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Article

Generalized Vector-Valued Hardy Functions

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Abstract: We consider analytic functions in tubes $\mathbb{R}^n + iB \subset \mathbb{C}^n$ with values in Banach space or Hilbert space. The base of the tube B will be a proper open connected subset of $\mathbb{R}^n$, an open connected cone in $\mathbb{R}^n$, an open convex cone in $\mathbb{R}^n$, and a regular cone in $\mathbb{R}^n$, with this latter cone being an open convex cone which does not contain any entire straight lines. The analytic functions satisfy several different growth conditions in L^p norm, and all of the resulting spaces of analytic functions generalize the vector valued Hardy space H^p in $\mathbb{C}^n$. The analytic functions are represented as the Fourier–Laplace transform of certain vector valued L^p functions which are characterized in the analysis. We give a characterization of the spaces of analytic functions in which the spaces are in fact subsets of the Hardy functions H^p. We obtain boundary value results on the distinguished boundary $\mathbb{R}^n + i\{\overline{0}\}$ and on the topological boundary $\mathbb{R}^n + i\partial B$ of the tube for the analytic functions in the L^p and vector valued tempered distribution topologies. Suggestions for associated future research are given.

Keywords: analytic functions; vector valued Hardy functions; boundary values

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1. Introduction

In [1] and related work, we defined and analyzed vector-valued Hardy $H^p(T^B, \mathcal{X})$ functions on tubes $T^B = \mathbb{R}^n + iB \subset \mathbb{C}^n$ with values in Banach space $\mathcal{X}$. We showed that any Banach space $\mathcal{X}$ vector-valued analytic function on T^B which obtained a $\mathcal{X}$ vector-valued distributional boundary value was a $H^p(T^B, \mathcal{X})$, $1 \leq p \leq \infty$, function with values in Banach space $\mathcal{X}$ if the $\mathcal{X}$ vector-valued boundary value was a $L^p(\mathbb{R}^n, \mathcal{X})$, $1 \leq p \leq \infty$, function. We showed that the $H^p(T^B, \mathcal{X})$, $1 \leq p \leq \infty$, functions admitted a representation by the Poisson integral of $L^p(\mathbb{R}^n, \mathcal{X})$, $1 \leq p \leq \infty$, functions if the values of the analytic functions were in a certain type of Banach space and then obtained a pointwise growth estimate for the $H^p(T^B, \mathcal{X})$ functions for this Banach space. In additional analysis, we have obtained many general results concerning $H^p(T^B, \mathcal{X})$ functions with values in Banach space including representations as Fourier–Laplace, Cauchy, and Poisson integrals and the existence of boundary values.

Previously, we defined generalizations of $H^p(T^B)$ functions in the scalar-valued case by using several more general growth conditions on the L^p norm of the analytic functions. Some of these scalar-valued results are contained in [2] (Chapter 5); other such results in the scalar-valued case are contained in papers listed under the author's name in the references in [1,2]. In this paper, we build upon these scalar-valued generalizations of $H^p(T^B)$ functions by considering the vector-valued case of functions and distributions with values in Banach or Hilbert space. The generalizations of the vector-valued analytic functions in $H^p(T^B, \mathcal{X})$, $\mathcal{X}$ being a Banach space, which we consider here are defined in Section 4 of this paper. Our results are obtained for the base B of the tube T^B successively being a proper open connected subset of $\mathbb{R}^n$, an open connected cone in $\mathbb{R}^n$, an open convex cone in $\mathbb{R}^n$, and a regular cone in $\mathbb{R}^n$, with this latter cone being an open convex cone which does not contain any entire straight lines; as the base B of the tube T^B is specialized, increasingly precise results are obtained in the analysis. For B being a proper open connected subset of $\mathbb{R}^n$ we show, for example, that the growth condition that defines the functions which generalize the Hardy functions can, in certain circumstances, be extended to the boundary of the base B of the tube T^B. At the open convex cone stage in our analysis we are able to show the equivalence of two types of vector-valued functions which generate $H^p(T^B, \mathcal{X})$ functions. In the cone setting for base B we show that certain elements of the defined analytic functions are in fact $H^p(T^B)$ functions which leads to the representation of these functions as Fourier–Laplace, Cauchy, and Poisson integrals. In the case that B is a regular cone we study the boundary values on the topological boundary of the tube defined by the cone as points in B approach a point on its boundary through circular bands within B. In general, our goal in this paper is to obtain results for the functions defined in Section 4 treated as generalizations of $H^p(T^B, \mathcal{X})$ functions and as generalizations of the scalar-valued functions noted in [2] (Chapter 5) and in some of our papers referenced in [2] and hence to generalize results concerning $H^p(T^B, \mathcal{X})$ spaces and concerning the scalar-valued functions noted in [2] (Chapter 5) and in certain references of [2] to these new spaces of analytic functions. Additionally, our goal is to obtain additional new results for the analytic functions of Section 4 which we accomplish.

As noted above, the vector-valued analytic functions considered in this paper are defined in Section 4. In Section 5, we show that certain vector-valued measurable functions generate the analytic functions by the Fourier–Laplace transform; conversely, in Section 6, we generate the measurable functions from the analytic functions and show that the analytic functions are representable through the generated measurable functions. As the base B of the tube T^B is made more specific the analytic functions and measurable functions obtain more specific properties. In Section 7, we show that under specified conditions the analytic functions considered are in fact vector-valued Hardy H^2 functions which immediately results in Cauchy and Poisson integral representations. Section 8 concerns the existence of boundary values of the analytic functions in vector-valued L^p and in vector-valued $\mathcal{S}'$ topologies on both the distinguished boundary and the topological boundary of the tube. Problems for future research are considered in Section 9, and conclusions are provided in Section 10.

2. Definitions and Notation

Throughout, $\mathcal{X}$ will denote a Banach space, $\mathcal{H}$ will denote a Hilbert space, $\mathcal{N}$ will denote the norm of the specified Banach or Hilbert space, and Θ will denote the zero vector of the specified Banach or Hilbert space. We reference Dunford and Schwartz [3] for integration of vector-valued functions and for vector-valued analytic functions. For foundational information concerning vector-valued distributions we refer to Schwartz ([4,5]).

The n-dimensional notation used in this paper will be the same as that in [1,2]. The information concerning cones in $\mathbb{R}^n$ needed here is contained in [2] (Chapter 1). We recall some very important notation and concepts of cones here that are necessary for this paper. $C \subset \mathbb{R}^n$ is a cone (with vertex at $\overline{0} = (0, 0, ..., 0) \in \mathbb{R}^n$) if $y \in C$ implies $\lambda y \in C$ for all positive scalars λ. The intersection of C with the unit sphere $|y| = 1$ is called the projection of C and is denoted $pr(C)$. A cone C' such that $pr(\overline{C'}) \subset pr(C)$ is a compact subcone of C which we will denote as $C' \subset\subset C$. The function

$$u_C(t) = \sup_{y \in pr(C)} (-\langle t, y \rangle), \ t \in \mathbb{R}^n,$$

is the indicatrix of C. The dual cone C^* of C is defined as

$$C^* = \{t \in \mathbb{R}^n : \langle t, y \rangle \geq 0 \text{ for all } y \in C\}$$

and satisfies $C^* = \{t \in \mathbb{R}^n : u_C(t) \leq 0\}$. An open convex cone which does not contain any entire straight lines will be called a regular cone. See [2] (Section 1.2) for examples of cones in $\mathbb{R}^n$. In this paper, we will be concerned with the distance from a point in a cone to the boundary of the cone; for C being an open connected cone in $\mathbb{R}^n$, the distance from $y \in C$ to the topological boundary ∂C of C is

$$d(y) = \inf\{|y - y_1| : y_1 \in \partial C\}.$$

For an open connected cone $C \subset \mathbb{R}^n$, we know from [2] (p. 6, (1.14)) that

$$d(y) = \inf_{t \in pr(C^*)} \langle t, y \rangle,\ y \in C,$$

and $0 < d(y) \leq |y|,\ y \in C$. Additionally, $d(\lambda y) = \lambda d(y), \lambda > 0$.

The $L^p(\mathbb{R}^n, \mathcal{X})$ functions, $1 \leq p \leq \infty$, with values in $\mathcal{X}$ and their norm $|\mathbf{h}|_p$ are noted in [3] (Chapter III). The Fourier transform on $L^1(\mathbb{R}^n)$ or $L^1(\mathbb{R}^n, \mathcal{X})$ is given in [2] (p. 3). All Fourier (inverse Fourier) transforms on scalar or vector-valued functions will be denoted $\widehat{\phi} = \mathcal{F}[\phi(t); x]$ $(\mathcal{F}^{-1}[\phi(t); x])$. As stated in [6] the Plancherel theory is not true for vector-valued functions except when $\mathcal{X} = \mathcal{H}$, a Hilbert space. The Plancherel theory is complete in the $L^2(\mathbb{R}^n, \mathcal{H})$ setting in that the inverse Fourier transform is the inverse mapping of the Fourier transform with $\mathcal{F}^{-1}\mathcal{F} = I = \mathcal{F}\mathcal{F}^{-1}$ with I being the identity mapping.

As usual, we denote $\mathcal{S}(\mathbb{R}^n)$ as the tempered functions with associated distributions being $\mathcal{S}'(\mathbb{R}^n)$ or associated vector-valued distributions being $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$. The Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$ and on $\mathcal{S}'(\mathbb{R}^n, \mathcal{X})$ is the usual such definition and is given in [4] (p. 73).

Let B be an open subset of $\mathbb{R}^n$ and $\mathcal{X}$ be a Banach space. The Hardy space $H^p(T^B, \mathcal{X})$, $0 < p < \infty$, consists of those analytic functions $\mathbf{f}(z)$ on the tube $T^B = \mathbb{R}^n + iB \subset \mathbb{C}^n$ with values in the Banach space $\mathcal{X}$ such that for some constant $M > 0$ and every $y \in B$

$$\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{f}(x + iy)))^p dx \ \leq\ M;$$

the usual modification is made for the case $p = \infty$.

3. Cauchy and Poisson Kernels and Integrals

Let C be a regular cone in $\mathbb{R}^n$. C^* is the dual cone of C. The Cauchy kernel corresponding to $T^C = \mathbb{R}^n + iC$ is

$$K(z - t) = \int_{C^*} e^{2\pi i \langle z - t, \eta \rangle} d\eta,\ t \in \mathbb{R}^n,\ z \in T^C.$$

The Poisson kernel corresponding to T^C is

$$Q(z; t) = \frac{K(z - t)\overline{K(z - t)}}{K(2iy)} = \frac{|K(z - t)|^2}{K(2iy)},\ t \in \mathbb{R}^n,\ z \in T^C.$$

Referring to [2] (Chapters 1 and 4) for details, we know for $z \in T^C$ that $K(z - \cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p},\ 1 < p \leq \infty$; and $Q(z; \cdot) \in \mathcal{D}(*, L^p) \subset \mathcal{D}_{L^p},\ 1 \leq p \leq \infty$, where $*$ is Beurling (M_p) or Roumieu $\{M_p\}$. These ultradifferentiable functions are contained in the Schwartz space $\mathcal{D}_{L^p} = \mathcal{D}(L^p, \mathbb{R}^n)$. Because of the combined properties of the Cauchy and Poisson kernels from [2], we know that the Cauchy and Poisson integrals

$$\int_{\mathbb{R}^n} \mathbf{h}(t) K(z - t) dt,\ z \in T^C,$$

and

$$\int_{\mathbb{R}^n} \mathbf{h}(t) Q(z; t) dt,\ z \in T^C,$$

are well defined for $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X}),\ 1 \leq p < \infty$, and $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X}),\ 1 \leq p \leq \infty$, respectively, for $\mathcal{X}$ being a Banach space.

We conclude this section with a boundary value calculation concerning the integral which defines the Cauchy kernel. Our calculations here provide motivation and guidance for boundary value results concerning the analytic functions considered in this paper which we obtain subsequently. Let C be a regular cone and put

$$K(z) = \int_{C^*} e^{2\pi i \langle z,t\rangle} dt,\ z \in T^C.$$

We know that $K(z)$ is analytic in T^C and is a bounded function of $x \in \mathbb{R}^n$ for $y \in C$. Thus, $K(x+iy) \in \mathcal{S}'(\mathbb{R}^n)$ as a function of $x \in \mathbb{R}^n$ for $y \in C$. Let $I_{C^*}(t)$ denote the characteristic function of C^*. We have the following result concerning points on the boundary of C, ∂C.

Theorem 1. *Let $y_o \in \partial C$. We have*

$$\lim_{y \to y_o, y \in C} K(x+iy) = \mathcal{F}[I_{C^*}(t) e^{-2\pi \langle y_o,t\rangle}]$$

in the strong topology of $\mathcal{S}'(\mathbb{R}^n)$.

Proof. For $y_o \in \partial C$, choose a sequence of points $\{y_m\}$, $m = 1, 2, \ldots$, in C which converges to y_o. We have

$$\langle y_o, t\rangle = \lim_{y_m \to y_o} \langle y_m, t\rangle \geq 0,\ t \in C^*.$$

Thus, $e^{-2\pi\langle y_o,t\rangle} I_{C^*}(t) \in \mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{F}[e^{-2\pi\langle y_o,t\rangle} I_{C^*}(t)] \in \mathcal{S}'(\mathbb{R}^n)$. Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $y \in C$.

$$\begin{aligned} &\langle K(x+iy) - \mathcal{F}[e^{-2\pi\langle y_o,t\rangle} I_{C^*}(t)], \phi(x)\rangle \\ &= \langle \mathcal{F}[(e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle}) I_{C^*}(t)], \phi(x)\rangle \\ &= \langle (e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle}) I_{C^*}(t), \widehat{\phi}(t)\rangle. \end{aligned}$$

Now

$$|(e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle}) I_{C^*}(t) \widehat{\phi}(t)| \leq (e^{-2\pi\langle y,t\rangle)} + e^{-2\pi\langle y_o,t\rangle}) I_{C^*}(t) |\widehat{\phi}(t)| \leq 2|\widehat{\phi}(t)|.$$

By the Lebesgue dominated convergence theorem, we have

$$\lim_{y \to y_o, y \in C} K(x+iy) = \mathcal{F}[I_{C^*}(t) e^{-2\pi \langle y_o,t\rangle}]$$

in the weak topology of $\mathcal{S}'(\mathbb{R}^n)$. Since $\mathcal{S}(\mathbb{R}^n)$ is a Montel space we have this convergence in the strong topology of $\mathcal{S}'(\mathbb{R}^n)$ also. □

In Theorem 1, notice that $\overline{0}$ is on the boundary of C. Thus, for $y_o = \overline{0}$,

$$\lim_{y \to \overline{0}, y \in C} K(x+iy) = \mathcal{F}[I_{C^*}(t)]$$

in the strong topology of $\mathcal{S}'(\mathbb{R}^n)$ in the conclusion of Theorem 1.

4. The Analytic Functions

As previously noted, we have studied vector-valued Hardy spaces in [1]; previous to this analysis we had generalized scalar-valued Hardy spaces by placing a more general bound on the L^p norm of the scalar-valued analytic function. These main scalar-valued generalizations are contained in [2] with other related work referenced in [2]. In the scalar-valued generalizations, we obtained Fourier–Laplace transform representation of the analytic functions and characterized the measurable function which generated this representation along with related results.

Given our recent work in studying vector-valued Hardy spaces, we now desire to study vector-valued generalizations of vector-valued Hardy spaces.

In this section, we introduce and define the vector-valued analytic functions that we study here. Throughout this section, B will denote a proper open connected subset of $\mathbb{R}^n$ unless stated otherwise; and, as previously stated, $\mathcal{X}$ will denote a Banach space with norm $\mathcal{N}$.

Definition 1. *$H_A^p(T^B, \mathcal{X})$, $1 \le p < \infty$, is the set of analytic functions $\boldsymbol{f}(z)$ on T^B with values in $\mathcal{X}$ such that*

$$|\boldsymbol{f}(x+iy)|_p = (\int_{\mathbb{R}^n} (\mathcal{N}(\boldsymbol{f}(x+iy)))^p dx)^{1/p} \le M(1+(d(y))^{-r})^s e^{2\pi A|y|},\ y \in B,$$

where $r \ge 0, s \ge 0, A \ge 0$, and $M = M(\boldsymbol{f}, p, A, r, s) > 0$.

Definition 2. *$R_A^p(T^B, \mathcal{X})$, $1 \le p < \infty$, is the set of analytic functions $\boldsymbol{f}(z)$ on T^B with values in $\mathcal{X}$ such that*

$$|\boldsymbol{f}(x+iy)|_p \le M(1+|y|^{-r})^s e^{2\pi A|y|},\ y \in B,$$

where $r \ge 0, s \ge 0, A \ge 0$, and $M = M(\boldsymbol{f}, p, A, r, s) > 0$.

Definition 3. *$V_A^p(T^B, \mathcal{X})$, $1 \le p < \infty$, is the set of analytic functions $\boldsymbol{f}(z)$ on T^B with values in $\mathcal{X}$ such that*

$$|\boldsymbol{f}(x+iy)|_p \le M e^{2\pi A|y|},\ y \in B,$$

were $A \ge 0$ and $M = M(\boldsymbol{f}, p, A) > 0$.

We consider situations and examples which help emphasize containment of these spaces although the definitions of these sets of functions show the containment in many cases. If B is an open connected cone we know from Section 2 that $d(y) \le |y|,\ y \in B$; thus, $R_A^p(T^B, \mathcal{X}) \subseteq H_A^p(T^B, \mathcal{X})$ in general in this case. For specific examples which help show proper containment let us just consider scalar-valued analytic functions in half planes in $\mathbb{C}^1$. Let $B = (0, \infty)$; thus, $T^{(0,\infty)} = \mathbb{R}^1 + i(0, \infty)$. We have

$$f(z) = \frac{e^{-2\pi i z}}{z(i+z)} \in R_1^2(T^{(0,\infty)}, \mathbb{C}^1) \cap H_1^2(T^{(0,\infty)}, \mathbb{C}^1),\ y = Im(z) \in (0, \infty),$$

as

$$||f(x+iy||_{L^2(\mathbb{R}^1)} \le \pi^{1/2}(1+y^{-1})e^{2\pi y},\ y = Im(z) > 0;$$

but this $f(z)$ is not in $V_1^2(T^{(0,\infty)}, \mathbb{C}^1)$. We have

$$f(z) = \frac{e^{-2\pi i z}}{i+z} \in V_1^2(T^{(0,\infty)}, \mathbb{C}^1)$$

but is not in $H^2(T^{(0,\infty)}, \mathbb{C}^1)$. Of course $f(z) = 1/(i+z) \in H^2(T^{(0,\infty)}, \mathbb{C}^1)$ and hence is in all of $V_1^2(T^{(0,\infty)}, \mathbb{C}^1), R_1^2(T^{(0,\infty)}, \mathbb{C}^1)$, and $H_1^2(T^{(0,\infty)}, \mathbb{C}^1)$. These examples help to see the containment of the defined spaces and the Hardy functions for most of the specified conditions on the base B of the tube T^B in our analysis in this paper.

For our next set of analytic functions, we must remember properties of sequences $M_p,\ p = 0, 1, 2, \ldots$, with which ultradifferentiable functions and ultradistributions are defined. These sequences and properties are discussed in [2] (Section 2.1). In this paper, we are principally concerned with the properties $(M.1)$ and $(M.3')$ and with the associated function

$$M^*(\rho) = \sup_p log(\rho^p p! M_0/M_p),\ 0 < \rho < \infty.$$

With these facts in mind we define additional vector-valued analytic functions.

Definition 4. *For B, being a proper open connected subset of $\mathbb{R}^n$ which does not contain $\overline{0}$, $H_*^p(T^B, \mathcal{X})$, $1 \leq p < \infty$, is the set of analytic functions $\boldsymbol{f}(z)$ on T^B with values in $\mathcal{X}$ such that*

$$|\boldsymbol{f}(x+iy)|_p \leq K(1 + (d(y))^{-r})^s e^{M^*(w/|y|)},\ y \in B,$$

where $r \geq 0, s \geq 0, w > 0$, and $K = K(\boldsymbol{f}, p, r, s, w) > 0$.

With Definition 4 in place, we can now state definitions for $R_*^p(T^B, \mathcal{X})$ and $V_*^p(T^B, \mathcal{X})$ from Definition 4 similarly as we did for $R_A^p(T^B, \mathcal{X})$ and $V_A^p(T^B, \mathcal{X})$ from Definition 1. In the scalar-valued case, we have proved that the Cauchy integral of ultradistributions $U \in \mathcal{D}'(*, L^p)$, where $*$ is Beurling (M_p) or Roumieu $\{M_p\}$, is analytic in T^C and satisfies the growth of Definition 4 where C is a regular cone in $\mathbb{R}^n$; see [2] (Section 4.2). Additionally, we have obtained boundary value results for scalar-valued functions of the type in Definition 4 in [2] (Chapter 5).

Throughout this paper, results concerning $H_*^p(T^B, \mathcal{X})$ and its subsets and associated norm growth bounds are obtained under the assumption that the sequence of positive numbers $M_p,\ p = 0, 1, 2, ...,$ from which the associated function $M^*(\rho)$ is defined, will always be assumed to satisfy properties $(M.1)$ and $(M.3')$ in [2] (p. 13).

5. Measurable Functions Generating Analytic Functions

The results which we will prove in this paper are obtained for functions in $H_A^p(T^B, \mathcal{X})$ of Definition 1 and for functions in $H_*^p(T^B, \mathcal{X})$ of Definition 4 by very similar methods. The results corresponding to $H_A^p(T^B, \mathcal{X})$ however are somewhat more general in nature than the corresponding ones for $H_*^p(T^B, \mathcal{X})$. Thus, we will concentrate our proofs on the results corresponding to $H_A^p(T^B, \mathcal{X})$ and subsequently state the corresponding results for $H_*^p(T^B, \mathcal{X})$ which will be denoted by a * next to the result number.

We begin by obtaining properties on measurable functions which we will use to generate analytic functions in $H_A^p(T^B, \mathcal{X})$. Let B be a proper open connected subset of $\mathbb{R}^n$ and let $\mathcal{X}$ be a Banach space. Let $1 \leq p < \infty$ and $\mathbf{g}(t)$ be a $\mathcal{X}$ valued measurable function on $\mathbb{R}^n$ such that

$$|e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_p \leq M(1 + (d(y))^{-r})^s e^{2\pi A|y|},\ y \in B, \tag{1}$$

where $r \geq 0, s \geq 0, A \geq 0$, and $M = M(\mathbf{g}, p, A, r, s) > 0$ do not depend on $y \in B$.

Theorem 2. *For B, being a proper open connected subset of $\mathbb{R}^n$ and $\mathcal{X}$ being a Banach space let $\boldsymbol{g}(t)$ be a $\mathcal{X}$ valued measurable function on $\mathbb{R}^n$ such that (1) holds for $y \in B$ and for $1 \leq p < \infty$. We have*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{g}(t) e^{2\pi i\langle z,t\rangle} dt,\ z = x + iy \in T^B, \tag{2}$$

is a $\mathcal{X}$ valued analytic function of $z \in T^B$.

Proof. Let $y_o \in B$. Choose an open neighborhood $N(y_o; r),\ r > 0$, and a compact subset $S \subset B$ such that $y_o \in N(y_o; r) \subset S \subset B$. Decompose $\mathbb{R}^n$ into a union of a finite number of non-overlapping cones $C_1, C_2, ..., C_k$ each having vertex at $\overline{0}$ and such that whenever two points y_1 and y_2 belong to one of these cones the angle between the rays from $\overline{0}$ to y_1 and from $\overline{0}$ to y_2 is less than $\pi/4$ radians; and hence $\langle y_1, y_2\rangle = |y_1||y_2|cos(\theta) > |y_1||y_2|2^{1/2}/2$ where θ is the angle between the two rays. There is a $\delta > 0$ such that $0 < \delta < r$ and $\{y : |y - y_o| = \delta\} \subset N(y_o; r)$. Put $\epsilon = 2\pi p\delta/2^{1/2} > 0$. For each $j = 1, 2, ..., k$ choose a fixed y_j such that $y_o - y_j \in C_j$ and $|y_j - y_o| = \delta$. For each $j = 1, 2, ..., k$ let $t \in C_j$; we have

$$\langle y_o - y_j, t\rangle \geq |y_o - y_j||t|/2^{1/2},\ t \in C_j,\ j = 1, 2, ..., k.$$

Thus, for $t \in C_j,\ j = 1, 2, ..., k,$,

$$\epsilon|t| = (2\pi p\delta/2^{1/2})|t| = 2\pi p|y_o - y_j||t|/2^{1/2} \leq 2\pi p\langle y_o - y_j, t\rangle = -2\pi p\langle y_j - y_o, t\rangle.$$

Hence, for each $j = 1, 2, ..., k$, using (1) we have

$$\begin{aligned}
&\int_{C_j} e^{-2\pi p\langle y_o,t\rangle} e^{\epsilon|t|} (\mathcal{N}(\mathbf{g}(t)))^p dt \\
&\leq \int_{C_j} e^{-2\pi p\langle y_o,t\rangle} e^{-2\pi p\langle y_j - y_o,t\rangle} (\mathcal{N}(\mathbf{g}(t)))^p dt = \int_{C_j} e^{-2\pi p\langle y_j,t\rangle} (\mathcal{N}(\mathbf{g}(t)))^p dt \\
&\leq \int_{\mathbb{R}^n} (\mathcal{N}(e^{-2\pi\langle y_j,t\rangle}\mathbf{g}(t)))^p dt \leq M^p (1 + (d(y_j))^{-r})^{sp} e^{2\pi pA|y_j|}
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-2\pi p\langle y_o,t\rangle} e^{\epsilon|t|} (\mathcal{N}(\mathbf{g}(t)))^p dt = \sum_{j=1}^{k} \int_{C_j} e^{-2\pi p\langle y_o,t\rangle} e^{\epsilon|t|} (\mathcal{N}(\mathbf{g}(t)))^p dt \\
&\leq M^p \sum_{j=1}^{k} (1 + (d(y_j))^{-r})^{sp} e^{2\pi pA|y_j|} \qquad (3)
\end{aligned}$$

for arbitrary $y_o \in B$. For $p = 1$ and the fact that $(\epsilon|t|/2) \leq \epsilon|t|,\ t \in \mathbb{R}^n$, we have from (3) that

$$\int_{\mathbb{R}^n} e^{-2\pi\langle y_o,t\rangle} e^{\epsilon|t|/2} \mathcal{N}(\mathbf{g}(t)) dt \leq M \sum_{j=1}^{k} (1 + (d(y_j))^{-r})^s e^{2\pi A|y_j|}. \qquad (4)$$

For $1 < p < \infty$, Hölder's inequality, the identity $e^{\epsilon|t|/2p} = e^{\epsilon|t|/p} e^{-\epsilon|t|/2p}$ and (3) yield

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-2\pi\langle y_o,t\rangle} e^{\epsilon|t|/2p} \mathcal{N}(\mathbf{g}(t)) dt \leq ||e^{-\epsilon|t|/2p}||_{L^q(\mathbb{R}^n)} |e^{-2\pi\langle y_o,t\rangle} e^{\epsilon|t|/p} \mathbf{g}(t)|_p \\
&\leq M||e^{-\epsilon|t|/2p}||_{L^q(\mathbb{R}^n)} \left(\sum_{j=1}^{k} (1 + (d(y_j))^{-r})^{sp} e^{2\pi pA|y_j|} \right)^{1/p} \qquad (5)
\end{aligned}$$

where $1/p + 1/q = 1$. If $|y - y_o| < \epsilon/4\pi p,\ y = \mathrm{Im}(z),\ 1 \leq p < \infty$, then for $z = x + iy$

$$\begin{aligned}
&\mathcal{N}(\mathbf{g}(t)e^{2\pi i\langle z,t\rangle}) = e^{-2\pi\langle y,t\rangle} \mathcal{N}(\mathbf{g}(t)) = e^{-2\pi\langle y - y_o,t\rangle} e^{-2\pi\langle y_o,t\rangle} \mathcal{N}(\mathbf{g}(t)) \\
&\leq e^{2\pi|y - y_o||t|} e^{-2\pi\langle y_o,t\rangle} \mathcal{N}(\mathbf{g}(t)) \leq e^{-2\pi\langle y_o,t\rangle} e^{\epsilon|t|/2p} \mathcal{N}(\mathbf{g}(t)) \qquad (6)
\end{aligned}$$

for all $t \in \mathbb{R}^n$. (4) and (5) now show that the right side of (6) is a $L^1(\mathbb{R}^n)$ function which is independent of $y = \mathrm{Im}(z)$ such that $|y - y_o| < \epsilon/4\pi p$ for all cases $1 \leq p < \infty$. Since $y_o \in B$ is arbitrary we conclude from (6) that $f(z)$ defined by (2) is a $\mathcal{X}$ valued analytic function of $z \in T^B$. Further, (6) proves that $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n, \mathcal{X}),\ y \in B$, for all cases $1 \leq p < \infty$ in addition to the fact that $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^p(\mathbb{R}^n, \mathcal{X}),\ y \in B,$ for each of the specific cases for $p,\ 1 \leq p < \infty$, because of the assumption (1). The proof is complete. □

The exact same method of proof used for Theorem 2 yields the following result corresponding to the growth for $H^p_*(T^B, \mathcal{X})$.

Theorem 3. *Let B be a proper open connected subset of $\mathbb{R}^n$ which does not contain $\overline{0} \in \mathbb{R}^n$, and let $\mathcal{X}$ be a Banach space. Let $1 \leq p < \infty$ and $\boldsymbol{g}(t)$ be a $\mathcal{X}$ valued measurable function on $\mathbb{R}^n$ such that*

$$|e^{-2\pi\langle y,t\rangle}\boldsymbol{g}(t)|_p \leq M(1 + (d(y))^{-r})^s e^{M^*(w/|y|)},\ y \in B,$$

where $r \geq 0, s \geq 0, w > 0$, and $M = M(\boldsymbol{g}, p, r, s, w) > 0$ are independent of $y \in B$. We have

$$f(z) = \int_{\mathbb{R}^n} g(t) e^{2\pi i \langle z,t \rangle} dt, \; z \in T^B,$$

is a $\mathcal{X}$ valued analytic function of $z \in T^B$.

The Fourier transform of vector-valued functions $L^p(\mathbb{R}^n, \mathcal{X})$ with the Plancherel theory and Parseval identity holding occurs only if $p = 2$ and $\mathcal{X} = \mathcal{H}$, a Hilbert space. For $p = 2$ in order to have an isomorphism of the Fourier transform of $L^2(\mathbb{R}^n, \mathcal{X})$ onto itself with the Parseval identity holding it is necessary and sufficient that $\mathcal{X} = \mathcal{H}$, a Hilbert space [6] (pp. 45, 61). We use the Fourier transform considerably in this paper, and its use is the reason we sometimes restrict the result to $p = 2$ and $\mathcal{X} = \mathcal{H}$. We obtain a corollary to Theorem 2.

Corollary 1. *Let B be a proper open connected subset of $\mathbb{R}^n$ and $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{g}(t)$ be a $\mathcal{H}$ valued measurable function on $\mathbb{R}^n$ such that (1) holds for $p = 2$. We have $\boldsymbol{f}(z) \in H^2_A(T^B, \mathcal{H})$ for $\boldsymbol{f}(z)$ defined in (2).*

Proof. $\mathbf{f}(z)$ is analytic in T^B by Theorem 2. By the assumption (1) for $p = 2$ and the proof of Theorem 2, $e^{-2\pi\langle y,t \rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H})$ for $y \in B$. Thus, $\mathbf{f}(x + iy) = \mathcal{F}[e^{-2\pi\langle y,t \rangle}\mathbf{g}(t); x]$, $y \in B$, with the Fourier transform being in the $L^1(\mathbb{R}^n, \mathcal{H})$ and the $L^2(\mathbb{R}^n, \mathcal{H})$ cases. By the Parseval equality $|\mathbf{f}(x + iy)|_2 = |e^{-2\pi\langle y,t \rangle}\mathbf{g}(t)|_2$ for $y \in B$. From (1) the desired growth on $\mathbf{f}(x + iy)$ of Definition 1 is obtained, and $\mathbf{f}(z) \in H^2_A(T^B, \mathcal{H})$. $\square$

Under certain circumstances, the growth on the $L^2(\mathbb{R}^n, \mathcal{H})$ function $e^{-2\pi\langle y,t \rangle}\mathbf{g}(t)$, $y \in B$, in Corollary 1 can be extended to hold for $y \in \overline{B}$.

Corollary 2. *Assume the hypotheses of Corollary 1 with the addition that (1) holds for $p = 2$ with $r = 0$ or $s = 0$. We have $\boldsymbol{f}(z) \in V^2_A(T^B, \mathcal{H})$ for $\boldsymbol{f}(z)$ defined in (2) and*

$$|e^{-2\pi\langle y,t \rangle}\boldsymbol{g}(t)|_2 \leq M e^{2\pi A|y|}, \; y \in \overline{B}.$$

Further if $\overline{0} \in \partial B$ then $\boldsymbol{g} \in L^2(\mathbb{R}^n, \mathcal{H})$.

Proof. From the proof of Corollary 1 and Definition 3, we have $\mathbf{f}(z) \in V^2_A(T^B, \mathcal{H})$ for $r = 0$ or $s = 0$ in (1). Let $y_o \in \partial B$ and let $\{y_m\}$ be a sequence of points in B which converges to y_o. By Fatou's lemma we have

$$\int_{\mathbb{R}^n} e^{-4\pi\langle y_o,t \rangle}(\mathcal{N}(\mathbf{g}(t)))^2 dt \leq \limsup_{m\to\infty} \int_{\mathbb{R}^n} e^{-4\pi\langle y_m,t \rangle}(\mathcal{N}(\mathbf{g}(t)))^2 dt$$
$$\leq \limsup_{m\to\infty} M^2 e^{4\pi A|y_m|} = M^2 e^{4\pi A|y_o|}$$

and

$$|e^{-2\pi\langle y_o,t \rangle}\mathbf{g}(t)|_2 \leq M e^{2\pi A|y_o|}.$$

Thus, (1) holds with $r = 0$ or $s = 0$ for $y \in \overline{B}$. If $\overline{0} \in \partial B$ then $\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ from the above inequality for $y_o = \overline{0}$. $\square$

For B being a proper open connected subset of $\mathbb{R}^n$ and $\mathcal{X}$ being a Banach space, assume (1) holds for $1 \leq p < \infty$ with $r = 0$ or $s = 0$ and for $\mathbf{g}$ having values in $\mathcal{X}$. The proof of Corollary 2 shows that (1) will hold for $y \in \overline{B}$ in this situation.

We study the extension of $\mathbf{f}(z)$ or $e^{-2\pi\langle y,t \rangle}\mathbf{g}(t)$, $y \in B$, in norm to the ∂B in greater detail later in this paper in section 8.

The proof of the following result is the same as that of Corollary 1 using Theorem 3.

Corollary 3. *Let B be a proper open connected subset of $\mathbb{R}^n$ which does not contain $\overline{0} \in \mathbb{R}^n$, and let $\mathcal{H}$ be a Hilbert space such that the growth of Theorem 3 holds for $p = 2$. We have $\boldsymbol{f}(z) \in H^2_*(T^B, \mathcal{H})$ for $\boldsymbol{f}(z)$ defined in (2).*

In several following results, we restrict the base B of the tube T^B to cones and obtain additional properties of the function $\mathbf{g}(t)$ in the results. Throughout $\text{supp}(\mathbf{g})$ denotes the support of $\mathbf{g}$.

Theorem 4. *Let C be an open connected cone in $\mathbb{R}^n$ and $1 \leq p < \infty$. Let $\boldsymbol{g}(t)$ be a Banach space $\mathcal{X}$ valued measurable function on $\mathbb{R}^n$ such that (1) holds for $y \in C$. We have $supp(\boldsymbol{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ almost everywhere (a.e.).*

Proof. Assume $\mathbf{g}(t) \neq \Theta$ on a set of positive measure in $\{t \in \mathbb{R}^n : u_C(t) > A\}$; there is a point $t_o \in \{t \in \mathbb{R}^n : u_C(t) > A\}$ such that $\mathbf{g}(t) \neq \Theta$ on a set of positive measure in the neighborhoods $N(t_o, \eta) = \{t \in \mathbb{R}^n : |t - t_o| < \eta\}$ for arbitrary $\eta > 0$. Since $t_o \in \{t \in \mathbb{R}^n : u_C(t) > A\}$ there is a point $y_o \in pr(C) \subset C$ such that $(-\langle t_o, y_o\rangle) > A \geq 0$. Using the continuity of $(-\langle t, y_o\rangle)$ at t_o as a function of t, there is a fixed $\sigma > 0$ and a fixed neighborhood $N(t_o; \eta')$ such that $-\langle t, y_o\rangle) > A + \sigma > 0$ for all $t \in N(t_o; \eta')$. Choose η above to be η'. For any $\lambda > 0$ we have

$$-\langle \lambda y_o, t\rangle = -\lambda\langle y_o, t\rangle > \lambda A + \lambda\sigma > 0,\ t \in N(t_o; \eta'),\ \lambda > 0. \tag{7}$$

$y_o \in pr(C) \subset C$ and C being a cone imply $\lambda y_o \in C, \lambda > 0$. From (7) and (1) with $y = \lambda y_o$ we have for all $\lambda > 0$ that

$$\begin{aligned}
&e^{2\pi p(\lambda A + \lambda\sigma)} \int_{N(t_o;\eta')} (\mathcal{N}(\mathbf{g}(t)))^p dt \leq \int_{N(t_o;\eta')} e^{-2\pi\langle \lambda y_o, t\rangle} (\mathcal{N}(\mathbf{g}(t)))^p dt \\
&\leq \int_{\mathbb{R}^n} e^{-2\pi p\langle \lambda y_o, t\rangle} (\mathcal{N}(\mathbf{g}(t)))^p dt \leq M^p (1 + (d(\lambda y_o))^{-r})^{sp} e^{2\pi p A|\lambda y_o|} \\
&= M^p (1 + \lambda^{-r}(d(y_o))^{-r})^{sp} e^{2\pi p\lambda A}
\end{aligned} \tag{8}$$

since $y_o \in pr(C)$ and $d(\lambda y_o) = \lambda d(y_o)$. The integral on the left of (8) is finite. From (8) we have

$$(1 + \lambda^{-r}(d(y_o))^{-r})^{-sp} e^{2\pi p\lambda\sigma} \int_{N(t_o,\eta')} (\mathcal{N}(\mathbf{g}(t)))^p dt \leq M^p \tag{9}$$

for all $\lambda > 0$ with $\sigma > 0$ being fixed and independent of λ. Recall that y_o depends only on t_o. The constants $d(y_o), r, s, p, \sigma, \eta'$, and M are all independent of $\lambda > 0$. We have $(1 + \lambda^{-r}(d(y_o))^{-r})^{-sp} = 1$ if $r = 0$ or $s = 0$, and $(1 + \lambda^{-r}(d(y_o))^{-r})^{-sp} \to 1$ as $\lambda \to \infty$ if $r > 0$ and $s > 0$. We let $\lambda \to \infty$ in (9) and conclude that $\mathbf{g}(t) = \Theta$ almost everywhere in $N(t_o; \eta')$ which contradicts the fact that $\mathbf{g}(t) \neq \Theta$ on a set of positive measure in $N(t_o, \eta')$. Thus, $\mathbf{g}(t) = \Theta$ a.e. in $\{t \in \mathbb{R}^n : u_C(t) > A\}$, and $\text{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. since $\{t \in \mathbb{R}^n : u_C(t) \leq A\}$ is a closed set in $\mathbb{R}^n$. $\square$

The proof of the corresponding result for the growth of Theorem 3 can be obtained by similar techniques as in Theorem 4.

Theorem 5. *Let C be an open connected cone in $\mathbb{R}^n$ and $1 \leq p < \infty$. Let $\boldsymbol{g}(t)$ be a Banach space $\mathcal{X}$ valued measurable function on $\mathbb{R}^n$ such that the growth of Theorem 3 holds for $y \in C$. We have $supp(\boldsymbol{g}) \subseteq C^*$ a.e.*

In [7,8], Vladimirov introduced a space of measurable functions on $\mathbb{R}^n$, denoted $\mathcal{S}_0'$, which when multiplied by a polynomial raised to a suitable negative power become $L^2(\mathbb{R}^n)$ functions. Analysis concerning the space $\mathcal{S}_0'$ can also be found in [9,10]. We now extend this space to the vector-valued case and for p such that $1 \leq p < \infty$. We then show that these new spaces of functions become equivalent to the measurable functions $\mathbf{g}$ of the preceding results in this section for each p and for the base of the tube being open convex cones in $\mathbb{R}^n$.

Definition 5. *Let $\mathcal{X}$ be a Banach space. $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$, $1 \le p < \infty$, is the set of all measurable functions $\mathbf{g}(t)$, $t \in \mathbb{R}^n$, with values in $\mathcal{X}$ such that there exists a real number $m \ge 0$ for which $(1 + |t|^p)^{-m}\mathbf{g}(t) \in L^p(\mathbb{R}^n, \mathcal{X})$.*

First note that $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{X})$, $1 \le p < \infty$. In our first result concerning the spaces $\mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$ the base of the tube T^C will be an open connected cone.

Theorem 6. *Let C be an open connected cone in $\mathbb{R}^n$ and $1 \le p < \infty$. Let $\mathbf{g}(t)$ be a measurable function on $\mathbb{R}^n$ with values in a Banach space $\mathcal{X}$ such that (1) holds for $y \in C$. We have $\mathbf{g} \in \mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$ and supp$(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \le A\}$ a.e.*

Proof. The support property of $\mathbf{g}$ has been proved in Theorem 4. We now prove that $\mathbf{g} \in \mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$. Choose a fixed point $y_o \in pr(C)$ and put $Y = \{y : y = \lambda y_o,\ 0 < \lambda \le 1\} \subset C$; choose a fixed compact subcone $C' \subset\subset C$ such that $y_o \in C'$. We have $Y \subset C' \subset\subset C$. Let $y \in Y$ be arbitrary; using (1) we have

$$\int_{\mathbb{R}^n} e^{-2\pi p\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^p dt \le M^p(1 + (d(y))^{-r})^{sp} e^{2\pi pA|y|},\ y \in C,$$

and hence

$$(d(y))^{rsp} \int_{\mathbb{R}^n} e^{-2\pi p\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^p dt \le M^p(1 + (d(y))^{r})^{sp} e^{2\pi pA|y|},\ y \in C. \tag{10}$$

(10) holds in particular for $y \in Y$ for which $|y| = \lambda|y_o| = \lambda$, $0 < \lambda \le 1$, since $y_o \in pr(C)$; and $\langle y,t\rangle \le |y||t|$, $t \in \mathbb{R}^n$, implies $(-|y||t|) \le -\langle y,t\rangle, t \in \mathbb{R}^n$. Corresponding to $C' \subset\subset C$ we use [7] (p. 6, (1.14)) and obtain $\delta = \delta(C') > 0$ depending only on C' and not on $y \in C'$ such that

$$0 < \delta|y| \le d(y) \le |y|,\ y \in C' \subset\subset C. \tag{11}$$

Using (11) and (10), we have

$$\begin{aligned}(\delta\lambda)^{rsp} \int_{\mathbb{R}^n} e^{-2\pi p\lambda|t|}(\mathcal{N}(\mathbf{g}(t)))^p dt &\le (d(y))^{rsp}\int_{\mathbb{R}^n} e^{-2\pi p\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^p dt \\ &\le M^p(1 + (d(y)))^r)^{sp} e^{2\pi pA|y|} \le M^p(1+\lambda^r)e^{2\pi p\lambda A}\end{aligned} \tag{12}$$

for $y = \lambda y_o \in Y \subset C' \subset\subset C$, $0 < \lambda \le 1$, with δ being independent of C' and hence independent of $y \in Y$ and independent of λ, $0 < \lambda \le 1$. Let $\epsilon > 1$ be fixed. Multiply both sides of (12) by $\lambda^{-1+\epsilon}$ and integrate the result from (12) over $0 < \lambda \le 1$ with respect to λ to obtain

$$\int_0^1 \lambda^{-1+\epsilon}(\delta\lambda)^{rsp}\int_{\mathbb{R}^n} e^{-2\pi p\lambda|t|}(\mathcal{N}(\mathbf{g}(t)))^p dt d\lambda \le M^p \int_0^1 \lambda^{-1+\epsilon}(1+\lambda^r)^{sp} e^{2\pi p\lambda A} d\lambda.$$

Now multiply this inequality by δ^{-rsp} and use Fubini's theorem on the left to obtain

$$\int_{\mathbb{R}^n} (\mathcal{N}(\mathbf{g}(t)))^p \int_0^1 \lambda^{rsp-1+\epsilon} e^{-2\pi p\lambda|t|} d\lambda dt \le M^p\delta^{-rsp}\int_0^1 \lambda^{-1+\epsilon}(1+\lambda^r)^{ps} e^{2\pi p\lambda A} d\lambda. \tag{13}$$

We note that all constants $M, \delta, r, s, p, \epsilon$, and A are independent of $y = \lambda y_o \in Y$ and hence independent of λ, $0 < \lambda \le 1$. Using the change of variable $u = 2\pi p\lambda|t|$ in the inner integral on the left of (13) and considering the cases $0 < |t| \le 1/2\pi p$ and $|t| > 1/2\pi p$ we obtain

$$\int_0^1 \lambda^{rsp-1+\epsilon} e^{-2\pi p\lambda|t|} d\lambda = (2\pi p|t|)^{-rsp-\epsilon}\int_0^{2\pi p|t|} u^{rsp-1+\epsilon} e^{-u} du \ge \tag{14}$$

$$\left\{ \begin{array}{ll} (ersp+\epsilon e)^{-1}(1+|t|^p)^{-rs-\epsilon/p} & \text{for} \quad 0<|t|\le 1/2\pi p \\ (2\pi p)^{-rsp-\epsilon}\int_0^1 u^{rsp-1+\epsilon}e^{-u}du(1+|t|^p)^{-rs-\epsilon/p} & \text{for} \quad |t|>1/2\pi p \end{array} . \right\}$$

Put

$$K=\min\left\{(ersp+\epsilon e)^{-1},(2\pi p)^{-rsp-\epsilon}\int_0^1 u^{rsp-1+\epsilon}e^{-u}du\right\}>0.$$

From (14), we have

$$\int_0^1 \lambda^{rsp-1+\epsilon}e^{-2\pi p\lambda|t|}d\lambda \ge K(1+|t|^p)^{-rs-\epsilon/p},\ |t|>0, \tag{15}$$

with this inequality holding also at $t=\overline{0}$ by adjusting the constant K if needed. Putting (15), which holds for all $t\in\mathbb{R}^n$ now, into (13) and recalling $\epsilon>1$, we have

$$\begin{aligned} &K\int_{\mathbb{R}^n}(1+|t|^p)^{-rs-\epsilon/p}(\mathcal{N}(\mathbf{g}(t)))^p dt \le M^p\delta^{-rsp}\int_0^1 \lambda^{-1+\epsilon}(1+\lambda^r)^{ps}e^{2\pi p\lambda A}d\lambda \\ &\le M^p\delta^{-rsp}2^{ps}e^{2\pi pA} \end{aligned}$$

with the right side being a fixed constant. Thus, $(1+|t|^p)^{-rs/p-\epsilon/p^2}\mathbf{g}(t)\in L^p(\mathbb{R}^n,\mathcal{X})$, and $\mathbf{g}\in\mathcal{S}_p'(\mathbb{R}^n,\mathcal{X})$ since $(rs/p+\epsilon/p^2)\ge 0$. □

We similarly obtain the following result from Theorem 5.

Theorem 7. *Let C be an open connected cone in $\mathbb{R}^n$ and $1\le p<\infty$. Let $\mathbf{g}(t)$ be a Banach space $\mathcal{X}$ valued measurable function on $\mathbb{R}^n$ such that the growth of Theorem 3 holds for $y\in C$. We have $\mathbf{g}\in\mathcal{S}_p'(\mathbb{R}^n,\mathcal{X})$ and supp(g) $\subseteq C^*$ a.e.*

In order for the converse implication of Theorem 6 to hold we need the cone C to be convex as well as open.

Theorem 8. *Let C be an open convex cone in $\mathbb{R}^n$, $1\le p<\infty$, and $A\ge 0$. Let $\mathbf{g}\in\mathcal{S}_p'(\mathbb{R}^n,\mathcal{X})$ with supp(g) $\subseteq\{t\in\mathbb{R}^n: u_C(t)\le A\}$ a.e. where $\mathcal{X}$ is a Banach space. We have $\mathbf{g}$ is a measurable function with values in $\mathcal{X}$ such that (1) holds for all $y\in C$.*

Proof. From [9] (p. 74, Lemma 3), $\{t\in\mathbb{R}^n: u_C(t)\le A\}=C^*+\overline{N(\overline{0};A)}, N(\overline{0};A)=\{t\in\mathbb{R}^n: |t|<A\}$, since the cone C is open and convex here. Thus, $t\in\{t\in\mathbb{R}^n: u_C(t)\le A\}$ yields $t=t_1+t_2,\ t_1\in C^*,\ t_2\in\overline{N(\overline{0};A)}$. Since $\mathbf{g}\in\mathcal{S}_p'(\mathbb{R}^n,\mathcal{X}),\ \mathbf{g}$ is measurable on $\mathbb{R}^n$ and $(1+|t|^p)^{-m}\mathbf{g}(t)\in L^p(\mathbb{R}^n,\mathcal{X})$ for some $m\ge 0$; thus

$$\int_{\mathbb{R}^n}(1+|t|^p)^{-mp}(\mathcal{N}(\mathbf{g}(t)))^p dt\le K<\infty$$

for a constant $K>0$. Let $y\in C$ be arbitrary. We have

$$\begin{aligned} &\int_{\mathbb{R}^n}e^{-2\pi p\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^p dt \\ &=\int_{C^*+\overline{N(\overline{0};A)}}e^{-2\pi p\langle y,t\rangle}(1+|t|^p)^{mp}(1+|t|^p)^{-mp}(\mathcal{N}(\mathbf{g}(t))))^p dt \\ &\le \sup_{t\in C^*+\overline{N(\overline{0};A)}}((1+|t|^p)^{mp}e^{-2\pi p\langle y,t\rangle})\int_{\mathbb{R}^n}(1+|t|^p)^{-mp}(\mathcal{N}(\mathbf{g}(t)))^p dt \\ &\le K\sup_{t\in C^*+\overline{N(\overline{0};A)}}(1+|t|^p)^{mp}e^{-2\pi p\langle y,t\rangle} \\ &\le K\sup_{t_1\in C^*,t_2\in\overline{N(\overline{0};A)}}(1+(|t_1|+|t_2|)^p)^{mp}e^{-2\pi p\langle y,t_1+t_2\rangle}. \end{aligned} \tag{16}$$

For $t_2 \in \overline{N(\bar{0}; A)}$, we have $|t_2| \leq A$ and

$$e^{-2\pi p \langle y, t_2 \rangle} \leq e^{2\pi p |t_2||y|} \leq e^{2\pi p A|y|}, \ t_2 \in \overline{N(\bar{0}; A)}, \ y \in C. \tag{17}$$

For $t_1 \in C^*$, we have $t_1 = \lambda_1 t_1^*$ where $\lambda_1 \geq 0$ and $t_1^* \in pr(C^*)$. From Section 2 we have

$$d(y) = \inf_{u \in pr(C^*)} \langle u, y \rangle = - \sup_{u \in pr(C^*)} (-\langle u, y \rangle), \ y \in C. \tag{18}$$

For $y \in C$, using (17) and (18) we continue (16) as

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-2\pi p \langle y, t \rangle} (\mathcal{N}(\mathbf{g}(t)))^p dt \\
&\leq K e^{2\pi p A|y|} \sup_{\lambda_1 \geq 0, t_1^* \in pr(C^*)} ((1 + (\lambda_1 + A)^p)^{mp} e^{-2\pi p \lambda_1 \langle t_1^*, y \rangle}) \\
&\leq K e^{2\pi p A|y|} \sup_{\lambda_1 \geq 0} ((1 + (\lambda_1 + A)^p)^{mp} e^{-2\pi p \lambda_1 d(y)}) \\
&\leq K(1 + (1 + A)^p)^{mp} e^{2\pi p A|y|} \sup_{\lambda_1 \geq 0} ((1 + \lambda_1^p)^{mp} e^{-2\pi p \lambda_1 d(y)}) \\
&\leq K(1 + (1 + A)^p)^{mp} e^{2\pi p A|y|} \sup_{\lambda_1 \geq 0} ((1 + \lambda_1)^{mp^2} e^{-2\pi p \lambda_1 d(y)}).
\end{aligned} \tag{19}$$

The supremum in the last line of (19) is a maximum which can be obtained using the first derivative test. If $(mp^2 - 2\pi p d(y)) > 0$ then $m > 0$ and the supremum occurs at $\lambda_1 = (mp^2 - 2\pi p d(y))/2\pi p d(y)$, and in this case

$$\begin{aligned}
&\sup_{\lambda_1 \geq 0} ((1 + \lambda_1)^{mp^2} e^{-2\pi p \lambda_1 d(y)}) \leq \left(1 + \frac{mp^2 - 2\pi p d(y)}{2\pi p d(y)}\right)^{mp^2} \\
&\leq \left(1 + \frac{mp^2}{2\pi p d(y)}\right)^{mp^2} = \left(\frac{mp}{2\pi}\right)^{mp^2} \left(\frac{2\pi}{mp} + \frac{1}{d(y)}\right)^{mp^2} \\
&\leq max\{1, \left(\frac{mp}{2\pi}\right)^{mp^2}\}(1 + (d(y))^{-1})^{mp^2}.
\end{aligned}$$

If $(mp^2 - 2\pi p d(y)) \leq 0$, the supremum in the last line of (19) occurs at $\lambda_1 = 0$ and

$$\sup_{\lambda_1 \geq 0} ((1 + \lambda_1)^{mp^2} e^{-2\pi p \lambda_1 d(y)}) = 1 \leq (1 + (d(y))^{-1})^{mp^2}.$$

Combining (19) with the above two estimates on the supremum over $\lambda_1 \geq 0$ we have for $y \in C$

$$\begin{aligned}
&\int_{\mathbb{R}^n} e^{-2\pi p \langle y, t \rangle} (\mathcal{N}(\mathbf{g}(t)))^p dt \\
&\leq K(1 + (1 + A)^p)^{mp} max\{1, \left(\frac{mp}{2\pi}\right)^{mp^2}\}(1 + (d(y))^{-1})^{mp^2} e^{2\pi p A|y|}.
\end{aligned}$$

Taking the pth root of this inequality, we obtain (1) holding for all $y \in C$ with $r = 1$ and $s = mp$. $\square$

For C being an open convex cone in $\mathbb{R}^n$ Theorems 6 and 8 show that $\mathbf{g}$ being a Banach space $\mathcal{X}$ valued measurable function with (1) holding for $y \in C, \ A \geq 0$, and $1 \leq p < \infty$ is an equivalent statement to $\mathbf{g} \in \mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$ with $\text{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. for $A \geq 0$ and $1 \leq p < \infty$. Thus, for any future result concerning open convex cones C, these two statements are interchangeable in hypotheses.

If $A = 0$, $\{t \in \mathbb{R}^n : u_C(t) \leq 0\} = C^*$. In this case we have the following corollary to Theorem 8.

Corollary 4. *Let C be an open convex cone in $\mathbb{R}^n$ and $1 \leq p < \infty$. Let $g \in \mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$ for $\mathcal{X}$ being a Banach space and $supp(g) \subseteq C^*$ a.e. We have*

$$|e^{-2\pi\langle y,t\rangle} g(t)|_p \leq M(1 + (d(y))^{-1})^{mp},\ y \in C,$$

for constants $M > 0$ and $m \geq 0$.

6. Analytic Functions Generating Measurable Functions

In this section, we consider generalized vector-valued Hardy functions and construct measurable functions which yield Fourier–Laplace transform representations. This material is followed in Section 7 by representing the analytic functions, in particular cases, by Cauchy and Poisson integrals.

We use the Fourier transform on $L^2(\mathbb{R}^n, \mathcal{H})$ considerably in this section and in Section 7. This causes us to restrict the results to $p = 2$ and functions having values in Hilbert space $\mathcal{H}$ as previously discussed in Section 2 in relation to the function Fourier transform.

To prove the Fourier–Laplace representation of functions in $H^2_A(T^B, \mathcal{H})$ in terms of a constructed measurable function we first need the following lemma.

Lemma 1. *Let B be a proper open connected subset of $\mathbb{R}^n$. Let $f(z) \in H^2_A(T^B, \mathcal{H})$),where $\mathcal{H}$ is Hilbert space, and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of B. Let $\epsilon > 0$. Put*

$$g_{\epsilon,y}(t) = \int_{\mathbb{R}^n} e^{-\epsilon \sum_{j=1}^n z_j^2} f(x+iy) e^{-2\pi i\langle x+iy,t\rangle} dx,\ y \in B, \tag{20}$$

and

$$g_y(t) = \mathcal{F}^{-1}[e^{2\pi\langle y,t\rangle} f(x+iy); t],\ y \in B,\ t \in \mathbb{R}^n, \tag{21}$$

in $L^2(\mathbb{R}^n, \mathcal{H})$. We have $g_{\epsilon,y}(t)$ is independent of $y \in B$ for any $\epsilon > 0$;

$$\lim_{\epsilon \to 0+} |g_{\epsilon,y}(t) - g_y(t)|_2 = 0,\ y \in B; \tag{22}$$

and $g_y(t)$ is independent of $y \in B$.

Proof. For $y \in B$ and $t \in \mathbb{R}^n$, $\mathbf{f}(x+iy) \in L^2(\mathbb{R}^n, \mathcal{H})$ and $e^{2\pi\langle y,t\rangle}\mathbf{f}(x+iy) \in L^2(\mathbb{R}^n, \mathcal{H})$ as functions of $x \in \mathbb{R}^n$. Further, $(e^{2\pi\langle y,t\rangle} e^{-\epsilon \sum_{j=1}^n z_j^2} \mathbf{f}(x+iy)) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H})$ for $y \in B$ and $t \in \mathbb{R}^n$. Thus, both $\mathbf{g}_{\epsilon,y}(t)$ and $\mathbf{g}_y(t)$ are well defined for $y \in B$ and both are in $L^2(\mathbb{R}^n, \mathcal{H})$. We assume here that $0 < \epsilon \leq 1$ since we are letting $\epsilon \to 0+$ in (22). We have for $y \in B$

$$\begin{aligned} &|\mathbf{g}_{\epsilon,y}(t) - \mathbf{g}_y(t)|_2 = |\mathcal{F}^{-1}[e^{2\pi\langle y,t\rangle}(e^{-\epsilon \sum_{j=1}^n z_j^2} - 1)\mathbf{f}(x+iy); t]|_2 \\ &= |e^{2\pi\langle y,t\rangle}(e^{-\epsilon \sum_{j=1}^n z_j^2} - 1)\mathbf{f}(x+iy)|_2. \end{aligned} \tag{23}$$

For $0 < \epsilon \leq 1$

$$\begin{aligned} &(\mathcal{N}(e^{2\pi\langle y,t\rangle}(e^{-\epsilon \sum_{j=1}^n z_j^2} - 1)\mathbf{f}(x+iy)))^2 \\ &= |e^{-\epsilon \sum_{j=1}^n z_j^2} - 1|^2 e^{4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{f}(x+iy)))^2 \\ &\leq (|e^{-\epsilon z_1^2}|...|e^{-\epsilon z_n^2}| + 1)^2 e^{4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{f}(x+iy)))^2 \\ &\leq (e^{|y|^2} + 1)^2 e^{4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{f}(x+iy)))^2, \end{aligned}$$

and the right side of this inequality is independent of $0 < \epsilon \leq 1$ and is integrable as a function of $x \in \mathbb{R}^n$. By the Lebesgue dominated convergence theorem (22) follows from (23).

To show that $\mathbf{g}_{\epsilon,y}(t)$ is independent of $y \in B$ let S be any compact subset of B, and let $y \in S \subset B$. We have

$$|e^{-\epsilon \sum_{j=1}^{n} z_j^2}| \leq e^{\epsilon n a^2} e^{-\epsilon |x|^2},\ x \in \mathbb{R}^n,\ y \in S,$$

where $a = \max_{y \in S}\{|y_1|, |y_2|, ..., |y_n|\}$. For $y \in S \subset B$ and $t \in \mathbb{R}^n$

$$\begin{aligned}
&\int_S \mathcal{N}(e^{-\epsilon \sum_{j=1}^{n} z_j^2} \mathbf{f}(x+iy) e^{-2\pi i \langle x+iy,t \rangle} dy \\
&= \int_S |e^{-\epsilon \sum_{j=1}^{n} z_j^2}||e^{-2\pi i \langle x+iy,t \rangle}| \mathcal{N}(\mathbf{f}(x+iy)) dy \\
&\leq A_S e^{\epsilon n a^2} e^{-\epsilon |x|^2} \int_S e^{2\pi |y||t|} dy
\end{aligned} \tag{24}$$

where A_S is a bound on $\mathcal{N}(\mathbf{f}(x+iy))$ for $x \in \mathbb{R}^n$ and $y \in S$; and the right side of (24) approaches 0 as $|x| \to \infty$. An application of the Caucyh-Poincare theorem yields $\mathbf{g}_{\epsilon,y}$ is independent of $y \in S$ for any $\epsilon > 0$ and hence independent of $y \in B$ for any $\epsilon > 0$ since S is any arbitrary compact subset of B. In the future we refer to $\mathbf{g}_{\epsilon,y}, y \in B$, as $\mathbf{g}_\epsilon$ since this function is independent of $y \in B$ for any $\epsilon > 0$.

Now to prove that $\mathbf{g}_y(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ is independent of $y \in B$ let y_1 and y_2 both be points of B. Since $\mathbf{g}_\epsilon = \mathbf{g}_{\epsilon,y}$ is independent of $y \in B$, for any $\epsilon > 0$ we have

$$\begin{aligned}
&|\mathbf{g}_{y_1}(t) - \mathbf{g}_{y_2}(t)|_2 = |\mathbf{g}_{y_1}(t) - \mathbf{g}_{\epsilon,y_1}(t) + \mathbf{g}_{\epsilon,y_2}(t) - \mathbf{g}_{y_2}(t)|_2 \\
&\leq |\mathbf{g}_{y_1}(t) - \mathbf{g}_{\epsilon,y_1}(t)|_2 + |\mathbf{g}_{y_2}(t) - \mathbf{g}_{\epsilon,y_2}(t)|_2.
\end{aligned} \tag{25}$$

Letting $\epsilon \to 0+$ in (25) and using (22), the right side of (25) approaches 0 while the left side is independent of $\epsilon > 0$. Thus, $\mathbf{g}_{y_1}(t) = \mathbf{g}_{y_2}(t)$ a.e., $t \in \mathbb{R}^n$, and $\mathbf{g}_y(t)$ defined in (21) is independent of $y \in B$. We write $\mathbf{g}_y(t)$ defined in (21) as $\mathbf{g}(t),\ y \in B,\ t \in \mathbb{R}^n$, in the future; and recall that $\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$. □

We obtain a Fourier–Laplace representation of elements in $H_A^2(T^B, \mathcal{H})$ now.

Theorem 9. *Let B be a proper open connected subset of $\mathbb{R}^n$. Let $\boldsymbol{f}(z) \in H_A^2(T^B, \mathcal{H})$, where $\mathcal{H}$ is Hilbert space, and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of B. There is a measurable function $\boldsymbol{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ for which*

$$|e^{-2\pi \langle y,t \rangle} \boldsymbol{g}(t)|_2 \leq M(1 + (d(y))^{-r})^s e^{2\pi A |y|},\ y \in B, \tag{26}$$

where $r \geq 0,\ s \geq 0,\ A \geq 0$, and $M = M(\boldsymbol{g}, r, s, A) > 0$ are independent of $y \in B$; and

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{g}(t) e^{2\pi i \langle z,t \rangle} dt,\ z \in T^B. \tag{27}$$

Proof. From Lemma 1 the function $\mathbf{g}(t) = \mathbf{g}_y(t)$ defined in (21) is independent of $y \in B$ and is in $L^2(\mathbb{R}^n, \mathcal{H})$. From (21)

$$e^{-2\pi \langle y,t \rangle} \mathbf{g}(t) = \mathcal{F}^{-1}[\mathbf{f}(x+iy); t],\ y \in B, \tag{28}$$

and by the Parseval equality

$$|e^{-2\pi \langle y,t \rangle} \mathbf{g}(t)|_2 = |\mathbf{f}(x+iy)|_2,\ y \in B,$$

where $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H}),\ y \in B$. Thus, (26) holds from the norm growth on $\mathbf{f}(z) \in H^2_A(T^B, \mathcal{H})$. Using the now obtained Equation (26), by the proof of Theorem 2 for $p = 2$ we have $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H}),\ y \in B$, and

$$\int_{\mathbb{R}^n} \mathbf{g}(t)e^{2\pi i\langle z,t\rangle}dt = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{g}(t); x],\ z = x + iy \in T^B,$$

is analytic in T^B with the Fourier transform being the $L^1(\mathbb{R}^n, \mathcal{H})$ transform. Thus, from (28),

$$\mathbf{f}(z) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{g}(t); x] = \int_{\mathbb{R}^n} \mathbf{g}(t)e^{2\pi i\langle z,t\rangle}dt,\ z = x + iy \in T^B,$$

with the Fourier transform being in both the $L^1(\mathbb{R}^n, \mathcal{H})$ and $L^2(\mathbb{R}^n, \mathcal{H})$ sense, and (27) is obtained. □

The structure of the proofs of Lemma 1 and Theorem 9 can be used to prove a result like Theorem 9 for functions in $H^2_*(T^B, \mathcal{H})$; we state this result now.

Theorem 10. *Let B be an open connected subset of $\mathbb{R}^n$ which does not contain $\bar{0} \in \mathbb{R}^n$. Let $f(z) \in H^2_*(T^B, \mathcal{H})$, where $\mathcal{H}$ is Hilbert space, and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of B. There is a measurable function $\boldsymbol{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ for which*

$$|e^{-2\pi\langle y,t\rangle}\boldsymbol{g}(t)|_2 \leq M(1 + (d(y))^{-r})^s e^{M^*(w/|y|)},\ y \in B,$$

where $r \geq 0, s \geq 0, w > 0$, and$M = M(\boldsymbol{g}, r, s, w) > 0$ are independent of $y \in B$; and

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{g}(t)e^{2\pi i\langle z,t\rangle}dt,\ z \in T^B.$$

By restricting the base B in Theorem 9, further information is obtained.

Corollary 5. *Let C be an open connected cone in $\mathbb{R}^n$. Let $\boldsymbol{f}(z) \in H^2_A(T^C, \mathcal{H})$, where $\mathcal{H}$ is Hilbert space, and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of C. There is a measurable function $\boldsymbol{g} \in L^2(\mathbb{R}^n, \mathcal{H}) \cap S'_2(\mathbb{R}^n, \mathcal{H})$ with $supp(\boldsymbol{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. such that (26) and (27) hold. Further, if C is an open convex cone in $\mathbb{R}^n$ we have*

$$\lim_{y \to \bar{0}, y \in C} |\boldsymbol{f}(x + iy) - \mathcal{F}[\boldsymbol{g}(t); x]|_2 = 0, \tag{29}$$

and

$$\lim_{y \to \bar{0}, y \in C} \boldsymbol{f}(x + iy) = \mathcal{F}[\boldsymbol{g}(t); x] \tag{30}$$

in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$.

Proof. The existence of $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ such that (26) and (27) hold follow from Theorem 9. The facts that $\mathbf{g} \in \mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$ with $\mathrm{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. now follow by Theorem 6. Let us further assume that the cone C is open and convex. From the proof of Theorem 8 we know that $\{t \in \mathbb{R}^n : u_C(t) \leq A\} = C^* + \overline{N(\bar{0}; A)}$ where C^* is the dual cone of C and $N(\bar{0}; A) = \{t \in \mathbb{R}^n : |t| < A\}$ since C is assumed to be convex now. Thus, $t \in \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ yields $t = t_1 + t_2,\ t_1 \in C^*,\ t_2 \in \overline{N(\bar{0}; A)}$ as in the proof of Theorem 8. Returning to the proof of Theorem 9 we have for $y \in C$

$$\begin{aligned} &|\mathbf{f}(x + iy) - \mathcal{F}[\mathbf{g}(t); x]|_2 = |\mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{g}(t); x] - \mathcal{F}[\mathbf{g}(t); x]|_2 \\ &= |\mathcal{F}[(e^{-2\pi\langle y,t\rangle} - 1)\mathbf{g}(t); x]|_2 = |(e^{-2\pi\langle y,t\rangle} - 1)\mathbf{g}(t)|_2. \end{aligned} \tag{31}$$

In (29) and (30), we prove limit properties as $y \to \overline{0},\ y \in C$; so we assume that $|y| \leq 1,\ y \in C$, in the remainder of this proof. For $t = t_1 + t_2 \in C^* + \overline{N(\overline{0};A)}$ we have

$$\begin{aligned}
&(\mathcal{N}((e^{-2\pi\langle y,t\rangle} - 1)\mathbf{g}(t)))^2 = |e^{-2\pi\langle y,t\rangle} - 1|^2(\mathcal{N}(\mathbf{g}(t)))^2 \\
&\leq (e^{-2\pi\langle y,t\rangle} + 1)^2(\mathcal{N}(\mathbf{g}(t)))^2 = (e^{-2\pi\langle y,t_1\rangle}e^{-2\pi\langle y,t_2\rangle} + 1)^2(\mathcal{N}(\mathbf{g}(t)))^2 \\
&\leq (1 + e^{2\pi A})^2(\mathcal{N}(\mathbf{g}(t)))^2
\end{aligned} \tag{32}$$

for $|y| \leq 1, y \in C$, where $\langle y, t_1\rangle \geq 0,\ y \in C$ and $t_1 \in C^*$, and $|t_2| \leq A$ for $t_2 \in \overline{N(\overline{0};A)}$. Since $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ and $\mathrm{supp}(\mathbf{g}) \subseteq C^* + \overline{N(\overline{0};A)}$, (32) and the Lebesgue dominated convergence theorem combined with (31) prove (29). For (30), let $\phi \in \mathcal{S}(\mathbb{R}^n)$. Using the Hölder inequality we have

$$\begin{aligned}
&\mathcal{N}(\langle \mathbf{f}(x+iy), \phi(x)\rangle - \langle \mathcal{F}[\mathbf{g}(t);x], \phi(x)\rangle \\
&\leq \int_{\mathbb{R}^n} \mathcal{N}((\mathbf{f}(x+iy) - \mathcal{F}[\mathbf{g}(t);x])\phi(x))dx \\
&\leq |\mathbf{f}(x+iy) - \mathcal{F}[\mathbf{g}(t);x]|_2 ||\phi||_{L^2(\mathbb{R}^n)},
\end{aligned}$$

and the use of (29) now shows (30) in the weak topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. But $\mathcal{S}(\mathbb{R}^n)$ is a Montel space; thus, (30) also holds in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. □

We now desire a converse result to Corollary 5 in the setting of tubes T^C where C is an open connected cone in $\mathbb{R}^n$.

Corollary 6. *Let C be an open connected cone in $\mathbb{R}^n$ and $\mathcal{H}$ be a Hilbert space. Let $\mathbf{g}(t)$ be a $\mathcal{H}$ valued measurable function on $\mathbb{R}^n$ such that* (26) *holds. We have $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ with $supp(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e., and $\mathbf{f}(z) \in H_A^2(T^C, \mathcal{H})$ for $\mathbf{f}(z)$defined as in* (27) *for $z \in T^C$. Further, if C is an open convex cone in $\mathbb{R}^n$ we have* (30) *holding in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$.*

Proof. We apply Theorem 6 and Corollary 1 to obtain $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ with $\mathrm{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. and to obtain that $\mathbf{f}(z)$ defined as in (27) for $z \in T^C$ is an element of $H_A^2(T^C, \mathcal{H})$. Now assume that C is an open convex cone in the remainder of this proof to obtain (30) here. Since $\mathbf{g} \in \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H}) \subset \mathcal{S}'(\mathbb{R}^n, \mathcal{H})$, the Fourier transform $\mathcal{F}[\mathbf{g}]$ is well defined in $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$. From the proof of Corollary 1 we have $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbf{R}^n, \mathcal{H})$ for $y \in C$. Thus, $\mathbf{f}(x+iy) = \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{g}(t);x],\ y \in C$, with the Fourier transform being in the $L^1(\mathbb{R}^n, \mathcal{H})$, the $L^2(\mathbb{R}^n, \mathcal{H})$, and the $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ cases. Recalling that $\mathrm{supp}(\mathbf{g}) \subseteq \{t \in \mathbf{R}^n : u_C(t) \leq A\}$ a.e. and referring to [9] (p. 119), we choose a function $\lambda(t) \in C^\infty,\ t \in \mathbb{R}^n$, such that for any n-tuple α of nonnegative integers $|D^\alpha\lambda(t)| \leq M_\alpha,\ t \in \mathbb{R}^n$, where M_α is a constant which depends only on α; and for $\epsilon > 0,\ \lambda(t) = 1$ for t on an ϵ neighborhood of $\{t \in \mathbb{R}^n : u_C(t) \leq A\}$, and $\lambda(t) = 0$ for $t \in \mathbb{R}^n$ but not on a 2ϵ neighborhood of $\{t \in \mathbb{R}^n : u_C(t) \leq A\}$. For $\phi \in \mathcal{S}(\mathbb{R}^n)$ we have for $y \in C$

$$\langle \mathbf{f}(x+iy), \phi(x)\rangle = \langle \mathcal{F}[e^{-2\pi\langle y,t\rangle}\mathbf{g}(t);x], \phi(x)\rangle = \langle \lambda(t)e^{-2\pi\langle y,t\rangle}\mathbf{g}(t), \mathcal{F}[\phi(x);t]\rangle.$$

For C being convex we apply [9] (p. 74, Lemma 3) as in our proof of Theorem 8 to obtin $\{t \in \mathbb{R}^n : u_C(t) \leq A\} = C^* + \overline{N(\overline{0};A)}$. The result (30) in this corollary now follows from the above equality, $\phi \in \mathcal{S}(\mathbb{R}^n)$, by the same analysis in [9] (p. 119, lines 2–22) in the weak topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to \overline{0}, y \in C$; and the weak topology implies the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as in the proof of (30) in Corollary 5. The proof is complete. □

Note that we can not say that $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$ in Corollary 6 and hence can not obtain the convergence (29) in this converse of Corollary 5.

For B being a proper open connected subset of $\mathbb{R}^n$ and $\mathcal{X}$ being a Banach space, the spaces $V_A^p(T^B, \mathcal{X})$ follow as subspaces of $H_A^p(T^B, \mathcal{X})$ (or appropriately of $R_A^p(T^B, \mathcal{X})$)

by letting either $r = 0$ or $s = 0$ in the norm growth defining these other spaces. Thus, Theorem 9 holds for $\mathbf{f}(z) \in V_A^2(T^B, \mathcal{H})$; and by the proof of Theorem 9, (26) will hold for the obtained function $\mathbf{g}$ in the form

$$|e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 \leq e^{2\pi A|y|},\ y \in B.$$

Using the same proof as in Corollary 2 we then can extend the norm growth on $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)$ to hold for $y \in \overline{B}$. This is stated in the following corollary to Theorem 9.

Corollary 7. *Let B be a proper open connected subset of $\mathbb{R}^n$. Let $\mathbf{f}(z) \in V_A^2(T^B, \mathcal{H})$, where $\mathcal{H}$ is Hilbert space, and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of B. There is a measurable function $\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ for which*

$$|e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 \leq Me^{2\pi A|y|},\ y \in \overline{B},$$

where $A \geq 0$ and $M = M(\mathbf{g}, A) > 0$ are independent of $y \in \overline{B}$; and

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{g}(t)e^{2\pi i\langle z,t\rangle}dt,\ z \in T^B.$$

For the base of the tube being an open connected cone in $\mathbb{R}^n$ we have the following corollary of Theorem 10 by combining Theorems 7 and 10. The limit properties in the following corollary will hold for C being an open connected cone in $\mathbb{R}^n$ by similar techniques as in the proof of Corollary 5; C does not need to be convex here for these limit properties to hold because the support of $\mathbf{g}$ is in C^*.

Corollary 8. *Let C be an open connected cone in $\mathbb{R}^n$. Let $\mathbf{f}(z) \in H_*^2(T^C, \mathcal{H})$, where $\mathcal{H}$ is Hilbert space, and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of C. There is a measurable function $\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H}) \cap \mathcal{S}_2'(\mathbb{R}^n, \mathcal{H})$ with $supp(\mathbf{g}) \subseteq C^*$ a.e. such that the norm inequality for $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)$ and the representation of $\mathbf{f}(z)$ hold as in the conclusions of Theorem 10. Further we have*

$$\lim_{y\to\bar{0}, y\in C} |\mathbf{f}(x+iy) - \mathcal{F}[\mathbf{g}(t);x]|_2 = 0$$

and

$$\lim_{y\to\bar{0}, y\in C} \mathbf{f}(x+iy) = \mathcal{F}[\mathbf{g}(t);x]$$

in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$.

7. Subsets of $H^2(T^C, \mathcal{H})$

Let C be an open connected cone in $\mathbb{R}^n$, and $1 \leq p < \infty$. Let $\mathbf{g}(t)$ be a measurable function on $\mathbb{R}^n$ with values in a Banach space $\mathcal{X}$ such that

$$|e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_p \leq M(1 + (d(y))^{-r})^s e^{2\pi A|y|},\ y \in C, \tag{33}$$

where $A \geq 0,\ r \geq 0,\ s \geq 0$, and $M = M(\mathbf{g}, p, r, s, A) > 0$, or

$$|e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_p \leq M(1 + (d(y))^{-r})^s e^{M^*(w/|y|)},\ y \in C, \tag{34}$$

where $w > 0,\ r \geq 0,\ s \geq 0$, and $M = M(\mathbf{g}, p, w, r, s) > 0$ with all constants being independent of $y \in C$. We have from Theorems 4 and 5 that $\mathrm{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. and $\mathrm{supp}(\mathbf{g}) \subseteq C^*$ a.e. respectively. Restricting to $p = 2$ and letting $\mathcal{X} = \mathcal{H}$, a Hilbert space, now we have from Corollarys 1 and 3 that the function

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{g}(t)e^{2\pi i\langle z,t\rangle}dt,\ z \in T^C,$$

is an element of $H^2_A(T^C,\mathcal{H})$ or $H^2_*(T^C,\mathcal{H})$, respectively. Conversely, we have proved in Corollary 5 or Corollary 8 that if $\mathbf{f}(z) \in H^2_A(T^C,\mathcal{H})$ or $\mathbf{f}(z) \in H^2_*(T^C,\mathcal{H})$ and in each case $\mathbf{f}(z)$ is bounded for $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$ in any compact subset of C then in each case there exists a measurable function $\mathbf{g} \in L^2(\mathbb{R}^n,\mathcal{H}) \cap \mathcal{S}'_2(\mathbb{R}^n,\mathcal{H})$ with $\mathrm{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \le A\}$ a.e. and (33) holds for $p = 2$ or $\mathrm{supp}(\mathbf{g}) \subseteq C^*$ a.e. and (34) holds for $p = 2$ with

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{g}(t) e^{2\pi i\langle z,t\rangle} dt,\ z \in T^C,$$

in each case.

We will now show from these results that both spaces $H^2_0(T^C,\mathcal{H})$, $A = 0$, and $H^2_*(T^C,\mathcal{H})$ are subsets of the Hardy space $H^2(T^C,\mathcal{H})$ and obtain immediate results from these subset properties.

Theorem 11. *Let C be an open connected cone in $\mathbb{R}^n$ and $\mathcal{H}$ be a Hilbert space. Let $\boldsymbol{f}(z) \in H^2_0(T^C,\mathcal{H})$ or $\boldsymbol{f}(z) \in H^2_*(T^C,\mathcal{H})$ and in either case be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of C. In either case there is a measurable function $\boldsymbol{g}(t) \in L^2(\mathbb{R}^n,\mathcal{H}) \cap \mathcal{S}'_2(\mathbb{R}^n,\mathcal{H})$ with $supp(\boldsymbol{g}) \subseteq C^*$ a.e. such that*

$$\boldsymbol{f}(z) = \int_{\mathbb{R}^n} \boldsymbol{g}(t) e^{2\pi i\langle z,t\rangle} dt,\ z \in T^C;$$

$$\sup_{y\in C} |\boldsymbol{f}(x+iy)|_2 = \sup_{y\in C} |e^{-2\pi\langle y,t\rangle}\boldsymbol{g}(t)|_2 = |\boldsymbol{g}|_2;$$

and $\boldsymbol{f}(z) \in H^2(T^C,\mathcal{H})$.

Proof. As noted previously in this section a function $\mathbf{g} \in L^2(\mathbb{R}^n,\mathcal{H}) \cap \mathcal{S}'_2(\mathbb{R}^n,\mathcal{H})$ is obtained from previous results such that

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathbf{g}(t) e^{2\pi i\langle z,t\rangle} dt,\ z \in T^C.$$

Further from the analysis leading to Corollarys 5 and 8 we know $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n,\mathcal{H}) \cap L^2(\mathbb{R}^n,\mathcal{H})$, $y \in C$, in both cases. If $A = 0$, $\{t \in \mathbb{R}^n : u_C(t) \le 0\} = C^*$; thus, in both cases $\mathrm{supp}(\mathbf{g}) \subseteq C^*$ a.e. In both cases we have

$$|\mathbf{f}(x+iy)|_2 = |e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2,\ y \in C.$$

In both cases

$$\int_{\mathbb{R}^n} (\mathcal{N}(e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)))^2 dt = \int_{C^*} (e^{-4\pi\langle y,t\rangle}(\mathcal{N}(\mathbf{g}(t)))^2 dt \le \int_{C^*} (\mathcal{N}(\mathbf{g}(t)))^2 dt = |\mathbf{g}|_2^2$$

for all $y \in C$. We thus have for all $y \in C$

$$|\mathbf{f}(x+iy)|_2 = |e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 \le |\mathbf{g}|_2,\ y \in C,$$

which yields $\mathbf{f}(x+iy) \in H^2(T^C,\mathcal{H})$. Further,

$$\sup_{y\in C} |\mathbf{f}(x+iy)|_2 = \sup_{y\in C} |e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)|_2 \le (\int_{C^*} (\mathcal{N}(\mathbf{g}(t)))^2 dt)^{1/2} = |\mathbf{g}|_2. \tag{35}$$

But $\overline{0} \in C^* = \{t \in \mathbb{R}^n : \langle t,y\rangle \ge 0 \text{ for all } y \in C\}$. Hence, the inequality in (35) is an equality. □

Because of this result we have immediate consequences for $\mathbf{f}(x+iy)$ in either space in Theorem 11 from previously proven results. If C is an open convex cone in $\mathbb{R}^n$ which contains an entire straight line then $\mathbf{f}(z) = \Theta$, $z \in T^C$, for both cases of $\mathbf{f}(z)$ in Theorem 11. If C is a regular cone in $\mathbb{R}^n$ then

$$\mathbf{f}(z) = \int_{\mathbb{R}^n} \mathcal{F}[\mathbf{g}(u);t]K(z-t)dt = \int_{\mathbb{R}^n} \mathcal{F}[\mathbf{g}(u);t]Q(z;t)dt, \; z \in T^C,$$

for the function $\mathbf{g}(t)$ in Theorem 11 and for both cases of $\mathbf{f}(z)$ in Theorem 11. Further, we note that Vindas has proved using functional analysis techniques in [1] that for C being a regular cone in $\mathbb{R}^n$ and $\mathcal{X}$ being a dual Banach space having the Radon-Nikodým property, any $\mathbf{f}(z) \in H^p(T^C, \mathcal{X})$, $1 \leq p \leq \infty$, is the Poisson integral of some $\mathbf{h} \in L^p(\mathbb{R}^n, \mathcal{X})$, $1 \leq p \leq \infty$. We say more about the use of functional analysis techniques in obtaining results corresponding to those of this paper and those of [1] in Section 9 below.

8. Boundary Values on the Topological Boundary

In Corollary 5 we obtained boundary value properties of $H^2_A(T^C, \mathcal{H})$ functions on the distinguished boundary of the tube T^C where C is an open convex cone in $\mathbb{R}^n$. The boundary values were obtained in the $L^2(\mathbb{R}^n, \mathcal{H})$ and $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ topologies. We now investigate boundary value properties of a subset of $H^2_A(T^C, \mathcal{H})$ on the topological boundary of the tube.

Our basic result in this section depends on the cone C being regular. We consider the subset $R^2_A(T^C, \mathcal{H})$ of $H^2_A(T^C, \mathcal{H})$ consisting of analytic functions $\mathbf{f}(z)$ in T^C with values in $\mathcal{H}$ such that

$$|\mathbf{f}(x+iy)|_2 \leq M(1 + |y|^{-r})^s e^{2\pi A|y|}, \; y \in C, \tag{36}$$

where $A \geq 0$, $r \geq 0$, $s \geq 0$, and $M = M(\mathbf{f}, A, r, s) > 0$ are all independent of $y \in C$. We prove that $R^2_A(T^C, \mathcal{H})$ functions have boundary values on the topological boundary of T^C again in the $L^2(\mathbb{R}^n, \mathcal{H})$ and $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ topologies. We have $R^2_A(T^C, \mathcal{H}) \subseteq H^2_A(T^C, \mathcal{H})$ since $0 < d(y) \leq |y|$ for y in any open connected cone in $\mathbb{R}^n$ from [2] (p. 6, (1.14)); recall Section 2 above.

Before proving our main result in this section we focus on the growth bound as in (36). If we had used this growth bound of (36) in the inequality (1) for $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t)$ and in the inequality for $|\mathbf{f}(x+iy)|_p$ which defines $H^p_A(T^B, \mathcal{X})$, that is if we replace $d(y)$ by $|y|$ in the growth bound, then the results, proofs, and conclusions from Theorem 2 through Theorem 11 in Sections 5–7 will all hold as before. In any conclusion in these results that contains the growth bound, the growth bound in the conclusion will be that of (36). We state this to emphasize the content of our proofs in this section which deal with $R^2_A(T^C, \mathcal{H})$ instead of $H^2_A(T^C, \mathcal{H})$.

Theorem 12. *Let C be a regular cone in $\mathbb{R}^n$. Let $\mathbf{f}(z) \in R^2_A(T^C, \mathcal{H})$ and be bounded for $x = Re(z) \in \mathbb{R}^n$ and $y = Im(z)$ in any compact subset of C. Let $y_o \in \partial C$, $y_o \neq \bar{0}$. There exists a function $\mathbf{F}(x+iy_o) \in L^2(\mathbb{R}^n, \mathcal{H})$ such that*

$$\lim_{y \to y_o} |\mathbf{f}(x+iy) - \mathbf{F}(x+iy_o)|_2 = 0 \tag{37}$$

for $y \in \{y \in C : 0 < a < |y| < b\}$ where a and b are any constants such that $0 < a < |y_o| < b$; and

$$\lim_{y \to y_o} \mathbf{f}(x+iy) = \mathbf{F}(x+iy_o) \tag{38}$$

in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ with $y \in \{y \in C : 0 < a < |y| < b\}$ again where a and b are any constants such that $0 < a < |y_o| < b$.

Proof. As noted previously the growth (36) for $R^2_A(T^C, \mathcal{H})$ functions is a special case of the growth for $H^2_A(T^C, \mathcal{H})$ functions since $0 < d(y) \leq |y|$, $y \in C$. Thus, $\mathbf{f}(z)$, $z \in T^C$, in this theorem satisfies the hypotheses of Corollary 5; and the conclusions of Corollary 5 follow for the $\mathbf{f}(z),\ z \in T^C$, here. In fact the construction of proofs above leading to Corollary 5 for the growth bound of type

$$M(1+(d(y))^{-r})^s e^{2\pi A|y|},\ y \in C,$$

would be the same for the growth of type (36) with $d(y)$ replaced by $|y|$ in the analysis of the proofs as noted before. Thus, there is a measurable function $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H}) \cap \mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$ with $\operatorname{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ a.e. such that (26) and (27) hold with $d(y)$ replaced by $|y|$ in (26), and $z = x+iy \in T^C$. From the construction of $\mathbf{g}$ in Lemma 1 and the proof of Theorem 2, $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H}),\ y \in C$. Let $y_o \in \partial C$, the boundary of C, $y_o \neq \overline{0}$. Since $|y_o| > 0$ choose constants a and b such that $0 < a < |y_o| < b$ and consider the band $\{y \in C : 0 < a < |y| < b\} \subset C$. Let $\{y_m\},\ m = 1,2,...$, be a sequence of points in this band which converges to y_o. For each $y_m, m = 1,2,...$, in this band

$$\int_{\mathbb{R}^n} (\mathcal{N}(e^{-2\pi\langle y_m,t\rangle}\mathbf{g}(t)))^2 dt \leq M^2(1+|y_m|^{-r})^{2s} e^{4\pi A|y_m|} \leq M^2(1+a^{-r})^{2s} e^{4\pi bA}.$$

Using Fatou's lemma we have

$$\begin{aligned}\int_{\mathbb{R}^n} (\mathcal{N}(e^{-2\pi\langle y_o,t\rangle}\mathbf{g}(t)))^2 dt &\leq \limsup_{y_m \to y_o} \int_{\mathbb{R}^n} (\mathcal{N}(e^{-2\pi\langle y_m,t\rangle}\mathbf{g}(t)))^2 dt \\ &\leq M^2(1+a^{-r})^{2s} e^{4\pi bA};\end{aligned}$$

and $e^{-2\pi\langle y_o,t\rangle}\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ for $y_o \in \partial C$; further $e^{-2\pi\langle y_o,t\rangle}\mathbf{g}(t) \in L^2(\mathbb{R}^n, \mathcal{H})$ even if $y_o = \overline{0}$ since $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$. Recall $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H}) \cap \mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$ and $e^{-2\pi\langle y,t\rangle}\mathbf{g}(t) \in L^1(\mathbb{R}^n, \mathcal{H}) \cap L^2(\mathbb{R}^n, \mathcal{H}),\ y \in C$. Form

$$\mathbf{F}(x+iy_o) = \mathcal{F}[e^{-2\pi\langle y_o,t\rangle}\mathbf{g}(t); x],\ y_o \in \partial C,\ y_o \neq \overline{0};$$

thus, $\mathbf{F}(x+iy_o) \in L^2(\mathbb{R}^n, \mathcal{H}),\ y_o \in \partial C,\ y_o \neq \overline{0}$. From the definition of $\mathbf{F}(x+iy_o)$ and Corollary 5 we have

$$\begin{aligned}|\mathbf{f}(x+iy) - \mathbf{F}(x+iy_o)|_2 &= |\mathcal{F}[(e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle})\mathbf{g}(t); x]|_2 \\ &= |(e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle})\mathbf{g}(t)|_2,\end{aligned} \tag{39}$$

for $y \in C$ and $y_o \in \partial C,\ y_o \neq \overline{0}$. We consider

$$\int_{\mathbb{R}^n} (\mathcal{N}((e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle})\mathbf{g}(t)))^2 dt$$

and want to show that this integral approaches 0 as $y \to y_o,\ y \in \{y \in C : 0 < a < |y| < b\}$. We have $\operatorname{supp}(\mathbf{g}) \subseteq \{t \in \mathbb{R}^n : u_C(t) \leq A\} = C^* + \overline{N(\overline{0}; A)}$ since C is open and convex as noted before in the proof of Theorem 8; thus, $t \in \{t \in \mathbb{R}^n : u_C(t) \leq A\}$ implies $t = t_1 + t_2$ where $t_1 \in C^*$ and $t_2 \in \overline{N(\overline{0}, A)}$. For $y \in \{y \in C : 0 < a < |y| < b\}$ with $0 < a < |y_o| < b$ by definition of a and b we have for almost all $t \in \mathbb{R}^n$

$$(\mathcal{N}((e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle})\mathbf{g}(t)))^2 = |e^{-2\pi\langle y,t_1+t_2\rangle} - e^{-2\pi\langle y_o,t_1+t_2\rangle}|^2(\mathcal{N}(\mathbf{g}(t)))^2.$$

Since $t_1 \in C^*$, $\langle y, t_1\rangle \geq 0$ for all $y \in C$ which implies $\langle y_o, t_1\rangle \geq 0$ also. Continuing the preceding inequality we have for $t_1 \in C^*$, $t_2 \in \overline{N(\overline{0}, A)}$, and all $y \in \{y \in C : 0 < a < |y| < b\}$

$$\begin{aligned}&(\mathcal{N}((e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle})\mathbf{g}(t)))^2 \leq (e^{-2\pi\langle y,t_2\rangle} + e^{-2\pi\langle y_o,t_2\rangle})^2(\mathcal{N}(\mathbf{g}(t)))^2 \\ &\leq (e^{2\pi|y||t_2} + e^{2\pi|y_o||t_2|})^2(\mathcal{N}(\mathbf{g}(t)))^2 \leq 4e^{4\pi bA}(\mathcal{N}(\mathbf{g}(t)))^2\end{aligned}$$

with the bound being independent of $y \in \{y \in C : 0 < a < |y| < b\}$ and being in $L^1(\mathbb{R}^n)$ since $\mathbf{g} \in L^2(\mathbb{R}^n, \mathcal{H})$. Since $(e^{-2\pi\langle y,t\rangle} - e^{-2\pi\langle y_o,t\rangle})\mathbf{g}(t) \to \Theta$ as $y \to y_o,\ y \in \{y \in C : 0 <$

$a < |y| < b\}$ with $0 < a < |y_o| < b$, the Lebesgue dominated convergence theorem and (39) yield (37).

To prove (38) let $\phi \in \mathcal{S}(\mathbb{R}^n)$ and $y_o \in \partial C$, $y_o \neq \overline{0}$. As before choose constants a and b such that $0 < a < |y_o| < b$. For $y \in \{y \in C : 0 < a < |y| < b\}$ we have

$$\begin{aligned} &\mathcal{N}(\langle \mathbf{f}(x+iy), \phi(x)\rangle - \langle \mathbf{F}(x+iy_o), \phi(x)\rangle) \\ &\leq \int_{\mathbb{R}^n} \mathcal{N}((\mathbf{f}(x+iy) - \mathbf{F}(x+iy_o))\phi(x))dx \\ &\leq |\mathbf{f}(x+iy) - \mathbf{F}(x+iy_o)|_2 ||\phi||_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Using (37) we obtain (38) in the weak topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ as $y \to y_o$, $y \in \{y \in C : 0 < a < |y| < b\}$ with $0 < a < |y_o| < b$. Now (38) is obtained in the strong topology of $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ since $\mathcal{S}(\mathbb{R}^n)$ is a Montel space. The proof is complete. □

Since both $R^2_A(T^C, \mathcal{H})$ and $V^2_A(T^C, \mathcal{H})$ are subsets of $H^2_A(T^C, \mathcal{H})$, functions in both of these subset spaces satisfy (29) and (30) on the distinguished boundary of T^C with C being a regular cone. Also $V^2_A(T^C, \mathcal{H})$ functions will have the results of Theorem 12 since $V^2_A(T^C, \mathcal{H}) \subseteq R^2_A(T^C, \mathcal{H})$.

Boundary value results for the analytic functions on the topological boundary of the tube may be able to be obtained for various types of base sets C of the tube T^C. For example one could consider C to be an open polyhedron in $\mathbb{R}^n$ as defined in [11] and [12] (p. 97). One could follow this situation by considering an open convex subset B of $\mathbb{R}^n$ with y_o being a point on its boundary; consideration could be given then to constructing an open polyhedron in B with y_o as boundary point and approaching y_o within the open polyhedron as Stein and Weiss have done in [12] (p. 98) for functions in $H^2(T^B)$. Clearly the types of boundary values available will depend on the specifics of the analytic functions and on the base of the tube if boundary values exist at all. More will be stated in Section 9 concerning boundary values.

We have previously obtained boundary value results on the distinguished boundary of the tube for functions of type $V^p_*(T^C), 1 < p \leq 2$, in the scalar-valued ultradistribution sense where C is a regular cone in $\mathbb{R}^n$. That is, the norm growth on the analytic functions on T^C is

$$||\mathbf{f}(x+iy)||_{L^p(\mathbb{R}^n)} \leq K e^{M^*(w/|y|)}, \ y \in C,$$

where $w > 0$ and $K = K(\mathbf{f}, p, w)$ are independent of $y \in C$. We have proved that such functions obtain a boundary value at $\overline{0}$ in the ultradistribution space $\mathcal{D}'((M_p), L^1(\mathbb{R}^n))$. We refer to [2] (p. 106, Theorem 5.2.1) and the preceding analysis in [2] (Section 5.2).

9. Suggested Research

In this section, we suggest problems to consider in future research which are associated with the analysis of this paper.

Let B be an open connected subset of $\mathbb{R}^n$. Stein and Weiss use a bound condition on $H^p(T^B)$ obtained in [12] (p. 99, Lemma 2.12) to prove [12] (p. 93, Theorem 2.3), the representation theorem for functions in $H^2(T^B)$. The bound condition holds for z in a tube whose base is restricted uniformly away from the complement of B. We have used a similarly needed growth condition, obtained in [2] (p. 87, Lemma 5.1.3), on the analytic functions studied in [2] (Chapter 5) in relation to boundary values in ultradistribution spaces.

Starting with Lemma 1 in Section 6 of this paper we have used the following assumption on $\mathbf{f}(z) \in H^2_A(T^B, \mathcal{H})$ to obtain several results; the assumption on $\mathbf{f}(z)$ is that it "be bounded for $x = \text{Re}(z) \in \mathbb{R}^n$ and $y = \text{Im}(z)$ in any compact subset of B". We conjecture that a bound condition like [12] (p. 99, Lemma 2.12) holds for $\mathbf{f}(z) \in H^p_A(T^B, \mathcal{X})$; such a result will allow us to delete the above quoted assumption used in Sections 6–8.

Additionally we suggest research to obtain a bound condition like [12] (p. 99, Lemma 2.12) for functions in $H^p(T^B, \mathcal{X})$.

Throughout this paper we have obtained boundary value results both on the distinguished boundary of the tube and on the topological boundary of the tube. In every case a question that had to be considered was the method to approach a point on the boundary by points in the base in order to obtain a desired result. Our results before Section 8 concerned tubes with base being a regular cone, an open connected cone in $\mathbb{R}^n$, or a proper open connected subset of $\mathbb{R}^n$. In these cases we could approach a considered boundary point y_o on the boundary of the base by a sequence of points within the base. Because of the nature of the analytic functions considered in Section 8 we needed to approach any boundary point y_o, $y_o \neq \overline{0}$, on the boundary of the base, a regular cone, by a sequence of points inside a band contained in the cone in order to obtain the desired result. Indications of other boundary point approaches for consideration were stated at the end of Section 8.

Stein and Weiss [12] (pp. 94–98) discuss situations in which boundary values on the boundary of tubes can not be obtained as points within the base arbitrarily approach the point y_o on the boundary of the base. In the first case a specific type of analytic function was constructed in order to show the non-existence of a boundary value for arbitrary approach to a point on the boundary by points within the base. In the second case a $H^2(T^B)$ function was constructed for which no limit in the L^2 norm existed for arbitrary approach to $\overline{0}$ within B; but if the base B was suitably restricted, any function in $H^2(T^B)$ for the restricted base B was shown to have a boundary value at any point on ∂B. Considerations of the approach to the boundary by points within bases B of other types than those of this paper could be made concerning the types of analytic functions defined in this paper. Are there base sets B in which an analytic function will not have a boundary value at a specified point $y_o \in \partial B$ or such that there could be a boundary value if the base B is specialized?

The basic results of Section 5, Theorems 2, 4, 6 and 8, have all been proved for the most general appropriate situation. B was an open connected subset of $\mathbb{R}^n$ or open (or convex) connected cone in $\mathbb{R}^n$; values were in Banach space $\mathcal{X}$; results held for all p, $1 \leq p < \infty$, in Section 5. In Sections 6–8, the base B of the tube remained an open connected subset of $\mathbb{R}^n$ or a cone in $\mathbb{R}^n$ as appropriate; but all of the main results of these sections were proved for values in Hilbert space $\mathcal{H}$ with $p = 2$.

Of course the reason for the restrictions in these sections to $p = 2$ and values in $\mathcal{H}$ is that the primary tool in our proofs was the Fourier transform which, as previously noted, is available in its desired completeness to the specific cases of $p = 2$ and values in $\mathcal{H}$. We desire to extend the results of Sections 6–8 to $1 \leq p < \infty$ and values in Banach space $\mathcal{X}$ as appropriate by using different techniques. This has been done by Vindas in [1] where functional analysis techniques have been used to extend the Poisson integral representation of functions in $H^p(T^C, \mathcal{H})$ from $p = 2$ with values in $\mathcal{H}$ to $1 \leq p \leq \infty$ with values in $\mathcal{X}$. See [1] (Theorem 2); similarly see also [1] (Theorem 1). Use of functional analysis techniques and accumulated knowledge related to vector-valued fuctions to obtain the desired extensions of the results noted in this paragraph should be considered. Extensions of results from $p = 2$ to $1 \leq p < \infty$ could possibly also be obtained here for Hilbert space $\mathcal{H}$ by applying limit processes using the $p = 2$ case. We believe that the basic results of Sections 6–8 can be extended to $1 \leq p < \infty$ and values in Banach space $\mathcal{X}$ as appropriate. We suggest consideration of this extension in future research.

For $p = 2$ we have proved in previous work that the $\mathcal{S}'(\mathbb{R}^n)$ Fourier transform maps the distribution space $\mathcal{D}'_{L^2(\mathbb{R}^n)}$ one-one and onto $\mathcal{S}'_2$; further we have proved that the $\mathcal{S}'(\mathbb{R}^n)$ Fourier transform maps $\mathcal{D}'_{L^p(\mathbb{R}^n)}$, $1 \leq p < 2$, one-one and into $\mathcal{S}'_q$, $(1/p) + (1/q) = 1$. The proofs are obtained using the characterization results for the form of elements in $\mathcal{D}'_{L^p(\mathbb{R}^n)}$, $1 \leq p \leq 2$. With knowledge of a characterization of elements in the vector-valued distribution space equivalent to $\mathcal{D}'_{L^2(\mathbb{R}^n)}$ we conjecture that the $\mathcal{S}'(\mathbb{R}^n, \mathcal{H})$ Fourier transform maps this vector-valued distribution space one-one and onto $\mathcal{S}'_2(\mathbb{R}^n, \mathcal{H})$. Of course the values of the vector-valued distributions would need to be in Hilbert space $\mathcal{H}$ because of the probable use of the function Fourier transform on $L^2(\mathbb{R}^n, \mathcal{H})$ functions.

Results similar to those of this paper may be in order concerning the functions defined as $H(C)$ in [7]. We leave this for future research.

10. Conclusions

As stated in Section 1 our goal in this paper was to obtain results for the analytic functions defined in Section 4 treated as generalizations of $H^p(T^B, \mathcal{X})$ functions and as generalizations of the scalar-valued functions noted in [2] (Chapter 5) and in some of our papers referenced in [2] and hence to generalize results concerning $H^p(T^B, \mathcal{X})$ spaces and concerning the functions of [2] (Chapter 5) to these new spaces of analytic functions. Additionally, we stated that our goal also was to obtain additional new results for the analytic fuctions of Section 4.

We were successful in our goals in Section 5 for all of the results there that had as assumption that $\mathbf{g}(t)$ was a $\mathcal{X}$ valued measurable function for which the growth (1) held and for all of the results that had as assumption that $\mathbf{g} \in \mathcal{S}'_p(\mathbb{R}^n, \mathcal{X})$; these results held for $\mathcal{X}$ being a Banach space and for all $p, 1 \leq p < \infty$.

We were partially successful in our goals in Section 6 where the results depended on hypotheses on the analytic function concerning $\mathcal{X}$ and p. Because our proofs of these results depended on the Fourier transform we had to restrict $\mathcal{X}$ to $\mathcal{H}$, a Hilbert space, and $p = 2$ as described previously. But under these restrictions in Section 6 we were able to obtain Fourier–Laplace integral representation and boundary value results on the distinguished boundary of the tube for the analytic functions. In Section 7, we were able to prove containment of certain analytic functions from Definitions 1–4 in the Hardy space $H^2(T^C, \mathcal{H})$. In Section 8, we were able to obtain boundary value results on the topological boundary of the tube domain for the functions considered there. We desire to have the results of Sections 6–8 holding as well for $\mathcal{X}$ being a Banach space and for $1 \leq p < \infty$.

In our previous work concerning scalar-valued generalizations of $H^p(T^B)$ functions we have been able to obtain results under the assumption on the analytic functions of the type in Sections 6–8 for all $p,\ 1 \leq p < \infty$. That is we have obtained Fourier–Laplace integral representation and boundary value results for all $p,\ 1 \leq p < \infty,$ on the assumed scalar-valued analytic function. Additionally, we have obtained Cauchy and Poisson integral representations as appropriate. Because of the existence of these results for all p in the scalar-valued case we have emphasized in Section 9 our belief that the basic results of Sections 6–8 can be extended to $1 \leq p < \infty$ and to values in Banach space $\mathcal{X}$ under assumption on the analytic function in the results. We believe that new techniques apart from the Fourier transform will be used to obtained these desired results as described in Section 9. We pursue the analysis of these topics for the generalized setting in the future.

The author believes that there is considerable additional interesting analysis in the generalized format of the results in this paper that can be obtained in regards to integral representation, boundary values, and applications for the functions of Definitions 1–4.

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