymptotic behavior as $\delta \to \infty$ of the quantity

$$\mathcal{E}(W_{\beta,\infty}^r; P_\gamma(\delta))_C = \sup_{f \in W_{\beta,\infty}^r} \|f(\cdot) - P_\gamma(\delta, f, \cdot)\|_C. \quad (3)$$

If the function $g(\delta)$ is found in an explicit form such that

$$\mathcal{E}(W_{\beta,\infty}^r; P_\gamma(\delta))_C = g(\delta) + o(g(\delta)), \delta \to \infty,$$

then according to Stepanets [6] we say that the Kolmogorov–Nikol'skii problem is solved for the class $W_{\beta,\infty}^r$ and the generalized Abel-Poisson integral in the uniform metric.

The approximation properties of the generalized Poisson integrals have been studied only in the cases $\gamma = 1$ (Poisson integral) and $\gamma = 1$ (Weierstrass integral). In particular, the Kolmogorov–Nikol'skii problems for the Poisson integral on the different functional classes have been solved in [7–11]. Similar problems for Weierstrass integral have been solved in [5,12–14].

Regarding the results of estimating the approximation rate by the generalized Poisson integrals for $0 < \gamma \leq 2$ we note the work [2], where the approximation properties of the integrals (2) on Zygmund classes $Z_\alpha, 0 \leq \alpha \leq 2$, have been studied.

In this paper, we aim to find asymptotic equations for quantities (3) for arbitrary $0 < \gamma \leq 2$. This will allow us to find such γ for any r, so that the approximation rate of functions from the classes $W_{\beta,\infty}^r$ by the generalized Abel-Poisson integrals, i.e, the rate at which the quantity (3) tends to zero, is equal to $\frac{1}{\delta}$. This approximation rate could not be achieved when approximating by Poisson integrals and Weierstrass integrals.

At present, the extremal problems of the approximation theory, being related to the study of the approximation properties of linear methods for summing Fourier series, become increasingly relevant in applied mathematics, in particular, in the creation of mathematical models [15–19], in signal transmission [20,21], in the decision theory [22] and others. The problem considered in the paper, as well as those close to it [23–25] find practical application in the issues of coding, transmission and reproduction of images.

2. Main Result

Let us define the summing function for the generalized Abel-Poisson integral as follows

$$\tau(u) = \begin{cases} (1-e^{-u^\gamma})\left((\gamma - r - 1)\delta^{\frac{r+2}{\gamma}-1}u^{2-\gamma} + (2+r-\gamma)\delta^{\frac{r+1}{\gamma}-1}u^{1-\gamma}\right), 0 \leq u \leq \frac{1}{\sqrt[\gamma]{\delta}}, \\ (1-e^{-u^\gamma})u^{-r}, u \geq \frac{1}{\sqrt[\gamma]{\delta}}, \end{cases} \quad (4)$$

where $0<\gamma\le 2,\ \delta>0$.

Theorem 1. *Let $r>\gamma$. Then the following asymptotic equality holds as $\delta\to\infty$:*

$$\mathcal{E}\left(W^r_{\beta,\infty};P_\gamma(\delta)\right)_C=\frac{1}{\delta}\sup_{f\in W^r_{\beta,\infty}}\|f_0^\gamma(\cdot)\|_C+O(\Upsilon(\delta,r,\gamma)), \tag{5}$$

where $f_0^\gamma(x)$ is (r,β)—derivative in the Weyl-Nagy sense as $r=\gamma,\ \beta=0$ and

$$\Upsilon(\delta,r,\gamma)=\begin{cases}\frac{1}{(\sqrt[\gamma]{\delta})^r}, & \gamma<r<2\gamma,\\ \frac{\ln\delta}{\delta^2}, & r=2\gamma,\\ \frac{1}{\delta^2}, & r>2\gamma.\end{cases}$$

Proof. Let us rewrite the function $\tau(u)$ given by (4) in the form $\tau(u)=\varphi(u)+\mu(u)$ (see, e.g., [26]), where

$$\varphi(u)=\begin{cases}(\gamma-r-1)\delta^{\frac{r+2}{\gamma}-1}u^2+(2+r-\gamma)\delta^{\frac{r+1}{\gamma}-1}u, & 0\le u\le\frac{1}{\sqrt[\gamma]{\delta}},\\ u^{\gamma-r},\ u\ge\frac{1}{\sqrt[\gamma]{\delta}}.\end{cases} \tag{6}$$

$$\mu(u)=$$

$$=\begin{cases}(1-e^{-u^\gamma}-u^\gamma)\left((\gamma-r-1)\delta^{\frac{r+2}{\gamma}-1}u^{2-\gamma}+(2+r-\gamma)\delta^{\frac{r+1}{\gamma}-1}u^{1-\gamma}\right), 0\le u\le\frac{1}{\sqrt[\gamma]{\delta}},\\ (1-e^{-u^\gamma}-u^\gamma)u^{-r}, u\ge\frac{1}{\sqrt[\gamma]{\delta}},\end{cases} \tag{7}$$

Further we show a summability of the transformations of the form

$$\hat{\varphi}_\beta(t)=\hat{\varphi}(t,\beta)=\frac{1}{\pi}\int_0^\infty\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du, \tag{8}$$

$$\hat{\mu}_\beta(t)=\hat{\mu}(t,\beta)=\frac{1}{\pi}\int_0^\infty\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du. \tag{9}$$

First, prove a convergence of the integral

$$A(\varphi)=\frac{1}{\pi}\int_{-\infty}^\infty|\hat{\varphi}_\beta(t)|dt.$$

Integrating twice by parts and taking into account that $\varphi(0)=0$, $\lim\limits_{u\to\infty}\varphi(u)=\lim\limits_{u\to\infty}\varphi'(u)=0$ and $\varphi'(u)$ is continuous on $[0,\infty)$, we have

$$\int_0^\infty\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du=\frac{1}{t^2}\left(\varphi'(0)\cos\frac{\beta\pi}{2}-\int_0^\infty\varphi''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right).$$

In view of the fact, that the function $\varphi(u)$ is downward closed on $\left[\frac{1}{\sqrt[\gamma]{\delta}},\infty\right)$, the last relation yields

$$\left|\int_0^\infty\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\le\frac{1}{t^2}\left(|\varphi'(0)|+\left(\int_0^{\frac{1}{\sqrt[\gamma]{\delta}}}+\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^\infty\right)|\varphi''(u)|du\right)=$$

$$= \frac{1}{t^2}\left((2+r-\gamma)\delta^{\frac{r+1}{\gamma}-1} + 2(r+1-\gamma)\delta^{\frac{r+1}{\gamma}-1} + \int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty} \varphi''(u)du\right) =$$

$$= \frac{1}{t^2}\left(K_1\delta^{\frac{r+1}{\gamma}-1} - \varphi'\left(\frac{1}{\sqrt[\gamma]{\delta}}\right)\right) = K_2\delta^{\frac{r+1}{\gamma}-1}\frac{1}{t^2}. \tag{10}$$

Here and below we denote by symbols $K_i, i = 1, 2, \ldots,$ some positive constants. From the inequalities (10) it follows that

$$\int\limits_{|t|\geq \sqrt[\gamma]{\delta}} \left|\int\limits_0^{\infty} \varphi(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt = O\left(\delta^{\frac{r}{\gamma}-1}\right),\ \delta \to \infty. \tag{11}$$

By virtue of the equality (4.16) from ([1], p. 69), we obtain

$$\int\limits_0^{\sqrt[\gamma]{\delta}}\left|\int\limits_0^{\infty} \varphi(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt = \int\limits_0^{\sqrt[\gamma]{\delta}}\left|\int\limits_0^{\frac{1}{\sqrt[\gamma]{\delta}}} + \int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty} \varphi(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt \leq$$

$$\leq \sqrt[\gamma]{\delta}\int\limits_0^{\frac{1}{\sqrt[\gamma]{\delta}}}|\varphi(u)|du + \int\limits_0^{\sqrt[\gamma]{\delta}}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+\frac{2\pi}{t}} \varphi(u)dudt \leq K_3\delta^{\frac{r}{\gamma}-1} + \int\limits_0^{\sqrt[\gamma]{\delta}}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+\frac{2\pi}{t}} u^{\gamma-r}dudt. \tag{12}$$

Making a change of variables and integrating by parts in the last integral, we obtain

$$\int\limits_0^{\sqrt[\gamma]{\delta}}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+\frac{2\pi}{t}} u^{\gamma-r}dudt = 2\pi\int\limits_{\frac{2\pi}{\sqrt[\gamma]{\delta}}}^{\infty}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x} u^{\gamma-r}du\frac{dx}{x^2} =$$

$$= 2\pi\left(-\frac{1}{x}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x} u^{\gamma-r}du\Bigg|_{\frac{2\pi}{\sqrt[\gamma]{\delta}}}^{\infty} + \int\limits_{\frac{2\pi}{\sqrt[\gamma]{\delta}}}^{\infty}\frac{1}{x}\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma-r}dx\right) =$$

$$= 2\pi\left(-\lim_{x\to\infty}\frac{1}{x}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x} u^{\gamma-r}du + \frac{\sqrt[\gamma]{\delta}}{2\pi}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{(1+2\pi)}{\sqrt[\gamma]{\delta}}} u^{\gamma-r}du + \right.$$

$$\left. + \delta^{\frac{r}{\gamma}-1}\int\limits_{\frac{2\pi}{\sqrt[\gamma]{\delta}}}^{\infty}\frac{1}{x}\left(1+\sqrt[\gamma]{\delta}x\right)^{\gamma-r}dx\right). \tag{13}$$

In view of

$$\lim_{x\to\infty}\frac{1}{x}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x} u^{\gamma-r}du = 0,$$

$$\frac{\sqrt[\gamma]{\delta}}{2\pi}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{(1+2\pi)}{\sqrt[\gamma]{\delta}}} u^{\gamma-r}du = K_4\delta^{\frac{r}{\gamma}-1},$$

$$\delta^{\frac{r}{\gamma}-1}\int\limits_{\frac{2\pi}{\sqrt[\gamma]{\delta}}}^{\infty}\frac{1}{x}\left(1+\sqrt[\gamma]{\delta}x\right)^{\gamma-r}dx=\delta^{\frac{r}{\gamma}-1}\int\limits_{1+2\pi}^{\infty}\frac{y^{\gamma-r}}{y-1}dy=$$

$$=\delta^{\frac{r}{\gamma}-1}\int\limits_{1+2\pi}^{\infty}y^{\gamma-r-1}\left(1+\frac{1}{y-1}\right)dy\leq\left(1+\frac{1}{2\pi}\right)\delta^{\frac{r}{\gamma}-1}\int\limits_{1+2\pi}^{\infty}y^{\gamma-r-1}dy=K_5\delta^{\frac{r}{\gamma}-1},$$

from (13) and (12) we can write that

$$\int\limits_{0}^{\sqrt[\gamma]{\delta}}\left|\int\limits_{0}^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt=O\left(\delta^{\frac{r}{\gamma}-1}\right),\ \delta\to\infty. \tag{14}$$

One can analogously show that

$$\int\limits_{-\sqrt[\gamma]{\delta}}^{0}\left|\int\limits_{0}^{\infty}\varphi(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt=O\left(\delta^{\frac{r}{\gamma}-1}\right),\ \delta\to\infty. \tag{15}$$

From the formulas (11), (14) and (15) we obtain

$$A(\varphi)=O\left(\delta^{\frac{r}{\gamma}-1}\right),\ \delta\to\infty.$$

Now we show the convergence of the integral

$$A(\mu)=\frac{1}{\pi}\int\limits_{-\infty}^{\infty}|\hat{\mu}_{\beta}(t)|dt.$$

Integrating twice by parts and taking into account that $\mu(0)=\mu'(0)=0,\ \lim\limits_{u\to\infty}\mu(u)=\lim\limits_{u\to\infty}\mu'(u)=0$, we have

$$\int\limits_{0}^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du=-\frac{1}{t^2}\int\limits_{0}^{\infty}\mu''(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du,$$

and hence

$$\left|\int\limits_{0}^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|\leq\frac{1}{t^2}\int\limits_{0}^{\infty}|\mu''(u)|du=\frac{1}{t^2}\left(\left(\int\limits_{0}^{\frac{1}{\sqrt[\gamma]{\delta}}}+\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{1}+\int\limits_{1}^{\infty}\right)|\mu''(u)|du\right). \tag{16}$$

Further we use the notations

$$V(u)=\left(1-e^{-u^{\gamma}}-u^{\gamma}\right)u^{2-\gamma},\ \ W(u)=\left(1-e^{-u^{\gamma}}-u^{\gamma}\right)u^{1-\gamma}. \tag{17}$$

Let us differentiate twice the functions $V(u)$ and $W(u)$:

$$V'(u)=\gamma u(e^{-u^{\gamma}}-1)+(2-\gamma)u^{1-\gamma}(1-e^{-\gamma}-u^{\gamma}),$$

$$W'(u)=\gamma(e^{-u^{\gamma}}-1)+(1-\gamma)u^{-\gamma}\left(1-e^{-u^{\gamma}}-u^{\gamma}\right),$$

$$V''(u)=\gamma\left(e^{-u^{\gamma}}(1-\gamma u^{\gamma})-1\right)+(2-\gamma)\left((1-\gamma)u^{-\gamma}\left(1-e^{-u^{\gamma}}-u^{\gamma}\right)+\gamma(e^{-u^{\gamma}}-1)\right),$$

$$W''(u) = -\gamma^2 u^{\gamma-1} e^{-u^\gamma} - \gamma(\gamma-1)u^{-\gamma-1}\left(1 - e^{-u^\gamma} - u^\gamma\right) + \gamma(\gamma-1)u^{-1}\left(e^{-u^\gamma} - 1\right).$$

By virtue of the fact, that for $u \in \left[0, \frac{1}{\sqrt[\gamma]{\delta}}\right]$

$$\mu''(u) = (\gamma - r - 1)\delta^{\frac{r+2}{\gamma}-1} V''(u) + (2 + r - \gamma)\delta^{\frac{r+1}{\gamma}-1} W''(u), \tag{18}$$

we obtain

$$\int_0^{\frac{1}{\sqrt[\gamma]{\delta}}} |\mu''(u)| du \le (r + 1 - \gamma)\delta^{\frac{r+2}{\gamma}-1} \int_0^{\frac{1}{\sqrt[\gamma]{\delta}}} |V''(u)| du + (2 + r - \gamma)\delta^{\frac{r+1}{\gamma}-1} \int_0^{\frac{1}{\sqrt[\gamma]{\delta}}} |W''(u)| du.$$

Taking into account, that for $u \in \left[0, \frac{1}{\sqrt[\gamma]{\delta}}\right]$ $V''(u) \le 0$, $W''(u) \le 0$, and also the inequalities

$$1 - e^{-u^\gamma} \le u^\gamma, \;\; e^{-u^\gamma} + u^\gamma - 1 \le \frac{u^{2\gamma}}{2}, \tag{19}$$

we have

$$\int_0^{\frac{1}{\sqrt[\gamma]{\delta}}} |\mu''(u)| du \le (r + 1 - \gamma)\delta^{\frac{r+2}{\gamma}-1} \left(V'(0) - V'\left(\frac{1}{\sqrt[\gamma]{\delta}}\right)\right) +$$

$$+ (2 + r - \gamma)\delta^{\frac{r+1}{\gamma}-1} \left(W'(0) - W'\left(\frac{1}{\sqrt[\gamma]{\delta}}\right)\right) =$$

$$= (r + 1 - \gamma)\delta^{\frac{r+2}{\gamma}-1} \left(\frac{\gamma}{\sqrt[\gamma]{\delta}}\left(1 - e^{-\frac{1}{\delta}}\right) + \frac{2-\gamma}{\left(\sqrt[\gamma]{\delta}\right)^{1-\gamma}}\left(e^{-\frac{1}{\delta}} + \frac{1}{\delta} - 1\right)\right) +$$

$$+ (2 + r - \gamma)\delta^{\frac{r+1}{\gamma}-1} \left(\gamma\left(1 - e^{-\frac{1}{\delta}}\right) + \frac{1-\gamma}{\left(\sqrt[\gamma]{\delta}\right)^{-\gamma}}\left(e^{-\frac{1}{\delta}} + \frac{1}{\delta} - 1\right)\right) \le K_6 \delta^{\frac{r+1}{\gamma}-2}. \tag{20}$$

Noting, that for $u \ge \frac{1}{\sqrt[\gamma]{\delta}}$

$$\mu''(u) = r(r+1)\left(1 - e^{-u^\gamma} - u^\gamma\right)u^{-r-2} - 2\gamma u^{\gamma-r-2}\left(e^{-u^\gamma} - 1\right) +$$

$$+ \gamma\left((\gamma-1)u^{\gamma-2}(e^{-u^\gamma} - 1) - \gamma u^{2\gamma-2} e^{-u^\gamma}\right)u^{-r},$$

we can write

$$\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^1 |\mu''(u)| du \le r(r+1) \int_{\frac{1}{\sqrt[\gamma]{\delta}}}^1 (e^{-u^\gamma} + u^\gamma - 1)u^{-r-2} du +$$

$$+ 2\gamma r \int_{\frac{1}{\sqrt[\gamma]{\delta}}}^1 \left(1 - e^{-u^\gamma}\right)u^{\gamma-r-2} du + \gamma \int_{\frac{1}{\sqrt[\gamma]{\delta}}}^1 \left|(\gamma-1)u^{\gamma-2}(e^{-u^\gamma} - 1) - \gamma u^{2\gamma-2} e^{-u^\gamma}\right| u^{-r} du.$$

The inequality (19) in combination with

$$\left|(\gamma-1)u^{\gamma-2}(e^{-u^\gamma} - 1) - \gamma u^{2\gamma-2} e^{-u^\gamma}\right| \le (2\gamma - 1)u^{2\gamma-2}, \;\; u \in [0, \infty), \tag{21}$$

yields

$$\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{1} |\mu''(u)|du \le \left(\frac{r(r+1)}{2} + 2\gamma r + \gamma(2\gamma-1)\right) \int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{1} u^{2\gamma-r-2}du \le$$

$$\le \left(\frac{r(r+1)}{2} + 2\gamma r + \gamma(2\gamma-1)\right) \sqrt[\gamma]{\delta} \int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{1} u^{2\gamma-r-1}du = \begin{cases} K_7\delta^{\frac{1}{\gamma}} + K_8\delta^{\frac{r+1}{\gamma}-2}, & r \ne 2\gamma, \\ K_9\delta^{\frac{1}{\gamma}}\ln\delta, & r = 2\gamma, \end{cases} \tag{22}$$

In the case $u \in [1, \infty)$ we obtain

$$\int_1^{\infty} |\mu''(u)|du \le r(r+1)\int_1^{\infty}(e^{-u^{\gamma}} + u^{\gamma} - 1)u^{-r-2}du +$$

$$+2\gamma r \int_1^{\infty}\left(1 - e^{-u^{\gamma}}\right)u^{\gamma-r-2}du + \gamma\int_1^{\infty}\left|(\gamma-1)u^{\gamma-2}(e^{-u^{\gamma}} - 1) - \gamma u^{2\gamma-2}e^{-u^{\gamma}}\right|u^{-r}du.$$

Let $0 < \gamma < 1$, then using the inequalities (19) and (21), we obtain

$$\int_1^{\infty} |\mu''(u)|du \le \left(\frac{r(r+1)}{2} + 2\gamma r + \gamma(2\gamma-1)\right)\int_1^{\infty} u^{2\gamma-r-2}du = K_{10}. \tag{23}$$

Let further $1 \le \gamma \le 2$. By virtue of the inequalities

$$(e^{-u^{\gamma}} + u^{\gamma} - 1)u^{-2} \le 1,\ \left(1 - e^{-u^{\gamma}}\right)u^{\gamma-2} \le 1,$$

$$|(\gamma-1)u^{\gamma-2}(e^{-u^{\gamma}} - 1) - \gamma u^{2\gamma-2}e^{-u^{\gamma}}| \le 2\gamma - 1,$$

we have

$$\int_1^{\infty} |\mu''(u)|du \le \left(\frac{r(r+1)}{2} + 2\gamma r + \gamma(2\gamma-1)\right)\int_1^{\infty} u^{-r}du = K_{11}. \tag{24}$$

Therefore, combining the relations (23), (24), we obtain

$$\int_1^{\infty} |\mu''(u)|du = O(1),\ \delta \to \infty. \tag{25}$$

In view of (16), taking into account (20), (22) and (25), we obtain

$$\int_{|t| \ge \sqrt[\gamma]{\delta}} \left|\int_0^{\infty} \mu(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt = \begin{cases} O(1), & \gamma < r < 2\gamma, \\ O(\ln\delta), & r = 2\gamma, \\ O(\delta^{\frac{r}{\gamma}-2}), & r > 2\gamma. \end{cases} \tag{26}$$

Let us further consider

$$\int_0^{\sqrt[\gamma]{\delta}} \left|\int_0^{\infty} \mu(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt \le \int_0^{\sqrt[\gamma]{\delta}} \left|\int_0^{\frac{1}{\sqrt[\gamma]{\delta}}} \mu(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt +$$

$$+ \int_0^{\sqrt[\gamma]{\delta}} \left|\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty} \mu(u)\cos\left(ut + \frac{\beta\pi}{2}\right)du\right|dt. \tag{27}$$

By the inequality (19), one can easily verify that the following relations hold

$$\int_0^{\sqrt[\gamma]{\delta}}\left|\int_0^{\frac{1}{\sqrt[\gamma]{\delta}}}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt \le \int_0^{\sqrt[\gamma]{\delta}}\int_0^{\frac{1}{\sqrt[\gamma]{\delta}}}|\mu(u)|dudt = K_{12}\delta^{\frac{r}{\gamma}-2}. \tag{28}$$

The function $|\mu(u)|$ is monotonically decreasing on the interval $[u_0,\infty], u_0 \ge 1$, non-negative and tends to zero as $u \to \infty$. Then, by the equality (4.16) from ([1], p. 69), we obtain

$$\int_0^{\sqrt[\gamma]{\delta}}\left|\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt = \int_0^{\sqrt[\gamma]{\delta}}\left|\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty}|\mu(u)|\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt \le$$

$$\le \int_0^{\sqrt[\gamma]{\delta}}\left|\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{u_0}|\mu(u)|\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt + \int_0^{\sqrt[\gamma]{\delta}}\left|\int_{u_0}^{\infty}|\mu(u)|\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt \le$$

$$\le \int_0^{\sqrt[\gamma]{\delta}}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{u_0}|\mu(u)|dudt + \int_0^{\sqrt[\gamma]{\delta}}\int_{u_0}^{u_0+\frac{2\pi}{t}}|\mu(u)|dudt \le \int_0^{\sqrt[\gamma]{\delta}}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{u_0+\frac{2\pi}{t}}|\mu(u)|dudt. \tag{29}$$

Let $n \in \mathbb{N}$ is such that $\frac{1}{\sqrt[\gamma]{\delta}} + \frac{2\pi(n-1)}{t} \le u_0 \le \frac{1}{\sqrt[\gamma]{\delta}} + \frac{2\pi n}{t}$, then

$$\int_0^{\sqrt[\gamma]{\delta}}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{u_0+\frac{2\pi}{t}}|\mu(u)|dudt \le \int_0^{\sqrt[\gamma]{\delta}}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+\frac{2\pi(n+1)}{t}}|\mu(u)|dudt. \tag{30}$$

We transform the latter integral using a change of variable and integration by parts (assume that $\delta > (2\pi(n+1)+1)^{\gamma}$)

$$\int_0^{\sqrt[\gamma]{\delta}}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+\frac{2\pi(n+1)}{t}}|\mu(u)|dudt = 2\pi(n+1)\int_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{\infty}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x}|\mu(u)|du\frac{dx}{x^2} =$$

$$= 2\pi(n+1)\left(-\left(\frac{1}{x}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x}|\mu(u)|du\right)\Bigg|_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{\infty} + \int_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{\infty}\frac{1}{x}\left|\mu\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)\right|dx\right) =$$

$$= 2\pi(n+1)\left(-\lim_{x\to\infty}\frac{1}{x}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x}\left(e^{-u^{\gamma}}+u^{\gamma}-1\right)u^{-r}du+\right.$$

$$+\frac{\sqrt[\gamma]{\delta}}{2\pi(n+1)}\int_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1+2\pi(n+1)}{\sqrt[\gamma]{\delta}}}\left(e^{-u^{\gamma}}+u^{\gamma}-1\right)u^{-r}du+$$

$$+\int\limits_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{1-\frac{1}{\sqrt[\gamma]{\delta}}}\frac{1}{x}\left(e^{-\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}}+\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}-1\right)\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{-r}dx+$$

$$\left.+\int\limits_{1-\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty}\frac{1}{x}\left(e^{-\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}}+\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}-1\right)\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{-r}dx\right). \tag{31}$$

Obviously,

$$\lim_{x\to\infty}\frac{1}{x}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1}{\sqrt[\gamma]{\delta}}+x}\left(e^{-u^{\gamma}}+u^{\gamma}-1\right)\psi(\sqrt[\gamma]{\delta}u)du=0. \tag{32}$$

Since the second inequality from (19) holds, then

$$\frac{\sqrt[\gamma]{\delta}}{2\pi(n+1)}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1+2\pi(n+1)}{\sqrt[\gamma]{\delta}}}\left(e^{-u^{\gamma}}+u^{\gamma}-1\right)u^{-r}du\le$$

$$\le\frac{\sqrt[\gamma]{\delta}}{4\pi(n+1)}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1+2\pi(n+1)}{\sqrt[\gamma]{\delta}}}u^{2\gamma-r}du\le\frac{\delta^{\frac{r+1}{\gamma}}}{4\pi(n+1)}\int\limits_{\frac{1}{\sqrt[\gamma]{\delta}}}^{\frac{1+2\pi(n+1)}{\sqrt[\gamma]{\delta}}}u^{2\gamma}du\le K_{13}\delta^{\frac{r}{\gamma}-2}. \tag{33}$$

Using the second inequality from (19), we have

$$\int\limits_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{1-\frac{1}{\sqrt[\gamma]{\delta}}}\frac{1}{x}\left(e^{-\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}}+\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}-1\right)\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{-r}dx\le$$

$$\le\int\limits_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{1-\frac{1}{\sqrt[\gamma]{\delta}}}\frac{1}{x}\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{2\gamma-r}dx=\delta^{\frac{r}{\gamma}-2}\int\limits_{\frac{2\pi(n+1)}{\sqrt[\gamma]{\delta}}}^{1-\frac{1}{\sqrt[\gamma]{\delta}}}\frac{1}{x}\left(1+\sqrt[\gamma]{\delta}x\right)^{2\gamma-r}dx=$$

$$=\delta^{\frac{r}{\gamma}-2}\int\limits_{1+2\pi(n+1)}^{\sqrt[\gamma]{\delta}}\frac{y^{2\gamma-r}}{y-1}dy=\delta^{\frac{r}{\gamma}-2}\int\limits_{1+2\pi(n+1)}^{\sqrt[\gamma]{\delta}}y^{2\gamma-1}\psi(y)\left(1+\frac{1}{y-1}\right)dy\le$$

$$\le\left(1+\frac{1}{2\pi(n+1)}\right)\delta^{\frac{r}{\gamma}-2}\int\limits_{1+2\pi(n+1)}^{\sqrt[\gamma]{\delta}}y^{2\gamma-1-r}dy=\begin{cases}K_{14}+K_{15}\delta^{\frac{r}{\gamma}-2}, & r\ne 2\gamma,\\ K_{16}\ln\delta, & r=2\gamma.\end{cases} \tag{34}$$

Considering the inequality

$$e^{-u^{\gamma}}+u^{\gamma}-1\le u^{\gamma},$$

we have

$$\int\limits_{1-\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty}\frac{1}{x}\left(e^{-\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}}+\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma}-1\right)\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{-r}dx\le$$

$$\leq \int\limits_{1-\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty} \frac{1}{x}\left(\frac{1}{\sqrt[\gamma]{\delta}}+x\right)^{\gamma-r}dx = \delta^{\frac{r}{\gamma}-1}\int\limits_{1-\frac{1}{\sqrt[\gamma]{\delta}}}^{\infty} \frac{1}{x}\left(1+\sqrt[\gamma]{\delta}x\right)^{\gamma-r}dx =$$

$$= \delta^{\frac{r}{\gamma}-1}\int\limits_{\sqrt[\gamma]{\delta}}^{\infty} \frac{y^{\gamma-r}}{y-1}dy = \delta^{\frac{r}{\gamma}-1}\int\limits_{\sqrt[\gamma]{\delta}}^{\infty} y^{\gamma-1-r}\left(1+\frac{1}{y-1}\right)dy \leq$$

$$\leq \left(1+\frac{1}{\sqrt[\gamma]{\delta}-1}\right)\delta^{\frac{r}{\gamma}-1}\int\limits_{\sqrt[\gamma]{\delta}}^{\infty} y^{\gamma-1-r}dy \leq K_{17}\delta^{\frac{r}{\gamma}-1}\int\limits_{\sqrt[\gamma]{\delta}}^{\infty} y^{\gamma-1-r}dy = K_{18}. \tag{35}$$

From (27), taking into account (28) and (29)–(35), we can write the estimation

$$\int\limits_{0}^{\sqrt[\gamma]{\delta}}\left|\int\limits_{0}^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt = \begin{cases} O(1), & \gamma<r<2\gamma, \\ O(\ln\delta), & r=2\gamma, \\ O(\delta^{\frac{r}{\gamma}-2}), & r>2\gamma. \end{cases} \tag{36}$$

Similarly, we can show that

$$\int\limits_{-\sqrt[\gamma]{\delta}}^{0}\left|\int\limits_{0}^{\infty}\mu(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du\right|dt = \begin{cases} O(1), & \gamma<r<2\gamma, \\ O(\ln\delta), & r=2\gamma, \\ O(\delta^{\frac{r}{\gamma}-2}), & r>2\gamma. \end{cases} \tag{37}$$

Combining formulas (26), (36) and (37), we obtain

$$A(\mu) = \begin{cases} O(1), & \gamma<r<2\gamma, \\ O(\ln\delta), & r=2\gamma, \\ O(\delta^{\frac{r}{\gamma}-2}), & r>2\gamma. \end{cases} \tag{38}$$

Similarly to [27] we can show that the following equality holds

$$f(x)-P_\gamma(\delta,f,x) = \frac{1}{(\sqrt[\gamma]{\delta})^r}\int\limits_{-\infty}^{\infty} f_\beta^\psi\left(x+\frac{t}{\sqrt[\gamma]{\delta}}\right)\hat{\tau}_\beta(t)dt,$$

where

$$\hat{\tau}_\beta(t) = \hat{\tau}(t,\beta) = \frac{1}{\pi}\int\limits_{0}^{\infty}\tau(u)\cos\left(ut+\frac{\beta\pi}{2}\right)du.$$

Thence

$$\mathcal{E}\left(W^r_{\beta,\infty};P_\gamma(\delta)\right)_C = \sup_{f\in W^r_{\beta,\infty}}\left\|\frac{1}{(\sqrt[\gamma]{\delta})^r}\int\limits_{-\infty}^{\infty} f_\beta^r\left(x+\frac{t}{\sqrt[\gamma]{\delta}}\right)\hat{\tau}_\beta(t)dt\right\|_C =$$

$$= \sup_{f\in W^r_{\beta,\infty}}\left\|\frac{1}{(\sqrt[\gamma]{\delta})^r}\int\limits_{-\infty}^{\infty} f_\beta^r\left(x+\frac{t}{\sqrt[\gamma]{\delta}}\right)(\hat{\varphi}_\beta(t)+\hat{\mu}_\beta(t))dt\right\|_C \leq$$

$$\leq \sup_{f\in W^r_{\beta,\infty}}\left\|\frac{1}{(\sqrt[\gamma]{\delta})^r}\int\limits_{-\infty}^{\infty} f_\beta^r\left(x+\frac{t}{\sqrt[\gamma]{\delta}}\right)\hat{\varphi}_\beta(t)dt\right\|_C + \frac{1}{(\sqrt[\gamma]{\delta})^r}\int\limits_{-\infty}^{\infty}|\hat{\mu}_\beta(t)|dt.$$

Therefore,

$$\mathcal{E}\left(W^r_{\beta,\infty};P_\gamma(\delta)\right)_C = \sup_{f\in W^r_{\beta,\infty}}\left\|\frac{1}{(\sqrt[\gamma]{\delta})^r}\int\limits_{-\infty}^{\infty} f_\beta^r\left(x+\frac{t}{\sqrt[\gamma]{\delta}}\right)\hat{\varphi}_\beta(t)dt\right\|_C + O\left(\frac{1}{(\sqrt[\gamma]{\delta})^r}A(\mu)\right). \tag{39}$$

Similarly to the work [28] , we can show that the Fourier series of the function $f_\varphi(x) = \int\limits_{-\infty}^{\infty} f_\beta^r\Big(x + \frac{t}{\sqrt[\gamma]{\delta}}\Big)\hat{\varphi}_\beta(t)dt$ has the form:

$$S[f_\varphi] = \sum_{k=1}^{\infty} \frac{k^\gamma}{(\sqrt[\gamma]{\delta})^{\gamma-r}}(a_k \cos kx + b_k \sin kx),$$

where a_k, b_k are the Fourier coefficients of the function f. Therefore

$$\int\limits_{-\infty}^{\infty} f_\beta^r\left(x + \frac{t}{\sqrt[\gamma]{\delta}}\right)\hat{\varphi}_\beta(t)dt = \frac{1}{(\sqrt[\gamma]{\delta})^{\gamma-r}} f_0^\gamma(x), \tag{40}$$

where $f_0^\gamma(x)$ is (r,β)—derivative in the Weyl-Nagy sense for $r = \gamma,\ \beta = 0$.

Substituting (40) into (39), we obtain

$$\mathcal{E}\left(W_{\beta,\infty}^r; P_\gamma(\delta)\right)_C = \frac{1}{\delta} \sup_{f \in W_{\beta,\infty}^r} \|f_0^\gamma(\cdot)\|_C + O\left(\frac{1}{(\sqrt[\gamma]{\delta})^r} A(\mu)\right),\ \ \delta \to \infty. \tag{41}$$

Substituting (38) into (41), we obtain the equation (5). The theorem is proved. □

3. Conclusions

One of the extremal problems of approximation theory, namely the problem of studying the asymptotic properties of linear summation methods of Fourier series, has been considered in the paper. Among the linear summation methods, on the one hand, there are methods that are defined by infinite numerical matrices, and on the other hand, methods that are defined by the set of functions of the natural argument that depend on the real parameter δ. This work is devoted to the study of the approximation properties of the methods of the last type, namely, generalized Poisson integrals. The Kolmogorov–Nikol'skii problem takes a special place among the extremal problems of the approximation theory. We have considered the problem of asymptotic equalities finding for the value of the exact upper limits of deviations of generalized Abel-Poisson integrals from functions of the Weyl-Nagy classes in the uniform metric. In particular, the asymptotic equality (5) for arbitrary $r > \gamma,\ 0 < \gamma \leq 2$, has been written in the paper, providing the solution of the corresponding Kolmogorov–Nikol'skii problem. The importance of this type of problems in the theory of decision making, in signal transmission, in the study of mathematical models and in the coding and reproduction of images has been noted. Regarding further research in this direction, we note that similar problems can be considered in the broader classes of functions, such as Stepanets classes and classes of non-periodic locally summable functions.

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