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Nearly Cosymplectic Manifolds of Constant Type

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Abstract: Fundamental identities characterizing a nearly cosymplectic structure and analytical expressions for the first and second structural tensors are obtained in this paper. An identity that is satisfied by the first structural tensor of a nearly cosymplectic structure is proved as well. A contact analog of nearly cosymplectic manifolds' constancy of type is introduced in this paper. Pointwise constancy conditions of the type of nearly cosymplectic manifolds are obtained. It is proved that for nearly cosymplectic manifolds of dimension greater than three, pointwise constancy of type is equivalent to global constancy of type. A complete classification of nearly cosymplectic manifolds of constant type is obtained. It is also proved that a nearly cosymplectic manifold of dimension less than seven is a proper nearly cosymplectic manifold.

Keywords: nearly cosymplectic structure; nearly Kähler structure; type constancy; cosymplectic manifold; structure tensor

MSC: 53C21; 53C25; 53D15

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Citation: Rustanov, A. Nearly Cosymplectic Manifolds of Constant Type. *Axioms* **2022**, *11*, 152. https://doi.org/10.3390/axioms11040152

Academic Editor: Anna Maria Fino

Received: 15 February 2022
Accepted: 21 March 2022
Published: 25 March 2022

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1. Introduction

The concept of a nearly cosymplectic structure is one of the most interesting generalizations of the concept of cosymplectic structure and is a contact analog of the concept of an approximate Kähler structure in Hermitian geometry.

It seems that nearly cosymplectic structures were first considered by H. Proppe [1]. He showed that the structure tensors Φ and η of the induced almost contact metric structure $S = (\Phi, \xi, \eta, g)$ on totally geodesic hypersurfaces of nearly Kähler manifolds are Killing tensors, and the second fundamental form of the structure Ω is proportional to $\eta \otimes \eta$. In [2], Blair proved that Φ is Killing if and only if the second fundamental form of the structure Ω is proportional to $\eta \otimes \eta$.

Blair also showed that if Ω is proportional to $\eta \otimes \eta$, then η is Killing. Almost contact metric structures whose structural tensors Φ and η are Killing were called nearly cosymplectic ones by Blair D. E. [2]. Blair gave an example of a nearly cosymplectic structure which differs from cosymplectic structure on a five-dimensional sphere S^5 embedded in a six-dimensional sphere $S^6 \subset O$ as a totally geodesic submanifold [2]. Blair and Showers [3] introduced the notion of a closely cosymplectic structure, as a nearly cosympectic one with a closed contact form η, and studied these structures from a topological point of view. In particular, it was shown that every five-dimensional closest cosymplectic manifold is cosymplectic. It was also proved that there are no nearly cosymplectic structures that are contact metric structures. The above example of a nearly cosymplectic structure on a five-dimensional sphere is neither a cosymplectic nor closest cosymplectic one (we get back to this example in our further discussion).

In [4], Kirichenko V. F. introduced the concept of a generalized nearly cosymplectic structure including the classical case of a nearly cosymplectic structure (with a Riemannian metric). A local structure of a pseudosplit reductive generalized nearly cosymplectic manifold was obtained. In the case of a classical nearly cosymplectic, closest cosymplectic and cosymplectic structure, the result obtained was expressed as in [5].

Any nearly cosymplectic manifold M is locally equivalent either to the product of a nearly Kähler manifold and the real line, or to the product of a nearly Kähler manifold and a five-dimensional sphere provided with a canonical nearly cosymplectic structure. If M is simply connected, then these local equivalences can be chosen as global ones.

Every closest cosymplectic manifold is locally equivalent to the product of a nearly Kähler manifold and the real line. If M is simply connected, then these local equivalences can be chosen to be global ones.

Every cosymplectic manifold is locally equivalent to the product of a Kähler manifold and a real line. If M is simply connected, then these local equivalences can be chosen to be global ones.

The work in [6] proves that a pseudosplit, reductive, generalized, nearly cosymplectic manifold satisfies the axiom of Φ-holomorphic planes if and only if it is a generalized, nearly cosymplectic manifold (a generalized cosymplectic manifold in the case of $m > 1$) of pointwise constant Φ-holomorphic sectional curvature of a dimension surpassing three. Any such manifold is locally equivalent to one of the following manifolds provided with a canonical, nearly cosymplectic structure of classical or hyperbolic type: 1. $C_k^n \times R^1$; 2. $CP_k^n \times R^1$; 3. $CD_k^n \times R^1$; 4. $(R^m \ltimes R^m) \times R^1$; 5. $(RP^n \div RP^n) \times R^1$; 6. $C_k^n \times (R^m \ltimes R^m) \times R^1$; 7. $S_k^6 \times R^1$; 8. S_k^5. If $m > 1$, cases 7 and 8 are excluded. In the case of completeness, simply connectedness, and sign-definiteness of the metric of the manifold, these equivalences are defined globally. Here, the symbol $RP^n \div RP^n$ denotes the variety of zero-pairs of the real projective space RP^n, and the symbol $R^m \ltimes R^m$ denotes the double Euclidean space equipped with the canonical para-Kähler structure [6].

We also highlight the works of Smolensk geometers Banaru M. B. and Stepanova L. V., in which there is a connection between nearly cosymplectic and nearly Kählerian manifolds. In [7], Banaru M. B. proved that if $(N, \{\Phi, \xi, \eta, g\})$ is a nearly cosymplectic hypersurface of the approximate Kähler manifold M^{2n}, σ is the second quadratic form of the immersion of N in M^{2n}, then N is minimal if and only if $\sigma(\xi, \xi) = 0$. In addition, it was established in [7] that for a nearly cosymplectic hypersurface N of the nearly Kähler manifold M^{2n}, the following statements are equivalent: (1) N is a minimal submanifold of the manifold M^{2n}; (2) N is a geodesic submanifold of the manifold M^{2n}; (3) the type number of the hypersurface N is identically equal to zero. It was proved in [8] that the type number of a nearly cosymplectic hypersurface in an almost Kähler manifold is not greater than one. It was also proved that such a hypersurface is minimal if and only if it is completely geodesic. Almost contact metric structures on hypersurfaces of quasi-Kähler manifolds were considered by Stepanova and Banaru [9].

It was proved that if a η-quasi-umbilical quasi-Sasakian hypersurface passes through every point of a quasi-Kähler manifold M, then M is a Kähler manifold, and the structure induced on the hypersurface is either cosymplectic or homothetic to the Sasakian structure. It was proved in [10] that for an almost contact metric hypersurface of an nearly Kähler manifold, the condition that its type number is equal to zero or one is not only necessary but also sufficient for this almost contact metric structure to be nearly cosymplectic. Here, it is necessary to point out the works of J. Mikes [11–13]. A more detailed analysis of papers on almost contact manifolds can be found in [13,14].

In her thesis [15], as well as in the series of works [16–21], Kusova E. V. studied aspects of the geometry of nearly cosymplectic manifolds. In particular, contact analogs of Gray's identities were pointed out, additional identities for the Riemannian curvature tensor were obtained, and on their basis, subclasses of nearly cosymplectic manifolds were distinguished and their local structure was studied. On the space of the adjoint G-structure, the components of the Weyl tensor of conformal curvature were calculated and five subclasses of nearly cosymplectic manifolds corresponding to the reduction to zero of the nonzero components of the Weyl tensor were identified. Conformally flat, nearly cosymplectic manifolds were studied. The question of integrability of nearly cosymplectic structures was investigated.

It can be seen from the above analysis that the study of the geometry of cosymplectic manifolds attracts the attention of geometers. However, there are still many problems to be solved. One such problem is the study of the geometry of nearly cosymplectic manifolds of constant type. In this paper, we introduce a contact analog of type constancy for nearly cosymplectic manifolds and study the geometry of this class of manifolds.

2. Fundamental Definitions of a Nearly Cosymplectic Structure and Its Structural Equations

Let M be a smooth manifold of dimension $2n+1$, $\mathcal{X}(M)$ be the C^∞-module of smooth vector fields on the manifold M. In what follows, all manifolds, tensor fields, etc., objects are assumed to be smooth ones belonging to the class C^∞.

An almost contact structure on the manifold M is a triple (η, ξ, Φ) of tensor fields on this manifold, where η is a differential 1-form, called the contact form of the structure, ξ is a vector field, called *the characteristic one* and Φ is an endomorphism of the module $\mathcal{X}(M)$, called a structural endomorphism, where

$$(1)\ \eta(\xi) = 1;\ (2)\ \eta \circ \Phi = 0;\ (3)\ \Phi(\xi) = 0;\ (4)\ \Phi^2 = -id + \eta \otimes \xi. \tag{1}$$

If, in addition, a Riemannian structure $g = \langle \cdot, \cdot \rangle$ is fixed on M such as

$$\langle \Phi X, \Phi Y \rangle = \langle X, Y \rangle - \eta(X)\eta(Y),\ X, Y \in \mathcal{X}(M), \tag{2}$$

the quadruple $(\eta, \xi, \Phi, g = \langle \cdot, \cdot \rangle)$ is called an almost contact metric (in short, AC) structure. A manifold on which an almost contact (metric) structure is fixed is called an almost contact (metric (in short, AC)) manifold [5,22].

The skew-symmetric tensor $\Omega(X, Y) = \langle X, \Phi Y \rangle$, $X, Y \in \mathcal{X}(M)$ is called *the fundamental form of the AC-structure* [5,22].

As is well known [5,22], specifying an almost contact metric structure (η, ξ, Φ, g) on the manifold M^{2n+1} is equivalent to specifying an adjoint G-structure on M^{2n+1} with structure group $\{1\} \times U(n)$. The elements of this G-structure are called A-benchmarks and are characterized by the fact that the matrices of the components of the tensors Φ_p and g_p in the A-benchmark have the form, respectively:

$$(\Phi_i^j) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sqrt{-1}I_n & 0 \\ 0 & 0 & -\sqrt{-1}I_n \end{pmatrix},\ (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}, \tag{3}$$

where I_n is the identity matrix of order n.

Let $(M^{2n+1}, \eta, \xi, \Phi, g = \langle \cdot, \cdot \rangle)$ be an almost contact metric manifold. Let us agree that throughout the whole work, unless otherwise stated, the values of the indices $i, j, k, \ldots$ run from 0 to $2n$, and the indices $a, b, c, \ldots$ run from 1 to n, and let us assign $\hat{a} = a + n$, $\hat{\hat{a}} = a$, $\hat{0} = 0$.

An almost contact metric structure $S = (\eta, \xi, \Phi, g)$ is said to be nearly cosymplectic (in short, (nearly cosymplectic) NC_S) if $\nabla_X(\Phi)X = 0$; $X \in \mathcal{X}(M)$, that is,

$$\nabla_X(\Phi)Y + \nabla_Y(\Phi)X = 0;\ \ X, Y \in \mathcal{X}(M), \tag{4}$$

A nearly cosymplectic structure with a closed contact form is called closely (in short, CC_S) cosymplectic structure. An almost contact metric manifold equipped with a closely cosymplectic structure is called a closely cosymplectic manifold [5].

The first group of structural equations of NC_S-manifolds takes the following shape [5]:

$$\begin{aligned}(1)\ d\omega &= F_{ab}\omega^a \wedge \omega^b + F^{ab}\omega_a \wedge \omega_b;\\ (2)\ d\omega^a &= -\theta^a_b \wedge \omega^b + C^{abc}\omega_b \wedge \omega_c + \tfrac{3}{2}F^{ab}\omega_b \wedge \omega;\\ (3)\ d\omega_a &= \theta^b_a \wedge \omega_b + C_{abc}\omega^b \wedge \omega^c + \tfrac{3}{2}F_{ab}\omega^b \wedge \omega.\end{aligned} \tag{5}$$

where:

$$\begin{gathered}C^{abc} = \frac{\sqrt{-1}}{2}\Phi^a_{\hat{b},\hat{c}};\ C_{abc} = -\frac{\sqrt{-1}}{2}\Phi^{\hat{a}}_{b,c};\ C^{[abc]} = C^{abc};\ C_{[abc]} = C_{abc};\\ \overline{C^{abc}} = C_{abc};\ F^{ab} = \sqrt{-1}\Phi^0_{\hat{a},\hat{b}};\ F_{ab} = -\sqrt{-1}\Phi^0_{a,b};\\ F^{ab} + F^{ba} = 0;\ F_{ab} + F_{ba} = 0;\ \overline{F_{ab}} = F^{ab}.\end{gathered} \tag{6}$$

Definition 1 ([21]). *A nearly cosymplectic structure is called a proper nearly cosymplectic structure if* $C^{abc} = 0$ *and* $F^{ab} \neq 0$.

An example of a proper nearly cosymplectic structure is the nearly cosymplectic structure on the five-dimensional sphere mentioned in the introduction. This structure is neither cosymplectic nor closely cosymplectic.

The standard procedure for the differential continuation of relations (5) allows us to obtain the second group of structural equations of the NC_S structure:

$$d\theta^a_b = -\theta^a_c \wedge \theta^c_b + (A^{ad}_{bc} - 2C^{adh}C_{hbc} - \frac{3}{2}F^{ad}F_{bc})\omega^c \wedge \omega_d, \tag{7}$$

where

$$A^{ad}_{[bc]} = A^{[ad]}_{bc} = 0,\ \overline{A^{ad}_{bc}} = A^{bc}_{ad}. \tag{8}$$

In addition,

$$\begin{aligned}(1)\ dC^{abc} + C^{dbc}\theta^a_d + C^{adc}\theta^b_d + C^{abd}\theta^c_d &= C^{abcd}\omega_d;\\ (2)\ dC_{abc} - C_{dbc}\theta^d_a - C_{adc}\theta^d_b - C_{abd}\theta^d_c &= C_{abcd}\omega^d;\\ (3)\ dF^{ab} + F^{cb}\theta^a_c + F^{ac}\theta^b_c &= 0;\\ (4)\ dF_{ab} - F_{cb}\theta^c_a - F_{ac}\theta^c_b &= 0,\end{aligned} \tag{9}$$

where

$$\begin{gathered}C^{a[bcd]} = \tfrac{3}{2}F^{a[b}F^{cd]},\ C_{a[bcd]} = \tfrac{3}{2}F_{a[b}F_{cd]},\ C^{[abc]d} = C^{abcd},\\ C_{[abc]d} = C_{abcd},\ \overline{C^{abcd}} = C_{abcd},\ F_{ad}C^{dbc} = F^{ad}C_{dbc} = 0.\end{gathered} \tag{10}$$

By differentiating Equation (7) externally, we obtain:

$$dA^{ad}_{bc} + A^{hd}_{bc}\theta^a_h + A^{ah}_{bc}\theta^d_h - A^{ad}_{hc}\theta^h_b - A^{ad}_{bh}\theta^h_c = A^{ad}_{bch}\omega^h + A^{adh}_{bc}\omega_h, \tag{11}$$

where

$$\begin{gathered}(1)\ A^{ad}_{b[ch]} = 0;\ (2)\ A^{a[dh]}_{bc} = 0;\ (3)\ C^{abcg}C_{gdh} = C_{abcg}C^{gdh} = 0;\\ (4)\ C^{abch}F_{hd} = C_{abch}F^{hd} = 0;\ (5)\ \left(A^{ag}_{b[c} - 2C^{agf}C_{fb[c}\right)C_{|g|dh]} = 0;\\ (6)\ \left(A^{a[c}_{bg} - 2C^{a[c|f|}C_{fbg}\right)C^{|g|dh]} = 0;\ (7)\ \left(A^{ah}_{b[c} - \tfrac{3}{2}F^{ah}F_{b[c}\right)F_{|h|d]} = 0;\\ (8)\ \left(A^{a[d}_{bh} - \tfrac{3}{2}F^{a[d}F_{bh}\right)F^{|h|c]} = 0.\end{gathered} \tag{12}$$

The identity $C^{abcg}C_{gdh} = C_{abcg}C^{gdh} = 0$ is called the first fundamental identity of the NC_S-manifold [21].

The identity $\left(A^{ag}_{b[c} - 2C^{agf}C_{fb[c}\right)C_{|g|dh]} = 0$ is called the second fundamental identity of an NC_S-manifold [15].

The identity $\left(A^{ah}_{b[c} - \frac{3}{2}F^{ah}F_{b[c}\right)F_{|h|d]} = 0$ is called the third fundamental identity of the NC_S-manifold.

The identity $F_{ad}C^{dbc} = F^{ad}C_{dbc} = 0$ is called the fourth fundamental identity of the NC_S-manifold.

3. The Structural Tensors of the NC_S-Structure

The system of functions $\left(C^{abc}, C_{abc}\right)$ defines a tensor of type(2,0), called the first structural tensor of the NC_S-manifold and the system of functions $\left(C^{abc}, C_{abc}\right)$ defines a tensor of type (1,0), called the second structural tensor of NC_S-manifold.

Let us obtain an analytic expression for the structural tensors of the NC_S-structure. In [23,24], explicit expressions for structural tensors are given.

Since the first and fourth structural tensors of the AC-structure are equal to zero, the expressions for the structural tensors take the form:

$$(1)\ C(X,Y) = -\tfrac{1}{2}\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi X; \tag{13}$$
$$(2)\ F(X) = \Phi \circ \nabla_{\Phi^2 X}(\Phi)\xi = -\Phi \circ \nabla_X(\Phi)\xi = -\nabla_X \xi;\ \forall X, Y \in \mathcal{X}(M).$$

Theorem 1. *The structural tensors of a nearly cosymplectic structure have the following properties:*

$$(1)\ C(\xi, X) = C(X, \xi) = 0;\ \ (2)\ C(X,Y) = -C(Y,X);$$
$$(3)\ C(\Phi X, Y) = C(X, \Phi Y) = -\Phi \circ C(X,Y);\ \ (4)\ F(\xi) = 0;\ \ (5)\ \langle F(X), Y\rangle = -\langle X, F(Y)\rangle;$$
$$(6)\ \Phi \circ F = -F \circ \Phi;\ \ (7)\ \eta \circ F = 0;\ \ \forall X, Y \in \mathcal{X}(M) \tag{14}$$

Proof. (1) $C(\xi, X) = -\frac{1}{2}\Phi \circ \nabla_\Phi X(\Phi)\Phi\xi = 0$, $C(X,\xi) = -\frac{1}{2}\Phi \circ \nabla_{\Phi\xi}(\Phi)\Phi X = 0$.

(2) From (9), taking into account (1), we have $C(X,Y) = -\frac{1}{2}\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi X = \frac{1}{2}\Phi \circ \nabla_{\Phi X}(\Phi)\Phi Y = -C(Y,X); \forall X, Y \in \mathcal{X}(M)$.

(3) By covariantly differentiating the equality $\Phi^2 = -id + \eta \otimes \xi$, we obtain $\nabla_Y(\Phi)\Phi X + \Phi \circ \nabla_{\Phi Y}(\Phi)X = \xi\nabla_Y(\eta)X + \eta(X)\nabla_Y\xi$.

In the last equality, first we make the replacement $X \to \Phi X$, then we act on the resulting identity with the operator Φ^2, and finally we obtain $\Phi \circ \nabla_Y(\Phi)\Phi X = \Phi^2 \circ \nabla_Y(\Phi)\Phi^2 X$. In the obtained set, we make a replacement $Y \to \Phi Y$, then

$$\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi X = \Phi^2 \circ \nabla_{\Phi Y}(\Phi)\Phi^2 X;\ \ \forall X, Y \in \mathcal{X}(M). \tag{15}$$

In the resulting identity, we make the replacement $X \longrightarrow \Phi X$, then we get $\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi^2 X = -\Phi^2 \circ \nabla_{\Phi Y}(\Phi)\Phi X; \forall X, Y \in \mathcal{X}(M)$. Taking into account the obtained result, from (9), taking into account (15), we obtain $\Phi \circ C(X,Y) = -\frac{1}{2}\Phi^2 \circ \nabla_{\Phi Y}(\Phi)\Phi X = \frac{1}{2}\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi^2 X = -C(\Phi X, Y)$ and $C(\Phi X, \Phi) = -\frac{1}{2}\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi^2 X = \frac{1}{2}\Phi \circ \nabla_{\Phi^2 X}(\Phi)\Phi Y = \frac{1}{2}\Phi^2 \circ \nabla_{\Phi^2 X}(\Phi)\Phi^2 Y = -\frac{1}{2}\Phi^2 \circ \nabla_{\Phi^2 Y}(\Phi)\Phi^2 X = -\frac{1}{2}\Phi \circ \nabla_{\Phi^2 Y}(\Phi)\Phi X = C(X, \Phi Y)$.

(4) Since $\Phi(\xi) = 0$, then $F(\xi) = \nabla_{\Phi\xi}(\Phi)\xi = 0$.

(5) $\langle F(X), Y\rangle = -\langle \nabla_X\xi, Y\rangle = -\nabla_X(\eta)Y = \nabla_Y(\eta)X = \langle \nabla_Y\xi, X\rangle = -\langle X, F(Y)\rangle$.

(6) From the analytical expression of the second structural tensor we have $\Phi \circ F(X) = -\Phi \circ \nabla_X\xi = \nabla_{\Phi X}\xi = -(F \circ \Phi)(X); \forall X \in \mathcal{X}(M)$, that is, $\Phi \circ F = -F \circ \Phi$.

(7) Since the identities $\eta \circ \Phi = 0$ and $\eta(\nabla_X\xi) = 0$ hold on an almost contact metric manifold, then $\eta(F(X)) = \eta(\nabla_{\Phi X}(\Phi)\xi) = 0$, i.e., $\eta \circ F = 0$. □

4. Q-Algebras of Nearly Cosymplectic Manifolds

In this section, we consider a Q-algebra attached to a nearly cosymplectic manifold.

Definition 2 ([25,26]). *A Q-algebra is a triple $\{V, \langle\!\langle\cdot,\cdot\rangle\!\rangle, *\}$, where V is a module over a commutative associative ring* $\mathbf{K}$ *with a (nontrivial) involution; $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ is a nondegenerate Hermitian form on V; "$*$" is a binary operation $* : V \times V \longrightarrow V$, antilinear in each argument, for which the Q-algebra axiom $\langle\!\langle X * Y, Z\rangle\!\rangle + \overline{\langle\!\langle Y, X * Z\rangle\!\rangle} = 0$, $X, Y, Z \in V$ is completed.*

If $\mathbf{K} = \mathbf{C}$, then the Q-algebra V is called complex.

Definition 3 ([25]). *The Q-algebra V is called:*
- Abelian or commutative Q-algebra if $X * Y = 0$, $(X, Y \in V)$;
- K-algebra, or anticommutative Q-algebra, if $X * Y = -Y * X$, $(X, Y \in V)$;
- A-algebra, or pseudocommutative Q-algebra, if $\langle X * Y, Z\rangle + \langle Y * Z, X\rangle + \langle Z * X, Y\rangle = 0$, $(X, Y, Z \in V)$.

Let us recall from [27] that the module $\mathcal{X}(M)$ of an almost contact metric manifold naturally introduces the structure of a Q-algebra $\Re$ over the ring of complex-valued smooth functions with the operation

$$X * Y = T(X, Y) = \frac{1}{4}\Big\{\Phi\nabla_{\Phi X}(\Phi)\Phi Y - \Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y\Big\};\ X, Y \in \mathcal{X}(M) \tag{16}$$

and metric

$$\langle\!\langle X, Y\rangle\!\rangle = \langle X, Y\rangle + \sqrt{-1}\langle X, \Phi Y\rangle;\ \ X, Y \in \mathcal{X}(M). \tag{17}$$

This Q-algebra is called an ***adjoint one***.

Let M be an NC_S-manifold. In the $C^\infty(M)$-module $\mathcal{X}(M)$ of smooth vector fields of the manifold M, the binary operation "$*$" is introduced by the formula $X * Y = T(X, Y) = \frac{1}{4}\{\Phi\nabla_{\Phi X}(\Phi)\Phi Y - \Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y\}$; $X, Y \in \mathcal{X}(M)$.

Theorem 2 ([25]). *The NC_S-structure has an anticommutative adjoint Q-algebra, i.e., K-algebra.*

Proof. From (4) we can easily derive that
$X * Y = T(X, Y) = \frac{1}{4}\{\Phi\nabla_{\Phi X}(\Phi)\Phi Y - \Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y\}$;
$X, Y \in \mathcal{X}(M)$, that is, $\Phi\nabla_{\Phi X}(\Phi)\Phi Y = -\Phi\nabla_{\Phi Y}(\Phi)\Phi X$; $X, Y \in \mathcal{X}(M)$. This means that $\Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y = -\Phi\nabla_{\Phi^2 Y}(\Phi)\Phi^2 X$; $X, Y \in \mathcal{X}(M)$. Then, $T(X, Y) = \frac{1}{4}\{\Phi\nabla_{\Phi X}(\Phi)\Phi Y - \Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y\} = -\frac{1}{4}\{\Phi\nabla_{\Phi Y}(\Phi)\Phi X - \Phi\nabla_{\Phi^2 Y}(\Phi)\Phi^2 X\} = -T(Y, X)$; $X, Y \in \mathcal{X}(M)$. The associated Q-algebra of an NC_S-structure is a K-algebra. □

Corollary 1. *CC_S-manifolds have an Abelian associated Q-algebra.*

Theorem 3. *The first structural tensor of the NC_S-structure satisfies the identity:*

$$\langle\!\langle C(X, Y), Z\rangle\!\rangle + \overline{\langle\!\langle Y, C(X, Z)\rangle\!\rangle} = 0; \tag{18}$$

Proof. Considering (9) and (17), we obtain

$$\begin{aligned}
\langle\!\langle C(X, Y), Z\rangle\!\rangle &= \langle C(X, Y), Z\rangle + \sqrt{-1}\langle C(X, Y), \Phi Z\rangle = \\
&= \Big\langle -\tfrac{1}{2}\Phi\circ\nabla_{\Phi Y}(\Phi)\Phi X, Z\Big\rangle + \sqrt{-1}\Big\langle -\tfrac{1}{2}\Phi\circ\nabla_{\Phi Y}(\Phi)\Phi X, \Phi Z\Big\rangle = \\
&= -\tfrac{1}{2}\langle \Phi X, \nabla_{\Phi Y}(\Phi)\Phi Z\rangle + \tfrac{1}{2}\sqrt{-1}\langle \Phi X, \nabla_{\Phi Y}(\Phi)\Phi^2 Z\rangle = \\
&= \tfrac{1}{2}\langle X, \Phi\circ\nabla_{\Phi Y}(\Phi)\Phi Z\rangle - \tfrac{1}{2}\sqrt{-1}\langle X, \Phi\circ\nabla_{\Phi Y}(\Phi)\Phi^2 Z\rangle = \\
&= \tfrac{1}{2}\langle X, \Phi\circ\nabla_{\Phi Y}(\Phi)\Phi Z\rangle - \tfrac{1}{2}\sqrt{-1}\langle X, \Phi^2\circ\nabla_{\Phi Y}(\Phi)\Phi Z\rangle = \\
&= -\langle X, C(Y, Z)\rangle - \sqrt{-1}\langle X, \Phi C(Y, Z)\rangle = -\overline{\langle\!\langle X, C(Y, Z)\rangle\!\rangle}.
\end{aligned}$$

□

5. Nearly Cosymplectic Manifolds of Constant Type

In this section, we consider the contact analog of type constancy and study it in detail for nearly cosymplectic manifolds.

Definition 4 ([28]). *A complex K-algebra $\Re$ is called a K-algebra of constant type if* $\exists c \in C$ $\forall X, Y \in \Re : \langle\langle X, Y\rangle\rangle = 0 \Longrightarrow \|X * Y\|^2 = c\|X\|^2\|Y\|^2$.

Definition 5. *An NC_S-manifold M is called a manifold of pointwise constant type if its associated K-algebra has a constant type at every point of the manifold M. The function c, if it exists, is called a constant of the type of the NC_S-manifold. If $c = const$, then M is called an NC_S-manifold of globally constant type.*

Theorem 4. *An NC_S-manifold M is a manifold of pointwise constant type c if and only if*

$$\forall X, Y \in \mathcal{X}(M) \langle\langle X, Y\rangle\rangle = 0 \Longrightarrow \|C(X, Y)\|^2 = c\|X\|^2\|Y\|^2. \tag{19}$$

Proof. Let M be an NC_S-manifold. Let us consider a Q-algebra $\Re$ attached to a manifold M with the operation $* : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$ defined by the identity $X * Y = \frac{1}{4}\{\Phi\nabla_{\Phi X}(\Phi)\Phi Y - \Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y\}$; $X, Y \in \mathcal{X}(M)$. It follows from (1) that on an NC_S-manifold $X * Y = \frac{1}{4}\{\Phi\nabla_{\Phi X}(\Phi)\Phi Y - \Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y\}$; $X, Y \in \mathcal{X}(M)$. Therefore $X * Y = \frac{1}{2}\Phi\nabla_{\Phi X}(\Phi)\Phi Y = -\frac{1}{2}\Phi\nabla_{\Phi^2 X}(\Phi)\Phi^2 Y$; $X, Y \in \mathcal{X}(M)$. Due to (1) of (13) and (16) we have $C(X, Y) = -\frac{1}{2}\Phi \circ \nabla_{\Phi Y}(\Phi)\Phi X = \frac{1}{2}\Phi \circ \nabla_{\Phi X}(\Phi)\Phi Y = X * Y$; $X, Y \in \mathcal{X}(M)$. Due to (17), the condition $\langle X, Y\rangle = \langle X, \Phi Y\rangle = 0$ is equivalent to the condition $\langle\langle X, Y\rangle\rangle = 0$. □

Theorem 5. *An NC_S-manifold is a manifold of point constant type c if and only if*

$$\mathcal{C}(X, Y, Z, W) = \langle\langle C(X, Y), C(Z, W)\rangle\rangle = c\{\langle\langle W, Y\rangle\rangle\langle\langle Z, X\rangle\rangle - \langle\langle W, X\rangle\rangle\langle\langle Z, Y\rangle\rangle\}.$$

Proof. Let us introduce the four-form $\mathcal{C}(X, Y, Z, W) = \langle\langle X * Y, Z * W\rangle\rangle = \langle\langle C(X, Y), C(Z, W)\rangle\rangle$ It is directly verified that it possesses the properties:

(1) Antilinearity in the first pair of arguments $\sqrt{-1}\mathcal{C}(X, Y, Z, W) = -\mathcal{C}(\Phi X, Y, Z, W) = -\mathcal{C}(X, \Phi Y, Z, W)$.

(2) Linearity in the second pair of arguments $\sqrt{-1}\mathcal{C}(X, Y, Z, W) = \mathcal{C}(X, Y, \Phi Z, W) = -\mathcal{C}(X, Y, Z, \Phi W)$.

(3) Skew symmetry in the first and second pairs of arguments $\mathcal{C}(X, Y, Z, W) = -\mathcal{C}(Y, X, Z, W) = -\mathcal{C}(X, Y, W, Z)$.

(4) $\mathcal{C}(X, Y, Z, W) = \overline{\mathcal{C}(Z, W, X, Y)}$, $X, Y, Z, W \in \mathcal{X}(M)$ is Hermitian.

Since $\mathcal{C}(X, Y, X, Y) = \langle\langle X * Y, X * Y\rangle\rangle = \langle\langle C(X, Y), C(X, Y)\rangle\rangle = \|C(X, Y)\|^2$, then a GK-manifold M is of pointwise constant type c if and only if

$$\mathcal{C}(X, Y, X, Y) = c\|X\|^2\|Y\|^2, \; X, Y \in \mathcal{X}(M), \; \langle\langle X, Y\rangle\rangle = 0. \tag{20}$$

We polarize this identity by replacing Y by $Y + Z$, where $Z \in \mathcal{X}(M)$, $\langle\langle X, Z\rangle\rangle = 0 : \mathcal{C}(X, Y + Z, X, Y + Z) = c\|X\|^2\|Y + Z\|^2$. Expanding it by linearity and making the necessary reductions, by taking into account (20), we obtain

$$\mathcal{C}(X, Y, X, Z) + \mathcal{C}(X, Z, X, Y) = c\|X\|^2(\langle\langle Y, Z\rangle\rangle + \langle\langle Z, Y\rangle\rangle). \tag{21}$$

Replacing Z here by ΦZ, and taking into account property (1) and the nondegeneracy of the endomorphism Φ, we obtain:

$$\mathcal{C}(X, Y, X, Z) + \mathcal{C}(X, Z, X, Y) = c\|X\|^2(-\langle\langle Y, Z\rangle\rangle + \langle\langle Z, Y\rangle\rangle). \tag{22}$$

By adding in an element-wise manner (21) to (22) we obtain:

$$\mathcal{C}(X,Y,X,Z) = c\|X\|^2 \langle\langle Z,Y\rangle\rangle. \tag{23}$$

Let now $Y, Z \in \mathcal{X}(M)$ be arbitrary vectors. Let us expand them in terms of the linear shell of the vector X and its orthogonal complement: $Y = \frac{\langle\langle Y,X\rangle\rangle}{\|X\|^2}X + Y'$; $Z = \frac{\langle\langle Z,X\rangle\rangle}{\|X\|^2}X + Z'$. Taking into account (23) and properties (3), after the necessary reductions we obtain: $\mathcal{C}(X,Y,X,Z) = \mathcal{C}(X,Y',X,Z') = c\|X\|^2\langle\langle Z',Y'\rangle\rangle = c\|X\|^2\left\langle\left\langle Z - \frac{\langle\langle Z,X\rangle\rangle}{\|X\|^2}X, Y - \frac{\langle\langle Y,X\rangle\rangle}{\|X\|^2}X\right\rangle\right\rangle = c\left\{\langle\langle Z,Y\rangle\rangle\|X\|^2 - \langle\langle Z,X\rangle\rangle\langle\langle X,Y\rangle\rangle\right\}$.

As a result:

$$\mathcal{C}(X,Y,X,Z) = c\left\{\langle\langle Z,Y\rangle\rangle\|X\|^2 - \langle\langle Z,X\rangle\rangle\langle\langle Z,X\rangle\rangle\right\}. \tag{24}$$

Let us replace Z by W in the resulting equality, then we get $\mathcal{C}(X,Y,X,W) = c\left\{\langle\langle W,Y\rangle\rangle\|X\|^2 - \langle\langle W,X\rangle\rangle\langle\langle X,Y\}\right\rangle$. In the last identity, we replace X by $X+Z$, and after expansion by linearity and the necessary reductions, taking into account (24), we obtain:

$$\mathcal{C}(X,Y,X,Z) = c\{\langle\langle W,Y\rangle\rangle\langle\langle W,Y\rangle\rangle - \langle\langle Z,X\rangle\rangle - \langle\langle W,X\rangle\rangle\langle\langle Z,Y\rangle\rangle\}. \tag{25}$$

Conversely, it is obvious that, on the basis of (25), (20) is fulfilled and hence M is an NC_S-manifold of pointwise constant type c. □

Theorem 6. *Let M be an NC_S-manifold. Then, the following statements are equivalent:*
(1) M is an NC_S-manifold of pointwise constant type c.
(2) The first structure tensor of an NC_S-manifold satisfies the identity

$$\langle\langle C(X,Y), C(Z,W)\rangle\rangle = c\{\langle\langle W,Y\rangle\rangle\langle\langle Z,X\rangle\rangle - \langle\langle W,X\rangle\rangle\langle\langle Z,Y\rangle\rangle\}.$$

(3) On the space of the associated G-structure, the following relation is fair

$$C^{abh}C_{hcd} = \frac{c}{2}\delta^{ab}_{cd}. \tag{26}$$

Proof. The equivalence of the first and second statements was proved in the previous theorem.

Let us find a representation of the identity (25) on the space of the associated G-structure. We fix a point $p \in M$, an orthonormal frame $r = (p, e_1, \ldots, e_n)$ of the space $T_p(M)$ considered as a C-module and the corresponding A-frame $r = (p, \epsilon_0, \epsilon_1, \ldots, \epsilon_n, \epsilon_{\hat{1}}, \ldots, \epsilon_{\hat{n}})$, where $\epsilon_a = \sqrt{2}\sigma_p(e_a)$, $\epsilon_{\hat{a}} = \sqrt{2}\overline{\sigma}_p(e_a)$, $\epsilon_0 = \xi_p$. Setting (25) as $X = e_a$, $Y = e_b$, $Z = e_c$, $W = e_d$ we obtain the equivalent (at the point p) identity

$$\mathcal{C}(e_a,e_b,e_c,e_d) = \langle\langle C(e_a,e_b), C(e_c,e_d)\rangle\rangle = c\{\langle\langle e_d,e_b\rangle\rangle\langle\langle e_c,e_a\rangle\rangle - \langle\langle e_d,e_a\rangle\rangle\langle\langle e_c,e_b\rangle\rangle\}. \tag{27}$$

Since $\langle\langle e_a,e_b\rangle\rangle = \langle\sigma e_a, \sigma e_b\rangle + \sqrt{-1}\langle\sigma e_a,\sigma e_b\rangle = 2\langle\sigma e_a, \overline{\sigma} e_b\rangle = \left\langle \epsilon_a, \epsilon^b\right\rangle = \delta^b_a$ which also follows from benchmark orthonormality $r = (p, e_1, \ldots, e_n)$. Since $C^{abc} = \frac{1}{2}C(\epsilon_{\hat{b}}, \epsilon_{\hat{c}})^a$, then $\langle\langle C(e_a,e_b), C(e_c,e_d)\rangle\rangle = \langle C(e_a,e_b), C(e_c,e_d)\rangle + \sqrt{-1}\langle C(e_a,e_b), C(e_c,e_d)\rangle = 2\langle\sigma C(e_a,e_b), \overline{\sigma}C(e_c,e_d)\rangle = 2\langle C(\overline{\sigma}e_a,\overline{\sigma}e_b), C(\sigma e_c,\sigma e_d)\rangle = \frac{1}{2}\langle C(\epsilon_{\hat{a}},\epsilon_{\hat{b}}), C(\epsilon_c,\epsilon_d)\rangle = 2\left\langle C^{hab}\epsilon_h, C_{gcd}\epsilon^g\right\rangle = 2C^{hab}C_{gcd}\langle\epsilon_h,\epsilon^g\rangle = 2C^{hab}C_{gcd} = 2C^{abh}C_{hcd}$. Then relations (26) are written down in the form of $C^{abh}C_{hcd} = \frac{c}{2}\delta^{ab}_{cd}$, where $\delta^{ab}_{cd} = \delta^a_c\delta^b_d - \delta^d_c\delta^a_b$ is a second-order Kronecker delta. □

Theorem 7. *The point constancy of the type of a connected NC_S-manifold of dimension greater than three is equivalent to the global constancy of its type.*

Proof. We differentiate the identity (26) externally in an element-wise way: $dC^{abh}C_{hcd} + C^{abh}dC_{hcd} = \frac{1}{2}\delta^{ab}_{cd}dc$. Taking into account the structural equations of NC_S-manifolds, we have $(-C^{gbh}\theta^a_g - C^{agh}\theta^b_g - C^{abg}\theta^h_g + C^{abhg}\omega_g)C_{hcd} + C^{abh}(C_{gcd}\theta^g_h + C_{hgd}\theta^g_c + C_{hcg}\theta^g_d + C_{hcdg}\omega^g) = \frac{1}{2}\delta^{ab}_{cd}dc$. Opening the brackets and bringing similar terms together, taking into account (26), (1) of (9), (2) of (9), (3) of (12), we obtain $\frac{1}{2}\delta^{ab}_{cd}dc = 0$. Therefore, if $dimM > 3$, then $dc = 0$, and if M is connected, then $c = const$. □

Theorem 8. *The class of zero constant-type NC_S-manifolds coincides with the class of proper nearly cosymplectic manifolds or with the class of cosymplectic manifolds.*

Proof. We perform a complete convolution (26):

$$\sum_{abc}\left|C^{abc}\right|^2 = C^{abc}C_{abc} = \frac{c}{2}n(n-1), \tag{28}$$

where $2n+1 = dimM$. This implies that $c \geq 0$, and $c = 0 \to C^{abc} = 0$, i.e., M is either a proper nearly cosymplectic manifold or a cosymplectic manifold. □

Theorem 9. *The class of NC_S-manifolds of nonzero constant type coincides with the class of seven-dimensional proper nearly cosymplectic manifolds.*

Proof. It remains for us to study NC_S-manifolds of nonzero constant type. Let M^{2n+1} be an NC_S-manifold of nonzero constant type c. Let us consider the expanded notation of the second fundamental identity, taking into account (26)

$$H^{ad}_{bc}C_{gfd} + H^{ad}_{bg}C_{fcd} + H^{ad}_{bf}C_{cgd} = 0, \tag{29}$$

where $H^{ag}_{bc} = A^{ad}_{bc} - c\delta^{ad}_{bc}$. We convolve Equality (29) with the object C^{gcf}, taking into account (26) and $c \neq 0$ we get:

$$A^{ac}_{bc} = c(n-1)\delta^a_b. \tag{30}$$

Further, convolving (29) with the object C^{gch} and taking into account (26) and (30), we obtain:

$$(3-n)(A^{ad}_{bc} - c\delta^{ad}_{bc}) = 0. \tag{31}$$

For $n = 3$, this relation is fulfilled identically. If $n \neq 3$, it follows from (31) that $A^{ad}_{bc} = c\delta^{ad}_{bc}$. Alternating this relation with respect to the indices a and d, we obtain that $c\delta^{ad}_{bc} = 0$. Furthermore, after a complete convolution of the resulting equality, we get: $cn(n-1) = 0$, i.e., $c = 0$, which contradicts the condition of nonzero type constancy. Hence, this case is impossible, i.e., $n = 3$, and $dimM = 7$.

Conversely, let $n = 3$. Then, the eigendistributions of the endomorphism Φ are three-dimensional, which means that in each A-frame $C_{abc} = C\epsilon_{abc}$, $C^{abc} = \bar{C}\epsilon^{abc}$, where ϵ_{abc} and ϵ^{abc} are the Kronecker symbols, $\epsilon_{abc} = (\omega^1 \wedge \omega^2 \wedge \omega^3)_{abc}$, $\epsilon^{abc} = (\omega_1 \wedge \omega_2 \wedge \omega_3)^{abc}$, where $\omega^a, \omega_a = \omega^{\hat{a}}$ are the elements of the dual cobenchmark. Bearing this in mind, $\omega^a, \omega_a = \omega^{\hat{a}}$. On the basis of Theorem 6, M is an NC_S-manifold of constant type $c = |C|^2$. □

Let us consider the three-form $C(X,Y,Z) = \langle\langle X, C(Y,Z)\rangle\rangle$, $X,Y,Z \in \mathcal{X}(M)$. Since the first structure tensor of the NC_S-structure is skew-symmetric, this form is a skew-symmetric three-form, C-linear in each argument. Therefore, it is identically equal to zero if $dimM < 7$. Since the Hermitian metric $\langle\langle\cdot,\cdot\rangle\rangle$ is nondegenerate, it follows that $C = 0$, and hence M is a proper nearly cosymplectic manifold. Thus, we get the following theorem.

Theorem 10. *An NC_S-manifold of dimension less than seven is a proper nearly cosymplectic manifold.*

6. Conclusions

In this paper, we have introduced the notion of nearly cosymplectic manifolds of constant type. The introduced notion turned out to be very useful in the study of the geometry of nearly cosymplectic manifolds. Furthermore, we have obtained a complete classification of nearly cosymplectic manifolds of constant type.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The author expresses their sincere gratitude to the referees for their valuable comments, which made it possible to significantly improve this paper.

Conflicts of Interest: The authors declare no conflict of interest.

Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
TLA	Three letter acronym
LD	Linear dichroism

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