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Statistical Inference of the Beta Binomial Exponential 2 Distribution with Application to Environmental Data

Osama H. Mahmoud Hassan [1,*], Ibrahim Elbatal [2], Abdullah H. Al-Nefaie[1] and Ahmed R. El-Saeed [3]

1 Department of Quantitative Methods, School of Business, King Faisal University, Al Hofuf 31982, Al-Ahsa, Saudi Arabia
2 Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh 11432, Saudi Arabia
3 Department of Basic Sciences, Obour High Institute for Management & Informatics, Obour 11848, Egypt
* Correspondence: ohhassan@kfu.edu.sa

Abstract: A new four-parameter lifetime distribution called the beta binomial exponential 2 ($BBE2$) distribution is proposed. Some mathematical features, including quantile function, moments, generating function and characteristic function, of the $BBE2$ distribution, are computed. When the life test is truncated at a predetermined time, acceptance sampling plans (ASP) are constructed for the $BBE2$ distribution. The truncation time is supposed to represent the median lifetime of the $BBE2$ distribution with predetermined factors for the smallest sample size required to guarantee that the prescribed life test is achieved at a given consumer's risk. Some numerical results for a given consumer's risk, $BBE2$ distribution parameters and truncation time are derived. Classical (maximum likelihood and maximum product of spacing estimation methods) and Bayesian estimation approaches are utilized to estimate the model parameters. The performance of the model parameters is examined through the simulation study by using the three different approaches of estimation. Subsequently, we examine real-world data applications to demonstrate the versatility and potential of the $BBE2$ model. A real-world application demonstrates that the new distribution can offer a better fit than other competitive lifetime models.

Keywords: beta family; acceptance sampling plan; binomial exponential 2; moments; Bayesian approach; maximum product spacing; maximum likelihood

MSC: 60E05; 62E15; 62F10; 62P99.

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1. Introduction

The monotonicity of the hazard rate of a life distribution is essential in modeling failure time data. Distributions with an increasing failure rate (IFR) are relevant in pricing and supply chain contracting problems. The IFR property is a well-established and useful concept in dynamic programming, reliability theory and other areas of statistics (see, for example, [1,2]). Ref. [3] proposed the binomial exponential 2 (BE2) distribution, a new two-parameter lifetime distribution with IFR that is designed and built as a random sum of independent exponential random variables (RVs) when the sample size has a zero truncated binomial distribution. As an alternative to the Weibull, exponentiated exponential, gamma and weighted exponential distributions, this distribution can be employed.

The cumulative distribution function (cdf) of $BE2$ distribution is provided via

$$G_{BE2}(x,\lambda,\theta) = 1 - \left(1 + \frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}, x > 0, \tag{1}$$

where $\lambda, \theta > 0$ are two scale parameters, where $, 0 \leq \theta \leq 1$. The corresponding probability density function (pdf) is provided below.

$$g_{BE2}(x, \lambda, \theta) = \left(1 + \frac{(\lambda x - 1)\theta}{2 - \theta}\right)\lambda e^{-\lambda x} = \frac{\lambda}{2 - \theta}[2(1 - \theta) + \lambda x \theta]e^{-\lambda x}. \tag{2}$$

Equation (2) could be expressed as

$$g_{BE2}(x, \lambda, \theta) = \pi \lambda e^{-\lambda x} + (1 - \pi)\lambda^2 x e^{-\lambda x}, \tag{3}$$

where $\pi = \frac{2(1-\theta)}{2-\theta}$ and the $BE2$ distribution is a mixture of an exponential (λ) distribution and a gamma (2, λ) distribution with mixing proportion π. We have realized that at $\theta = 0$, we have the standard exponential distribution, and at $\theta = 1$, the $BE2$ model reduces to the gamma (2, λ) distribution.

Ref. [4] invented the beta (B) generalized (B-G) distribution as a rich class of generalized distributions. This class has received a lot of attention in recent years. The distributions that have been investigated are as follows: the B normal, proposed by [5]; the B Gumbel distribution, suggested by [6]; the B Frechet distribution, investigated by [7]; the B Weibull distribution, introduced by [8]; the B Weibull geometric, presented by [9]; the B generalized exponential distribution, introduced by [10]; the B-modified Weibull distribution, investigated by [11]; the B inverse Weibull, proposed by [12]; the B generalized Pareto, discussed by [13]; the B exponentiated Weibull distribution, studied by [14], among others. The cdf of the B-G distribution has the below form.

$$F(x) = \frac{1}{B(a,b)} \int_0^{G(x)} w^{(a-1)}(1 - w)^{b-1} dw = \frac{B_{G(x)}(a,b)}{B(a,b)} = I_{G(x)}(a,b), a, b > 0, \tag{4}$$

where $G(x)$ is an arbitrary baseline cdf of an RV and $B_y(a,b) = \int_0^y w^{(a-1)}(1 - w)^{b-1} dw$ is the incomplete B function with $B(a,b) = B_1(a,b)$ and $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$ is the incomplete B function ratio. The associated pdf to (4) is provided via

$$f(x) = \frac{g(x)}{B(a,b)} G(x)^{a-1}\{1 - G(x)\}^{b-1}, a, b > 0, x \in R. \tag{5}$$

For example, some researchers have suggested techniques for adding probability models. This phenomenon of parameter addition creates more robust families of distributions, which are efficiently employed for modeling datasets in biological research, engineering, economics and environmental sciences. As a result, some well-known courses are the odd Fréchet$-G$ by [15] exponentiated generalized $-G$ proposed by [16], odd-generalized N-H $-G$ by [17], $T - X$ class by [18], exponentiated power-generalized Weibull power series $-G$ by [19], the Weibull-$\mathcal{G}$ by [20], Type-II half logistic class by [21], truncated Cauchy power Weibull $-G$ class of distributions by [22], odd Perks $-G$ class of distributions by [23], Type-I half logistic Burr X $-G$ family by [24], sine Topp–Leone $-G$ family of distributions by [25], a new power Topp–Leone-generated family of distributions by [26], truncated inverted Kumaraswamy-generated family of distributions by [27], alpha power transformation family of distributions introduced by [28], exponentiated version of the M class of distributions introduced by [29], transmuted odd Fréchet-G family of distributions proposed by [30], among others.

The aim, goal and novelty of this paper can be considered as the following items:

(i) Provide a generalization of the $BE2$ distribution by including two additional shape parameters that allow for larger adaptability in the form of the beta binomial exponential 2 ($BBE2$) distribution and, as a result, in modeling observed positive data.

(Ii) The pdf of the $BBE2$ distribution can take different shapes, such as decreasing, unimodal and right skewness, and the shapes of the hazard rate function (hrf) can be decreasing, increasing and constant.

(IIi) Some statistical and mathematical features of the $BBE2$ distribution are computed and investigated.
(iv) Develop an acceptance sampling plan (ASP), derive its operating characteristic function and give the corresponding decision rule by using the $BBE2$ distribution.
(v) Study two classical approaches of estimation; maximum likelihood (ML) and maximum product of spacing ($MPSP$). Further, the Bayesian approach of estimation is utilized to estimate the model parameters.
(vi) The significance of the $BBE2$ model is demonstrated through a study of real-world data applications, which demonstrates the flexibility and potential of the $BBE2$ model in comparison to other well-known competitive models.

The rest of the paper is organized as follows: In Section 2, we define the $BBE2$ distribution and some special cases of this model. Several structural properties of the $BBE2$ distribution, including quantile function, moments, moment-generating function and characteristic function, are discussed in Section 3. ASP using the $BBE2$ distribution is derived in Section 4. In Sections 5 and 6, we demonstrate the ML estimates ($MLEs$), $MPSP$ estimates ($MPSPEs$) and Bayesian estimates (BEs) of the unknown parameters. The performance of the model parameters is examined through the simulation results in Section 7. In Section 8, we demonstrate the significance of the new model by studying real-world data applications to demonstrate its versatility and potential. Finally, the concluding remarks are illustrated in Section 9.

2. Beta Binomial Exponential 2 Distribution

In this section, we will go over the four-parameter $BBE2$ distribution by taking $G(x)$ in (4) to be the cdf of the $BE2$ distribution. We assume that $\xi = (\lambda, \theta, a, b)$ are the parameters of the new model, where $\lambda, a, b > 0$, $0 \leq \theta \leq 1$. The cdf of the $BBE2$ distribution can indeed be expressed as

$$F_{BBE2}(x,\xi) = I_{\left\{1-\left(1+\frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}\right\}}(a,b) = \frac{1}{B(a,b)}\int_0^{1-\left(1+\frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}} w^{(a-1)}(1-w)^{b-1}dw, x > 0. \tag{6}$$

The pdf of the $BBE2$ distribution takes the next form

$$f_{BBE2}(x,\xi) = \frac{\lambda e^{-b\lambda x}}{B(a,b)}\left(1+\frac{(\lambda x-1)\theta}{2-\theta}\right)\left[1-\left(1+\frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}\right]^{a-1}\left(1+\frac{\lambda\theta x}{2-\theta}\right)^{b-1}. \tag{7}$$

The reliability function (RF) of the $BBE2$ distribution is defined as

$$\overline{F}_{BBE2}(x,\xi) = 1 - F_{BBE2}(x,\xi) = 1 - I_{\left\{1-\left(1+\frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}\right\}}(a,b).$$

For the $BBE2$ distribution, the hazard rate function takes the form

$$h_{BBE2}(x) = \frac{\frac{\lambda e^{-b\lambda x}}{B(a,b)}\left(1+\frac{(\lambda x-1)\theta}{2-\theta}\right)}{1 - I_{1-\left\{1-\left(1+\frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}\right\}}(a,b)} \times \left[1-\left(1+\frac{\lambda\theta x}{2-\theta}\right)e^{-\lambda x}\right]^{a-1}\left(1+\frac{\lambda\theta x}{2-\theta}\right)^{b-1}.$$

Figures 1 and 2 represent some plots of the probability density and survival function of the $BBE2$ distribution for some different values of the parameters. The pdf can take the following forms: decreasing, unimodal and right skewness. However, the hazard rate function can be decreasing, increasing and constant.

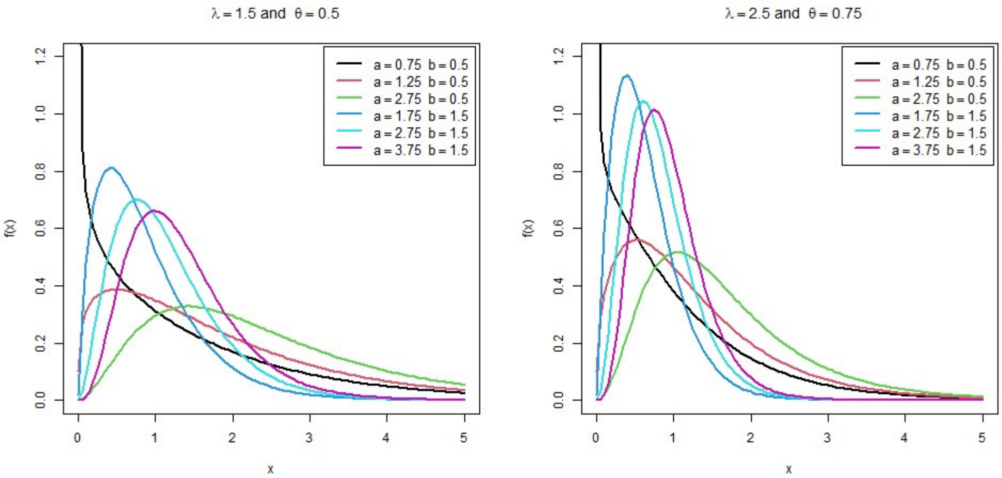

Figure 1. Plots of the pdf for the $BBE2$ distribution.

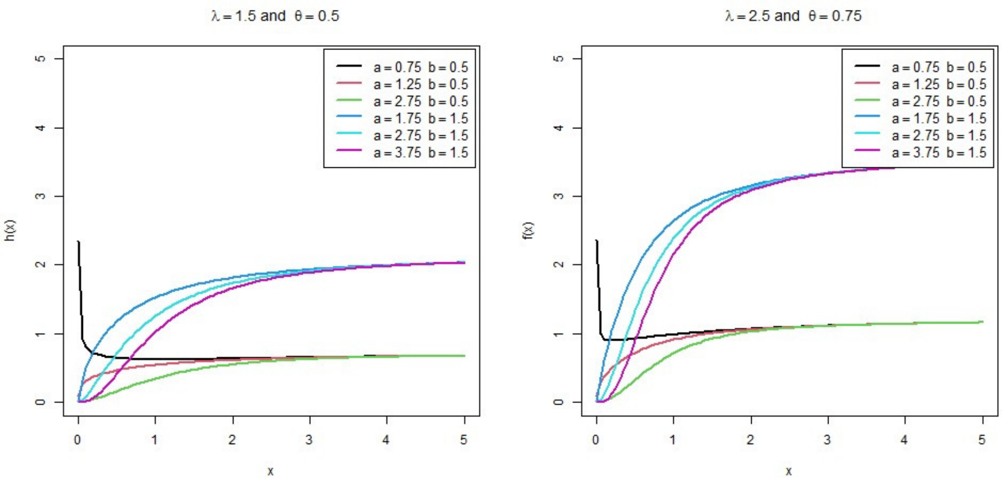

Figure 2. Plots of the hrf for the $BBE2$ distribution.

A Useful Representation

Here, we reveal the representation of pdf of the $BBE2$ distribution. The mathematical relation offered below will be relevant in this section. If $b > 0$, a positive, real non-integer, and $\mid z \mid < 1$, see [31]

$$(1-z)^{b-1} = \sum_{i=0}^{\infty} (-1)^i \binom{b-1}{i} z^i = \sum_{i=0}^{\infty} (-1)^i \frac{\Gamma(b)}{\Gamma(b-i)i!} z^i, \tag{8}$$

and

$$(1+z)^{b-1} = \sum_{j=0}^{\infty} \binom{b-1}{j} z^j = \sum_{j=0}^{\infty} \frac{\Gamma(b)}{\Gamma(b-j)j!} z^j, \tag{9}$$

and if b is a positive real integer, then the upper of this summation stops at $(b-1)$. This means that the $BBE2$ pdf can indeed be demonstrated as a mixture of $GBE2$ pdfs. As a result, numerous $BBE2$ characterizations can be obtained from those of the $GBE2$ model. By using (8) and (9) in (7), and after some algebraic manipulation, an alternative expression for the pdf is provided via

$$
\begin{aligned}
f_{BBE2}(x,\xi) &= \frac{\lambda^{j+1}\theta^j}{B(a,b)(2-\theta)^{j+1}} \sum_{i,j=0}^{\infty} (-1)^i \binom{a-1}{i}\binom{b+i-1}{j} x^j (2(1-\theta)+\lambda\theta x) e^{-\lambda(b+i)x} \\
&= \sum_{i,j=0}^{\infty} d_{i,j} \Big(2(1-\theta)x^j + \lambda\theta x^{j+1}\Big) e^{-\lambda(b+i)x}. \quad (10)
\end{aligned}
$$

where

$$
d_{i,j}(a,b) = \frac{\lambda^{j+1}\theta^j}{B(a,b)(2-\theta)^{j+1}} \sum_{i,j=0}^{\infty} (-1)^i \binom{a-1}{i}\binom{b+i-1}{j}
$$

The $BBE2$ distribution is a very flexible model because it contains well-known distributions as sub-models in Table 1.

Table 1. Some sub-models of the $BBE2$ distribution

λ	θ	a	b	Distribution	Authors
–	–	–	1	generalized $BE2$	[32]
–	–	1	1	$BE2$	[3]
–	0	–	–	B exponential	[33]
–	2	–	–	B gamma	[34]
–	0	1	1	exponential	
–	2	1	1	gamma	

3. Statistical Features of the *BBE2* Distribution

In this section, we started to look at some statistical features of the $BBE2$ distribution, including the quantile function, ordinary moments, moment-generating function and characteristic function.

3.1. Quantile Function

For an RV X with a cdf of the $BBE2$ distribution, the quantile function $Q(u)$ is computed by the relation $Q(u) = \inf\{x \in R : F(x) \geq u\}$ where $0 < u < 1$. This relation is utilized to get the quantile function of the $BBE2$ distribution from Equation (6) as below

$$
\left(1+\frac{\lambda\theta u}{2-\theta}\right)e^{-\lambda u} = 1 - I_u^{-1}(a,b), u \in (0,1). \quad (11)
$$

The above equation is non-linear, and we can solve it numerically.

3.2. Moments

In this subsection, we will look at the r^{th} moment of the $BBE2$ distribution.

Theorem 1. *If X has an RV $BBE2(\lambda,\theta,a,b)$ then the r^{th} moment of X is provided via*

$$
\mu_r' = \sum_{i,j=0}^{\infty} d_{i,j} \frac{\Gamma(r+j+1)}{(\lambda(b+i))^{r+j+2}} [2\lambda(1-\theta)(b+i) + \lambda\theta(r+j+1)]. \quad (12)
$$

Proof. We assume that X is an RV following the $BBE2$ distribution. The well-known formula for calculating the r^{th} ordinary moment can be employed as below

$$
\mu_r' = \int_0^{\infty} x^r f(x)dx = \sum_{i,j=0}^{\infty} d_{i,j} \int_0^{\infty} \Big(2(1-\theta)x^{r+j} + \lambda\theta x^{r+j+1}\Big) e^{-\lambda(b+i)x} dx.
$$

□

By setting $y = \lambda(b+i)x$, the r_{th} moment can be expressed as

$$\mu_r^{'} = \sum_{i,j=0}^{\infty} d_{i,j} \frac{\Gamma(r+j+1)}{(\lambda(b+i))^{r+j+2}} [2\lambda(1-\theta)(b+i) + \lambda\theta(r+j+1)],$$

where $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$ denotes the gamma function.

3.3. Moment-Generating Function

Theorem 2. *Suppose that X has the $BBE2(\lambda, \theta, a, b)$, then the moment-generating function of RV X is as below*

$$M_X(t) = \sum_{i,j=0}^{\infty} d_{i,j} \frac{\Gamma(j+1)}{[\lambda(b+i) - t]^{j+2}} [2(1-\theta)[\lambda(b+i) - t] + \lambda\theta(j+1)]. \quad (13)$$

Proof. We begin with a well-known concept of the moment-generating function, which is offered by

$$\begin{aligned} M_X(t) &= E(e^{tX}) = \int_0^\infty e^{tx} f(x) dx \\ &= \sum_{i,j=0}^{\infty} d_{i,j} \int_0^\infty \left(2(1-\theta)x^j + \lambda\theta x^{j+1}\right) e^{-[\lambda(b+i)-t]x} dx \\ &= \sum_{i,j=0}^{\infty} d_{i,j} \frac{\Gamma(j+1)}{[\lambda(b+i) - t]^{j+2}} [2(1-\theta)[\lambda(b+i) - t] + \lambda\theta(j+1)], \end{aligned}$$

which completes the proof. The characteristic function of X as characterized by $\phi(t) = E(e^{ztX})$ has the form

$$\phi(t) = \sum_{i,j=0}^{\infty} d_{i,j} \frac{\Gamma(j+1)}{[\lambda(b+i) - zt]^{j+2}} [2(1-\theta)[\lambda(b+i) - zt] + \lambda\theta(j+1)]$$

where $z = \sqrt{-1}$ is the complex number. □

Table 2 lists some numerical values of moments for numerous parameter values of the $BBE2$ distribution.

Table 2. Some numerical results for moments of the $BBE2$ distribution.

Moments ($\mu_r^{'}$)	**(a, b, θ, λ)**			
	(0.5,0.7,0.30,3.0)	**(2.5,2.7,1.0,3.0)**	**(1.5,2.7,0.70,3.0)**	**(1.5,2.7,1.00,3.0)**
$\mu_1^{'}$	0.362942	0.571999	2.92529	0.435659
$\mu_2^{'}$	0.358767	0.388811	9.50626	0.245108
$\mu_3^{'}$	0.554513	0.30641	76.2397	0.168439
$\mu_4^{'}$	1.14747	0.274943	550.099	0.136526
$\mu_5^{'}$	2.95521	0.277046	4696.42	0.127411
$\mu_6^{'}$	9.07734	0.310069	46206.3	0.1345
Variance	0.22704	0.061628	0.948911	0.0553086
Skewness	2.39872	0.882932	46.3885	1.03515
Kurtosis	164.972	13.4627	50623.0	3.9319

When parameter θ increases, the numerical values of $\mu_1^{'}$, $\mu_2^{'}$, $\mu_3^{'}$, $\mu_4^{'}$, $\mu_5^{'}$, $\mu_6^{'}$, variance, skewness and kurtosis decrease.

4. Acceptance Sampling Plans

We suppose that a product's lifetime follows the $BBE2$ distribution with parameters (λ, θ, a, b) described by (6), and that the prescribed median lifetime of the units assumed by a producer is m_0. Our goal is to draw a conclusion about whether the suggested lot should be accepted or rejected based on the criteria that the actual median lifetime, m, of the units is greater than the recommended lifetime, m_0. A common practice through life testing is to end the test at a specified time t_0 and record the number of failures. To notice the median lifetime, the experiment is run for $t_0 = km_0$ units of time, a multiple of the assumed median lifetime multiplied by any positive constant k.

Several studies have been proposed for ASP. Ref. [35] obtained the ASPs for the power-inverted Topp–Leone distribution based on a truncated life test and using the median life of the given distribution. Ref [36] studied a Fréchet Binomial distribution with applications to ASPs. For the three-parameter inverted Topp–Leone model, [37] proposed the ASP and studied the behavior of life median as a truncated lifetime.

According to [38], the concept of accepting the proposed lot based on the evidence that $m \geq m_0$, given a probability of at least α (consumer's risk), utilizing a single ASP is as described in the following:

1. Take n units at random from the suggested lot as a sample.
2. Run the following test for t_0 units of time:
   If c or fewer units (acceptance number) fail during the test, accept the entire lot; otherwise, the lot is rejected.

Under the suggested sampling plan, the probability of accepting a lot is offered by considering sufficiently large-sized lots so that the binomial distribution can indeed be implemented.

$$L(p) = \sum_{i=0}^{c} \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 1, 2, \ldots, n \tag{14}$$

where $p = F_{BBE2}(t_0; \lambda, \theta, a, b)$, as defined in (6). Function $L(p)$ is the operating characteristic function of the sampling plan, i.e., the acceptance probability of the lot as a function of the failure probability. In addition, using $t_0 = km_0$, thus p_0 can be written as

$$p_0 = F_{BBE2}(t_0 = km_0; \lambda, \theta, a, b) \tag{15}$$

Currently, the problem is finding the lowest positive integer n for given values of $\alpha(0 < \alpha < 1), km_0$ and c.

$$L(p_0) = \sum_{i=0}^{c} \binom{n}{i} p_0^i (1-p_0)^{n-i} \leq 1 - \alpha \tag{16}$$

where p_0 is described in Equation (15). The low values of n satisfying the inequality (16) and its corresponding operating characteristic probability are computed and mentioned in Tables 3 and 4 for the supposed parameters listed below:

1. $\alpha = 0.25, 0.75, 0.95,$
2. $c = 0, 1, 5, 10, 20,$
3. $k = 0.1, 0.2, 0.4, 0.8, 1$ (note that when $k = 1, t_0 = m_0 = 0.5 \quad \forall \quad \lambda, \theta, a, b$)
4. The parameters of the BBE2 distribution (λ, θ) are assumed to be:
   - Case 1: $(\lambda = 0.50, \theta = 0.25)$
   - Case 2: $(\lambda = 0.50, \theta = 0.75)$
5. Parameter (a, b) of the BBE2 distribution is assumed to be $(0.5, 1.5)$ and $(1.5, 0.5)$.

From the results obtained in Tables 3 and 4, We realize the following:

- For the parameters of ASP: When α and c are increasing, the required sample size n is increasing, but $L(p_0)$ is decreasing. While k is increasing, the required n is decreasing, but $L(p_0)$ is increasing.

- For the parameters of the $BBE2$ distribution: With increases in any parameters of λ, θ, a and b where the other parameters are fixed, the required n is increasing, but $L(p_0)$ is decreasing.

Eventually, we double-checked all of our findings, $L(p_0) \leq 1 - \alpha$. Further, when $a = 1$, we have $p_0 = 0.5$ as $t_0 = m_0$, and hence all numerical results $(n, L(p_0))$ for any vector of parameters (λ, θ, a, b) are the same.

Table 3. ASPs for the $BBE2$ distribution with parameters: $\lambda = 0.5, \theta = 0.25$ and different values for a and b.

α	c	$k \rightarrow$	**0.10**		**0.20**		**0.4**		**0.80**		**1**	
			n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
						$a = 0.5$ and $b = 1.5$						
0.25	0	2	0.8315	2	0.7634	1	1.0000	1	1.0000	1	1.0000	
	1		6	0.8002	5	0.7607	3	0.8912	3	0.7944	3	0.7500
	5		26	0.7640	19	0.7623	14	0.7673	10	0.8288	9	0.8555
	10		52	0.7683	38	0.7560	27	0.7912	20	0.8076	19	0.7597
	20		107	0.7579	77	0.7556	56	0.7533	41	0.7738	37	0.7975
0.75	0	8	0.2748	6	0.2593	4	0.3010	3	0.2987	3	0.2500	
	1		16	0.2536	11	0.2756	8	0.2699	6	0.2510	5	0.3125
	5		43	0.2669	31	0.2531	22	0.2595	16	0.2520	14	0.2905
	10		76	0.2612	54	0.2604	38	0.2804	28	0.2516	25	0.2706
	20		140	0.2585	99	0.2657	71	0.2581	51	0.2696	46	0.2757
0.95	0	17	0.0522	12	0.0513	8	0.0607	5	0.0892	5	0.0625	
	1		27	0.0517	19	0.0510	13	0.0567	9	0.0608	8	0.0625
	5		60	0.0529	42	0.0544	29	0.0616	21	0.0520	18	0.0717
	10		98	0.0502	69	0.0501	48	0.0566	34	0.0576	30	0.0680
	20		168	0.0526	119	0.0502	84	0.0515	59	0.0619	53	0.0632
						$a = 1.5$ and $b = 0.5$						
0.25	0	10	0.7641	4	0.7855	2	0.8107	1	1.0000	1	1.0000	
	1		33	0.7572	13	0.7635	6	0.7591	3	0.8343	3	0.7500
	5		144	0.7531	55	0.7626	23	0.7747	11	0.8217	9	0.8555
	10		294	0.7512	113	0.7515	47	0.7569	23	0.7506	19	0.7597
	20		604	0.7521	231	0.7545	96	0.7502	45	0.7874	37	0.7975
0.75	0	47	0.2528	18	0.2545	7	0.2840	3	0.3516	3	0.2500	
	1		91	0.2531	34	0.2644	14	0.2638	6	0.3249	5	0.3125
	5		251	0.2526	95	0.2568	38	0.2734	17	0.3081	14	0.2905
	10		441	0.2513	167	0.2556	68	0.2540	31	0.2650	25	0.2706
	20		809	0.2517	307	0.2543	125	0.2522	57	0.2681	46	0.2757
0.95	0	101	0.0503	38	0.0509	15	0.0530	6	0.0733	5	0.0625	
	1		160	0.0502	60	0.0515	24	0.0511	10	0.0650	8	0.0625
	5		355	0.0501	134	0.0503	53	0.0543	23	0.0635	18	0.0717
	10		573	0.0503	216	0.0515	86	0.0551	38	0.0610	30	0.0680
	20		983	0.0501	372	0.0502	149	0.0532	67	0.0536	53	0.0632

Table 4. ASPs for the $BBE2$ distribution with parameters: $\lambda = 0.5, \theta = 0.75$ and different values for a and b.

α	c	$k \rightarrow$	0.10		0.20		0.4		0.80		1	
			n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$	n	$L(p_0)$
						$a = 0.5$ and $b = 1.5$						
0.25	0	2	0.8413	2	0.7752	1	1.0000	1	1.0000	1	1.0000	
	1		7	0.7559	5	0.7801	4	0.7606	3	0.7986	3	0.7500
	5		28	0.7503	20	0.7589	14	0.7948	10	0.8360	9	0.8555
	10		56	0.7510	40	0.7532	28	0.7873	21	0.7541	19	0.7597
	20		114	0.7508	81	0.7542	58	0.7518	41	0.7913	37	0.7975
0.75	0	9	0.2509	6	0.2800	4	0.3169	3	0.3038	3	0.2500	
	1		17	0.2530	12	0.2546	8	0.2921	6	0.2580	5	0.3125
	5		46	0.2602	32	0.2724	23	0.2513	16	0.2638	14	0.2905
	10		81	0.2571	57	0.2576	40	0.2597	28	0.2674	25	0.2706
	20		149	0.2552	105	0.2536	73	0.2741	52	0.2516	46	0.2757
0.95	0	18	0.0530	12	0.0608	8	0.0684	6	0.0509	5	0.0625	
	1		28	0.0573	20	0.0516	13	0.0665	9	0.0640	8	0.0625
	5		64	0.0520	44	0.0569	31	0.0508	21	0.0565	18	0.0717
	10		104	0.0511	72	0.0550	50	0.0551	34	0.0640	30	0.0680
	20		179	0.0513	125	0.0525	87	0.0530	60	0.0576	53	0.0632
						$a = 1.5$ and $b = 0.5$						
0.25	0	17	0.7519	6	0.7622	2	0.8462	1	1.0000	1	1.0000	
	1		55	0.7530	19	0.7541	7	0.7674	3	0.8469	3	0.7500
	5		240	0.7509	81	0.7518	28	0.7730	12	0.7725	9	0.8555
	10		489	0.7515	164	0.7545	57	0.7643	23	0.7971	19	0.7597
	20		1007	0.7506	337	0.7540	117	0.7577	47	0.7766	37	0.7975
0.75	0	78	0.2536	26	0.2572	9	0.2628	3	0.3706	3	0.2500	
	1		152	0.2520	51	0.2507	17	0.2700	6	0.3523	5	0.3125
	5		420	0.2502	140	0.2510	48	0.2501	18	0.2888	14	0.2905
	10		736	0.2511	245	0.2534	84	0.2511	32	0.2779	25	0.2706
	20		1351	0.2504	450	0.2526	154	0.2526	59	0.2801	46	0.2757
0.95	0	169	0.0501	56	0.0504	18	0.0584	7	0.0509	5	0.0625	
	1		267	0.0505	88	0.0519	29	0.0567	11	0.0519	8	0.0625
	5		593	0.0503	197	0.0501	66	0.0525	24	0.0638	18	0.0717
	10		957	0.0504	318	0.0504	107	0.0526	40	0.0565	30	0.0680
	20		1642	0.0500	546	0.0501	185	0.0507	70	0.0526	53	0.0632

5. Non-Bayesian Estimation Methods

In this section, Non-Bayesian estimation (Non-BE) methods will be introduced for the $BBE2$ distribution with parameters (λ, θ, a, b). These methods are the maximum likelihood estimation method and the maximum product of the spacing estimation method.

5.1. Maximum Likelihood Estimation

In this subsection, we determine the $MLEs$ of the parameters of the $BBE2$ distribution from complete samples only. We assume that $X_1, X_2, \ldots, X_n$ is a random sample of size

n from $BBE2(\xi)$ where $\xi = (\lambda, \theta, a, b)$. Let $\xi = (\lambda, \theta, a, b)^T$ be the parameter vector. The likelihood (LL) function of ξ, given data x and $f_{BBE2}(x, \xi)$, can be expressed as:

$$L = \left(\frac{\lambda}{B(a,b)}\right)^n \prod_{i=1}^n \exp(-\lambda x_i)\left(1 + \frac{(\lambda x_i - 1)\theta}{2-\theta}\right)\left[1 - \left(1 + \frac{\lambda\theta x_i}{2-\theta}\right)\exp(-\lambda x_i)\right]^{a-1}$$
$$\left[\left(1 + \frac{\lambda\theta x_i}{2-\theta}\right)\exp(-\lambda x_i)\right]^{b-1}.$$

Let $w_i = \left(1 + \frac{\lambda\theta x_i}{2-\theta}\right)\exp(-\lambda x_i)$, then, the LL function is provided via

$$L = \left(\frac{\lambda}{(2-\theta)B(a,b)}\right)^n \exp\left(-\lambda \sum_{i=1}^n x_i\right) \prod_{i=1}^n (2 - 2\theta + \lambda\theta x_i)(1 - w_i)^{a-1} w_i^{b-1}. \tag{17}$$

The log LL function for the vector of parameters $\xi = (\lambda, \theta, a, b)$ can be computed as

$$\log L = n\log(\lambda) - n\log(2-\theta) - n\log[B(a,b)] - \lambda\sum_{i=1}^n x_i + \sum_{i=1}^n \log(2 - 2\theta + \lambda\theta x_i)$$
$$+(a-1)\sum_{i=1}^n \log(1 - w_i) + (b-1)\sum_{i=1}^n \log(w_i).$$

The associated score function is provided via

$$U_n(\xi) = \left[\frac{\partial \log L}{\partial\lambda}, \frac{\partial \log L}{\partial\theta}, \frac{\partial \log L}{\partial a}, \frac{\partial \log L}{\partial b}\right]^T.$$

The log-LL can be maximized either directly or by solving the non-linear LL equations computed by differentiating (18). The components of the score vector are provided via

$$\begin{aligned}
\frac{\partial \log L}{\partial\lambda} &= \frac{n}{\lambda} - b\sum_{i=1}^n x_i + b\sum_{i=1}^n \frac{\theta x_i}{2 - 2\theta + \lambda\theta x_i} + (a-1)\sum_{i=1}^n \frac{x_i w_i - \theta x_i \exp(-\lambda x_i)}{(2-\theta)[1 - w_i]},\\
\frac{\partial \log L}{\partial\theta} &= b\sum_{i=1}^n \frac{\lambda x_i - 2}{2 - 2\theta + \lambda\theta x_i} + \frac{nb}{2-\theta} - (a-1)\sum_{i=1}^n \frac{2\lambda x_i \exp(-\lambda x_i)}{(2-\theta)^2[1 - w_i]},\\
\frac{\partial \log L}{\partial a} &= \sum_{i=1}^n \log(1 - w_i) - n\psi(a) + n\psi(a+b),\\
\frac{\partial \log L}{\partial b} &= -\lambda\sum_{i=1}^n x_i + \sum_{i=1}^n \log(2 - 2\theta + \lambda\theta x_i) - n\log(2-\theta) - n\psi(b) + n\psi(a+b).
\end{aligned} \tag{18}$$

where ψ is a digamma function.

The ML estimation of ξ, $\hat{\xi}_{MLE}$, is computed by solving the above system of non-linear equations that have no closed form and can be solved using Newton–Raphson's iterative method.

The confidence intervals (cIs) of the vector of the unknown parameters ξ could be obtained from the asymptotic distribution of the MLE of the parameters, $(\hat{\xi}_{MLE} - \xi) \to N_4\big(0, I^{-1}(\hat{\xi}_{MLE})\big)$, where $I(\xi)$ is the Fisher information matrix. Under particular regularity conditions, the two-sided $100(1-\gamma)\%, 0 < \gamma < 1$, asymptotic CIs (Asy-CIs) for the vector of unknown parameters ξ can be acquired in the following ways: $\hat{\xi}_{MLE} \pm Z_{\frac{\gamma}{2}}\sqrt{\mathrm{Var}(\hat{\xi})}$, where $\mathrm{Var}(\hat{\xi}_{MLE})$ is the element of the main diagonal of $I^{-1}(\hat{\xi}_{MLE})$, and $Z_{\frac{\gamma}{2}}$ is the upper $\frac{\gamma}{2}^{\text{th}}$ percentile of the standard normal distribution. Finally, the corresponding coverage probabilities (CP)

$$CP_{\xi} = P\left[\left|\frac{\hat{\xi} - \xi}{\sqrt{\mathrm{Var}(\hat{\xi})}}\right| \le Z_{\gamma/2}\right],$$

can be calculated with the Monte Carlo simulations.

5.2. Maximum Product of Spacing Estimation

The $MPSP$ method, as an approximation of the Kullback–Leibler information measure, is a good alternative to the ML method. We assume that $D_i(\lambda,\theta,a,b) = F_{BBE2}(x_{i=1:n} \mid \lambda,\theta,a,b) - F_{BBE2}(x_{i-1:n} \mid \lambda,\theta,a,b)$, for $i = 1,2,\ldots\ldots,n+1$, is the uniform spacing of a random sample from the BBE2 distribution, where $F(x_{0:n} \mid \lambda,\theta,a,b) = 0, F(x_{n+1:n} \mid \lambda,\theta,a,b) = 1$ and $\sum_{i=1}^{n+1} D_i(\lambda,\theta,a,b) = 1$. The MPSP for $\hat{\lambda}_{MPSP}$, $\hat{\theta}_{MPSP}$, $\hat{a}_{MPSP}$, and $\hat{b}_{MPSP}$ can be obtained by maximizing the geometric mean of the spacing

$$G(\lambda,\theta,a,b) = \left[\prod_{i=1}^{n+1} D_i(\lambda,\theta,a,b)\right]^{\frac{1}{n+1}}, \tag{19}$$

or, equivalently, by maximizing the logarithm of the geometric mean of the sample spacing

$$H(\lambda,\theta,a,b) = \frac{1}{n+1}\sum_{i=1}^{n+1} \log D_i(\lambda,\theta,a,b). \tag{20}$$

To achieve the required estimators, the numerical technique is utilized.

6. Bayesian Estimation

In this section, the Bayesian estimation of the unknown parameters of a BBE2 distribution will be investigated. For Bayesian parameter estimation, different loss functions can indeed be regarded as: squared error (SE) loss function, LINEX loss and general entropy (GE) loss function. We recommend utilizing independent gamma priors for ξ with pdfs for prior distributions of $BBE2$ parameters.

$$\begin{array}{ll} \pi_1(\lambda) \propto \lambda^{s_1-1}\exp(-q_1\lambda) & \lambda > 0, s_1 > 0, q_1 > 0, \\ \pi_2(\theta) \propto \theta^{s_2-1}\exp(-q_2\theta) & \theta > 0, s_2 > 0, q_2 > 0, \\ \pi_3(a) \propto a^{s_3-1}\exp(-q_3 a) & a > 0, s_3 > 0, q_3 > 0, \\ \pi_4(b) \propto b^{s_4-1}\exp(-q_4 b) & b > 0, s_4 > 0, q_4 > 0, \end{array} \tag{21}$$

where the hyper-parameters $s_j, q_j, j = 1,2,\ldots,4$ are chosen to reflect the prior knowledge about the unknown parameters. The joint prior for $\xi = (\lambda,\theta,a,b)$ is provided via

$$\begin{aligned} \pi(\xi) &= \pi_1(\lambda)\pi_2(\theta)\pi_3(a)\pi_4(b) \\ \pi(\xi) &\propto \lambda^{s_1-1}\theta^{s_2-1}a^{s_3-1}b^{s_4-1}\exp(-q_1\lambda - q_2\theta - q_3 a - q_4 b). \end{aligned} \tag{22}$$

The corresponding posterior density given the observed data $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is provided via:

$$\pi(\xi \mid \mathbf{x}) = \frac{\pi(\xi)\mathrm{L}(\xi)}{\int_\xi \pi(\xi)\mathrm{L}(\xi)d\xi},$$

thus, the posterior density function can indeed be expressed as:

$$\pi(\xi \mid \mathbf{x}) \propto \frac{\lambda^{n+s_1-1}\theta^{s_2-1}a^{s_3-1}b^{s_4-1}}{[(2-\theta)B(a,b)]^n}\exp\left(-\lambda\left(\sum_{i=1}^n x_i + q_1\right) - q_2\theta - q_3 a - q_4 b\right)$$

$$\prod_{i=1}^n (2 - 2\theta + \lambda\theta x_i)(1-w_i)^{a-1}w_i^{b-1}.$$

The Bayes Estimator of any function, say $l(\xi)$ under the SE loss function, is provided via

$$\hat{\xi}_{BSE} = E[l(\xi)|\mathbf{x}] = \int_\xi l(\xi)\pi(\xi|x)d\xi. \tag{23}$$

The SE loss is an asymmetric loss function that puts equal weight on the underestimation and overestimation. In several real-world situations, underestimation may be more serious than overestimation, and vice versa. In such cases, a LINEX loss can be proposed as an alternative to the SE loss, which is provided via

$$\left(l(\xi),\hat{l}(\xi)\right) = \exp\left(\hat{l}(\xi) - l(\xi)\right) - v\left(\hat{l}(\xi) - l(\xi)\right) - 1,$$

where $v \neq 0$ is a shape parameter. Here $v > 1$ suggests that an overestimation is more serious than the underestimation, and vice versa for $v < 0$. Further, v approaching 0 replicates the SE loss function itself. One may refer to [39] and [40] for more information in this respect. The BE of $l(\xi)$ under this loss can be calculated as

$$\hat{\xi}_{BLINEX} = E[\exp(-vl(\xi))|\mathbf{x}] = -\frac{1}{v}\log\left[\int_{\xi}\exp(-vl(\xi))\pi(\xi|x)d\xi\right]. \tag{24}$$

Further, we also look at the general entropy (GE) loss function suggested by [41], which is described as follows:

$$\left(l(\xi),\hat{l}(\xi)\right) = \left(\frac{\hat{l}(\xi)}{l(\xi)}\right)^{\tau} - \tau\log\left(\frac{\hat{l}(\xi)}{l(\xi)}\right) - 1,$$

where $\tau \neq 0$ is a shape parameter, and it reflects the departure from symmetry. When $\tau > 0$, it considers overestimation to be more serious than underestimation and the converse for $\tau < 0$. The Bayes estimator regarding the GE loss function is provided by

$$\hat{\xi}_{BGE} = \left[E\left((l(\xi))^{-\tau}|\mathbf{x}\right)\right]^{-1/\tau} = \left[\int_{\xi}(l(\xi))^{-\tau}\pi(\xi|x)d\xi\right]^{-1/\tau} \tag{25}$$

It can be seen that the estimates provided by (23)–(25) cannot be simplified into closed-form expressions. Therefore, we next apply the Markov chain Monte Carlo (MCMC) approach and generate a posterior sample using Metropolis–Hasting (MH) Algorithm to obtain the desired Bayesian estimates (BEs).

6.1. Markov Chain Monte Carlo

The MCMC strategies are a general simulation procedure for sampling from posterior distributions and computing posterior quantities of interest. Indeed, the MCMC samples can be employed to completely summarize the posterior uncertainty about the parameters ξ as well as a kernel estimate of the posterior distribution; for further information on MCMC, see [42].

MCMC algorithms are founded on the idea of a discrete-time Markov chain. A Markov chain is a stochastic process

$$\xi^{(0)}, \xi^{(1)}, \xi^{(2)}, \ldots$$

Here $\xi^{(i)}$ is an RV whose values lie in a "state space". The state space, the state of the process at time i, is the same for all times i. Markov chains have the next Markov property: the distribution of the next state $\xi^{(i+1)}$ according to the history $\xi^{(0)}, \xi^{(1)}, \ldots, \xi^{(i)}$ only through the present state $\xi^{(i)}$. The Markov chains utilized in MCMC approaches are homogeneous, i.e., the conditional distribution of $\xi^{(i+1)}$ given $\xi^{(i)}$ does not depend on the index i. To draw samples from a statistical distribution employing MCMC:

1. Starting with an initial guess: just one value that could be gathered from the distribution.
2. Creating a series of new samples based on this first guess. Each new sample is created in two steps:
    - Proposal: A new sample proposal is produced by adding a small random disturbance to the most recent sample.
    - Acceptance: The suggestion is either accepted as the new sample or rejected (in which case the old sample is retained).

There are numerous methods for adding random noise to generated proposals, as well as specific methodologies for acceptance and rejection, such as Gibbs sampling and the Metropolis–Hastings algorithm.

6.2. Metropolis–Hasting Algorithm

To run the MH algorithm on the $BBE2$ distribution, a suggested distribution and initial values for the unknown parameters ξ must be characterized. For the suggested distribution, a multivariate normal distribution will be considered, that is $q\left(\xi^{'}|\xi\right) \equiv N_4(\xi, S_{\xi})$, where S_{ξ} comprises the variance–covariance matrix. Negative observations can indeed be acquired, which are unacceptable. For the initial values, the $MLEs$ for ξ are considered, that is $\xi^{(0)} = \hat{\xi}_{MLE}$. The choice of $S\xi$ is regarded as the asymptotic variance–covariance matrix. $I^{-1}(\hat{\xi}_{MLE})$, where $I(.)$ is the Fisher information matrix. It has been observed that the choice of S_{ξ} is a crucial issue in the MH algorithm, where the acceptance rate depends upon this. In this due consideration, the stages of the MH algorithm to draw a sample from the specified posterior density (24) would be as described in the following:

Step 1. Set initial value of ξ as $\xi^{(0)} = \left(\hat{\lambda}_{MLE}, \hat{\theta}_{MLE}, \hat{a}_{MLE}, \hat{b}_{MLE}\right)$.

Step 2. For $i = 1, 2, \ldots, M$ repeat the next stages:

2.1: Set $\xi = \xi^{(i-1)}$.

2.2: Generate a new candidate parameter value δ from $N_4(\log \xi, S_{\xi})$.

2.3: Set $\theta^{'} = \exp(\delta)$.

2.4: Calculate $\beta = \frac{\pi(\xi^{'}|x)}{\pi(\xi|x)}$, where $\pi(\cdot)$ is the posterior density in (23).

2.5: Generate a sample u from the uniform $U(0, 1)$ distribution.

2.6: Accept or reject the new candidate $\theta^{'}$

$$\begin{cases} \text{If} \quad u \le \beta \quad \text{set} \quad \xi^{(i)} = \xi^{'} \\ \text{otherwise} \quad \text{set} \quad \xi^{(i)} = \xi. \end{cases}$$

Eventually, a portion of the random samples of size M drawn from the posterior density can indeed be discarded (burn-in), and the remaining samples can be employed to determine BEs. More accurately, the BEs of $\xi^{(i)} = \left(\lambda^{(i)}, \theta^{(i)}, a^{(i)}, b^{(i)}\right)$ utilizing MCMC under SE, LINEX and GE loss functions can be estimated as

$$\hat{\xi}_{BEL} = \frac{1}{M - l_B} \sum_{i=l_B}^{M} \xi^{(i)}, \tag{26}$$

$$\hat{\xi}_{BLINEX} = -\frac{1}{v} \log\left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} \exp\left(-v\xi^{(i)}\right)\right], \tag{27}$$

$$\hat{\xi}_{BGE} = \left[\frac{1}{M - l_B} \sum_{i=l_B}^{M} \left(\xi^{(i)}\right)^{-\tau}\right]^{-1/\tau} \tag{28}$$

where l_B represents the number of burn-in samples.

6.3. Highest Posterior Density Intervals

In this sub-section, HPD credible intervals for the vector of the unknown parameters ξ of the $BBE2$ distribution are constructed utilizing the samples drawn from the suggested MH algorithm in the previous sub-section. Let us suppose that $\xi^{(\gamma)}$ is the γth quantile of ξ, that is,

$$\xi^{(\gamma)} = \inf\{\xi : \Pi(\xi|\mathbf{x}) \ge \gamma\},$$

where $0 < \gamma < 1$ and $\Pi(\cdot)$ is the posterior distribution function of ξ. Notice that for a given ξ^*, a consistent simulation estimator of $\pi(\xi|\mathbf{x})$ can be estimated as

$$\Pi(\xi^*|\mathbf{x}) = \frac{1}{M - l_B} \sum_{i=l_B}^{M} I_{(\xi) \leq \xi^*}$$

Here $I_{(\xi) \leq (\xi^*)}$ is the indicator function. Then the corresponding estimate is computed as

$$\hat{\Pi}(\xi^*|\mathbf{x}) = \begin{cases} 0 & \text{if } \xi^* < \xi_{(l_B)} \\ \sum_{j=l_B}^{i} \omega_j & \text{if } \xi_{(i)} < \xi^* < \xi_{(i+1)} \\ 1 & \text{if } \xi^* > \xi_{(M)} \end{cases}$$

where $\omega_j = \frac{1}{M - l_B}$ and $\xi_{(j)}$ are the ordered values of ξ_j. Now, for $i = l_B, \ldots, M$, $\xi^{(\gamma)}$ can indeed be estimated by

$$\tilde{\xi}^{(\gamma)} = \begin{cases} \xi_{(l_B)} & \text{if } \quad \gamma = 0 \\ \\ \xi_{(i)} & \text{if } \sum_{j=l_B}^{i-1} \omega_j < \gamma < \sum_{j=l_B}^{i} \omega_j. \end{cases}$$

Now, in order to acquire a $100(1-\gamma)\%$ HPD credible interval for ξ, let

$$HPD_j^{\xi} = \left(\tilde{\xi}^{\left(\frac{j}{M}\right)}, \quad \tilde{\xi}^{\left(\frac{j+(1-\gamma)M}{M}\right)} \right)$$

for $j = l_B, \ldots, [\gamma M]$, here $[m]$ denotes the largest integer less than or equal to m. Then, select HPD_{j^*} among all the HPD_j's such that it has the lowest width.

7. Simulation Study and Data Analysis

The purpose of this section is to investigate the performance of BE and Non-BE methods, specifically MLE, which were discussed in the previous section. A Monte Carlo study is used to test the performance of the proposed estimation methods, and a real dataset is examined for illustrative purposes. The R-statistical programming language will be utilized for computation. Further, one can utilize *bbmle*, *BMT* and *HDInterval* packages to calculate MLEs, s and HPD intervals, respectively, in R-language.

Simulation Study

The Monte Carlo simulation is executed using a variety of suggested estimation methods (Non-BE and BE). We generate 1000 sets of data from the $BBE2$ distribution with the following assumptions for Non-BE strategies:

1. Sample size generated from the $BBE2$ distribution is supposed to be $n = 100, 200$.
2. For the parameters (a, b) of the beta distribution, we assumed that: $a = 0.50, 0.75$ and $b = 0.50, 0.75$.
3. For parameter (λ) of the exponential distribution, we assumed that: $\lambda = 0.50, 1.50$.
4. For parameter (θ) of the binomial distribution, we assumed that: $\theta = 0.25, 0.50$.

Depending on the generated data, MLEs and associated 95% Asy-CI and s are computed. The average estimates (Avg.) and mean square errors (MSEs) are computed for the two methods, and interval estimates (lower and upper) and average interval lengths (AILs) with CP are computed based on MLEs. All results for Non-BE are reported in Tables 5–8.

Table 5. Avg. estimated values, MSEs, Asy-CI, AILs and CP (in %) of Non-BE (MLE and MPSP) for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 0.50, \theta = 0.25$

n		MLE		MPSP		Asy-CI			
		Avg.	**MSE**	**Avg.**	**MSE**	**Lower**	**Upper**	**AIL**	**CP (%)**
				$a = 0.50$ and $b = 0.50$					
100	λ	0.6629	0.1951	0.6599	0.2084	0.0000	1.4682	1.4682	95.80
	θ	0.4114	0.0382	0.3850	0.0399	0.1952	0.6275	0.4323	96.70
	a	0.5658	0.0083	0.5382	0.0052	0.4425	0.6891	0.2466	97.00
	b	0.5980	0.0918	0.6370	0.3456	0.0358	1.1602	1.1244	96.60
200	λ	0.5922	0.0733	0.5939	0.0931	0.0930	1.0915	0.9985	94.80
	θ	0.4029	0.0300	0.3880	0.0278	0.2436	0.5623	0.3187	96.90
	a	0.5760	0.0082	0.5618	0.0061	0.4804	0.6717	0.1913	97.50
	b	0.6139	0.0599	0.6115	0.0705	0.1891	1.0388	0.8497	95.50
				$a = 0.50$ and $b = 0.75$					
100	λ	0.7995	0.2320	0.7655	0.2944	0.0597	1.5393	1.4796	95.40
	θ	0.3798	0.0301	0.3414	0.0331	0.1540	0.6056	0.4515	95.20
	a	0.5613	0.0074	0.5365	0.0046	0.4433	0.6793	0.2360	97.20
	b	0.7402	0.1308	0.8397	0.5737	0.0311	1.4492	1.4181	97.30
200	λ	0.7841	0.1755	0.7697	0.2010	0.1803	1.3880	1.2077	96.10
	θ	0.3766	0.0238	0.3529	0.0224	0.2036	0.5496	0.3460	96.30
	a	0.5658	0.0064	0.5529	0.0048	0.4753	0.6562	0.1808	95.70
	b	0.7135	0.0713	0.7383	0.1529	0.1947	1.2323	1.0376	96.00
				$a = 0.75$ and $b = 0.50$					
100	λ	0.4720	0.0605	0.4387	0.2247	0.0000	0.9526	0.9526	96.20
	θ	0.3255	0.0336	0.3335	0.0785	0.0000	0.6540	0.6540	96.80
	a	0.8641	0.0276	0.8136	0.0160	0.6271	1.1012	0.4741	98.70
	b	0.8028	0.2435	1.0501	2.3357	0.0367	1.5689	1.5322	96.20
200	λ	0.4495	0.0474	0.3067	0.0991	0.0333	0.8657	0.8325	97.10
	θ	0.3199	0.0267	0.3942	0.0921	0.0294	0.6104	0.5810	96.60
	a	0.8921	0.0277	0.8650	0.0200	0.7220	1.0621	0.3401	98.90
	b	0.8125	0.2128	1.2654	2.6261	0.1456	1.4795	1.3339	95.40
				$a = 0.75$ and $b = 0.75$					
100	λ	0.5471	0.0587	0.4611	0.1206	0.0802	1.0141	0.9339	98.60
	θ	0.3169	0.0315	0.3026	0.0696	0.0000	0.6399	0.6399	95.40
	a	0.8618	0.0265	0.8149	0.0162	0.6289	1.0947	0.4658	96.80
	b	0.9678	0.2120	1.3863	4.5384	0.1709	1.7647	1.5937	97.20
200	λ	0.5292	0.0313	0.3797	0.0551	0.1862	0.8721	0.6859	95.50
	θ	0.2946	0.0252	0.3386	0.0752	0.0000	0.5937	0.5937	95.00
	a	0.8920	0.0275	0.8661	0.0203	0.7236	1.0603	0.3367	98.60
	b	0.9877	0.2839	1.9519	9.0478	0.0510	1.9245	1.8734	98.20

Table 6. Avg. estimated values, MSEs, Asy-CI, AILs and CP (in %) of Non-BE (MLE) for the MPSPr $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 0.50, \theta = 0.50$

n		MLE		MPSP		Asy-CI			
		Avg.	**MSE**	**Avg.**	**MSE**	**Lower**	**Upper**	**AIL**	**CP (%)**
				$a = 0.50$ and $b = 0.50$					
100	λ	0.6544	0.1423	0.6864	0.2201	0.0000	1.3295	1.3295	95.10
	θ	0.4545	0.0162	0.4375	0.0270	0.2217	0.6872	0.4655	95.20
	a	0.5817	0.0109	0.5499	0.0066	0.4537	0.7097	0.2560	97.50
	b	0.5521	0.0771	0.5499	0.1646	0.0174	1.0869	1.0695	97.00
200	λ	0.6190	0.0790	0.6332	0.1054	0.1195	1.1184	0.9989	95.20
	θ	0.4509	0.0107	0.4406	0.0146	0.2720	0.6297	0.3577	96.70
	a	0.5867	0.0099	0.5708	0.0073	0.4915	0.6820	0.1905	96.90
	b	0.5440	0.0441	0.5347	0.0664	0.1415	0.9464	0.8049	95.70
				$a = 0.50$ and $b = 0.75$					
100	λ	0.7933	0.2350	0.7753	0.2725	0.0364	1.5502	1.5139	95.90
	θ	0.4253	0.0198	0.4041	0.0357	0.1911	0.6594	0.4683	94.60
	a	0.5774	0.0101	0.5489	0.0061	0.4521	0.7026	0.2505	96.20
	b	0.6679	0.0789	0.7489	0.6881	0.1410	1.1948	1.0538	95.40
200	λ	0.7735	0.1568	0.7669	0.1753	0.2121	1.3350	1.1229	95.50
	θ	0.4165	0.0154	0.4043	0.0216	0.2369	0.5960	0.3591	95.60
	a	0.5826	0.0093	0.5682	0.0070	0.4854	0.6798	0.1944	96.20
	b	0.6460	0.0628	0.7268	1.3109	0.1987	1.0934	0.8946	95.60
				$a = 0.75$ and $b = 0.50$					
100	λ	0.4447	0.0640	0.4102	0.2185	0.0000	0.9304	0.9304	94.70
	θ	0.3149	0.0638	0.3539	0.1009	0.0000	0.6532	0.6532	96.20
	a	0.9530	0.0616	0.8937	0.0374	0.6723	1.2338	0.5615	95.50
	b	0.7806	0.1991	0.9517	0.8384	0.0980	1.4632	1.3652	96.20
200	λ	0.3823	0.0439	0.3623	0.2497	0.0407	0.7240	0.6833	95.20
	θ	0.3579	0.0514	0.4617	0.0898	0.0102	0.7056	0.6953	97.10
	a	0.9604	0.0540	0.9299	0.0431	0.7657	1.1551	0.3894	97.10
	b	0.9183	0.3579	1.5568	6.7943	0.0759	1.7607	1.6848	97.10
				$a = 0.75$ and $b = 0.75$					
100	λ	0.5287	0.0549	0.4691	0.1171	0.0719	0.9855	0.9136	95.80
	θ	0.3188	0.0304	0.2878	0.0581	0.0038	0.6338	0.6301	96.20
	a	0.8490	0.0205	0.8065	0.0132	0.6457	1.0522	0.4065	97.20
	b	0.9937	0.1904	1.2301	2.4442	0.2825	1.7048	1.4223	97.20
200	λ	0.5257	0.0405	0.3588	0.0671	0.1333	0.9181	0.7848	95.10
	θ	0.3012	0.0243	0.3547	0.0845	0.0117	0.5907	0.5790	96.20
	a	0.8875	0.0260	0.8625	0.0194	0.7216	1.0533	0.3317	97.30
	b	1.0043	0.2560	1.7036	6.4666	0.1448	1.8639	1.7192	98.40

Table 7. Avg. estimated values, MSEs, Asy-CI, AILs and CP (in %) of Non-BE (MLE and MPSP) for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 1.50, \theta = 0.25$

n		MLE		MPSP		Asy-CI			
		Avg.	**MSE**	**Avg.**	**MSE**	**Lower**	**Upper**	**AIL**	**CP (%)**
				$a = 0.50$ and $b = 0.50$					
100	λ	1.7799	0.7489	1.7769	1.0536	0.1736	3.3861	3.2125	95.00
	θ	0.4144	0.0392	0.3839	0.0388	0.1985	0.6304	0.4319	95.60
	a	0.5694	0.0080	0.5426	0.0048	0.4587	0.6801	0.2214	96.20
	b	0.6304	0.0884	0.6892	0.6447	0.1064	1.1545	1.0481	96.50
200	λ	1.6477	0.3536	1.6560	0.4899	0.5180	2.7774	2.2593	95.80
	θ	0.4057	0.0315	0.3886	0.0287	0.2387	0.5726	0.3338	96.50
	a	0.5816	0.0088	0.5676	0.0066	0.4911	0.6720	0.1809	97.00
	b	0.6432	0.0675	0.6343	0.0764	0.2182	1.0683	0.8501	95.50
				$a = 0.50$ and $b = 0.75$					
100	λ	1.8993	0.8923	1.7612	1.1563	0.2198	3.5787	3.3589	94.00
	θ	0.3611	0.0270	0.3165	0.0340	0.1231	0.5990	0.4758	93.10
	a	0.5877	0.0107	0.5627	0.0067	0.4794	0.6960	0.2167	96.50
	b	0.8848	0.1525	1.0652	1.7052	0.1658	1.6037	1.4379	96.70
200	λ	1.8545	0.5077	1.7559	0.6021	0.6422	3.0668	2.4247	96.30
	θ	0.3525	0.0202	0.3327	0.0254	0.1597	0.5453	0.3856	94.70
	a	0.5999	0.0117	0.5867	0.0092	0.5182	0.6815	0.1633	96.80
	b	0.8671	0.0944	0.8991	0.2018	0.3099	1.4243	1.1144	95.60
				$a = 0.75$ and $b = 0.50$					
100	λ	1.2553	0.2930	1.2813	2.8457	0.3053	2.2054	1.9001	96.10
	θ	0.3443	0.0746	0.3157	0.0662	0.0000	0.8488	0.8488	99.20
	a	0.8580	0.0259	0.8080	0.0155	0.6232	1.0927	0.4695	98.40
	b	0.8357	0.2413	0.9190	0.7613	0.1300	1.5413	1.4113	96.10
200	λ	1.2239	0.2137	0.9158	0.9213	0.4938	1.9539	1.4601	97.30
	θ	0.3127	0.0276	0.3944	0.1111	0.0096	0.6159	0.6063	98.20
	a	0.9045	0.0319	0.8972	0.0962	0.7280	1.0811	0.3531	99.10
	b	0.8640	0.3663	1.1613	2.6713	0.0000	1.8160	1.8160	97.30
				$a = 0.75$ and $b = 0.75$					
100	λ	1.4147	0.1615	1.2993	0.4753	0.6421	2.1872	1.5451	97.80
	θ	0.3186	0.0276	0.2755	0.0480	0.0213	0.6160	0.5946	94.80
	a	0.8478	0.0226	0.8056	0.0146	0.6233	1.0723	0.4490	97.80
	b	1.0633	0.2446	1.0911	0.3475	0.3104	1.8162	1.5058	94.80
200	λ	1.3527	0.1320	1.1632	0.8639	0.6995	2.0059	1.3064	96.60
	θ	0.2874	0.0204	0.3059	0.0629	0.0161	0.5586	0.5425	97.30
	a	0.8744	0.0220	0.8507	0.0167	0.7158	1.0329	0.3171	97.90
	b	1.0731	0.1735	1.4224	4.6088	0.5563	1.5900	1.0337	95.90

Table 8. Avg. estimated values, MSEs, Asy-CI, AILs and CP (in %) of Non-BE (MLE and MPSP) for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 1.5, \theta = 0.50$

n		MLE		MPSP		Asy-CI			
		Avg.	MSE	Avg.	MSE	Lower	Upper	AIL	CP (%)
					$a = 0.50$ and $b = 0.50$				
100	λ	1.7500	0.5951	1.8153	0.9300	0.3187	3.1813	2.8626	94.80
	θ	0.4586	0.0177	0.4411	0.0305	0.2107	0.7064	0.4956	94.90
	a	0.5835	0.0109	0.5516	0.0063	0.4603	0.7067	0.2464	97.20
	b	0.5822	0.0622	0.5949	0.4472	0.1204	1.0440	0.9236	95.60
200	λ	1.7314	0.4019	1.8013	0.5867	0.5739	2.8889	2.3150	95.40
	θ	0.4491	0.0102	0.4349	0.0126	0.2779	0.6203	0.3424	96.10
	a	0.5906	0.0107	0.5746	0.0079	0.4924	0.6889	0.1965	97.20
	b	0.5643	0.0416	0.5403	0.0452	0.1846	0.9440	0.7594	95.60
					$a = 0.50$ and $b = 0.75$				
100	λ	1.8728	0.6567	1.7872	1.0015	0.4613	3.2843	2.8230	95.00
	θ	0.3899	0.0256	0.3529	0.0496	0.1626	0.6171	0.4546	94.20
	a	0.6050	0.0155	0.5777	0.0100	0.4730	0.7369	0.2639	96.90
	b	0.8055	0.1052	0.8585	0.2975	0.1787	1.4322	1.2536	95.50
200	λ	1.8906	0.5142	1.8276	0.6019	0.7111	3.0700	2.3590	96.70
	θ	0.3899	0.0217	0.3766	0.0320	0.1979	0.5819	0.3840	94.30
	a	0.6078	0.0138	0.5935	0.0108	0.5169	0.6986	0.1817	96.80
	b	0.7688	0.0618	0.8135	0.6507	0.2827	1.2550	0.9724	95.30
					$a = 0.75$ and $b = 0.50$				
100	λ	1.3773	0.2912	1.3769	1.4309	0.3428	2.4119	2.0690	96.50
	θ	0.3389	0.0733	0.3603	0.1203	0.0000	0.7673	0.7673	93.90
	a	0.9531	0.0653	0.9171	0.1759	0.6474	1.2587	0.6113	97.40
	b	0.8073	0.4344	0.8034	2.2071	0.0000	1.9551	1.9551	94.70
200	λ	1.2145	0.2265	0.8009	0.8269	0.4630	1.9659	1.5029	97.60
	θ	0.3300	0.0692	0.4241	0.1110	0.0000	0.7264	0.7264	97.30
	a	0.9534	0.0508	0.9202	0.0383	0.7622	1.1447	0.3825	97.30
	b	0.8614	0.4029	1.2763	4.4615	0.0000	1.8912	1.8912	94.60
					$a = 0.75$ and $b = 0.75$				
100	λ	1.3441	0.1501	1.4643	3.0728	0.6448	2.0434	1.3986	96.50
	θ	0.3433	0.0897	0.3196	0.1223	0.0000	0.8464	0.8464	97.70
	a	0.9316	0.0510	0.8796	0.0350	0.6666	1.1966	0.5300	96.50
	b	1.0458	0.2340	1.0464	0.3567	0.2913	1.8004	1.5090	95.30
200	λ	1.2832	0.1969	1.0449	0.8917	0.5199	2.0466	1.5267	96.50
	θ	0.3119	0.0681	0.3704	0.1266	0.0000	0.6687	0.6687	95.30
	a	0.9621	0.0529	0.9304	0.0409	0.7866	1.1377	0.3510	97.70
	b	1.1514	0.5048	1.9029	0.1950	0.0000	2.3071	2.3071	94.20

The BEs are calculated for the Bayesian methodology utilizing MCMC and the MH algorithm with an informative prior. For the informative prior, we presumed that all gamma distribution hyperparameters are equal to 1.5. These values are then used to determine the estimated values. MLEs are taken into account as initial guess values when using the MH algorithm. In the end, 2000 burn-in samples are discarded among the overall 10,000 samples generated from the posterior density, and, subsequently, obtained BEs and HPD interval estimates under different loss functions, namely: SEL, LINEX at $v = 0.5$ and, finally, GE at $\tau = 0.5$. All results for Non-BE are reported in Tables 9–12.

From the tabulated results, one can indicate that:

- In general, the increasing n, MSEs and AILs are decreasing for all methods of Non-BEs and BEs. Further, two Non-BEs methods (MLE and MPSP) are competing well for estimating the parameters of the $BBE2$ distribution. For BE methods, the loss function GE estimates are better than BEs under other loss functions (LINEX and SE).
- For fixed (λ, θ, a) and as the value of b increases, the MSEs of λ and b estimates are increasing, but the MSEs of θ and a are decreasing.
- For fixed (λ, θ, b) and as the value of a increases, the MSEs of λ and θ estimates are decreasing, but the MSEs of a and b are increasing.
- For fixed (λ, a, b) and as the value of θ increases, the MSEs of λ estimates is decreasing, but the MSEs of a and b are increasing.
- For fixed (θ, a, b) and as the value of λ increases, the MSEs of all parameters are increasing.

Table 9. Avg. estimated values and MSEs of the BE using MCMC for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 0.50, \theta = 0.25$

n		BE: SEL		BE:LINEX		BE: GE		HPD			
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Lower	Upper	AIL	CP (%)
						$a = 0.50$ and $b = 0.50$					
100	λ	0.6513	0.1480	0.6469	0.1428	0.6356	0.1376	0.1306	1.3829	1.2523	95.30
	θ	0.4091	0.0570	0.4069	0.0557	0.3944	0.0511	0.0881	0.7280	0.6399	96.40
	a	0.5563	0.0091	0.5551	0.0089	0.5503	0.0084	0.4209	0.7235	0.3027	98.10
	b	0.6258	0.1299	0.6218	0.1252	0.6114	0.1205	0.1866	1.3006	1.1139	96.20
200	λ	0.6024	0.1001	0.5991	0.0972	0.5891	0.0937	0.1458	1.1631	1.0173	95.00
	θ	0.4108	0.0507	0.4087	0.0496	0.3973	0.0457	0.1547	0.7023	0.5476	97.00
	a	0.5647	0.0071	0.5640	0.0070	0.5611	0.0067	0.4672	0.6780	0.2109	98.50
	b	0.6530	0.1095	0.6488	0.1054	0.6381	0.0997	0.1881	1.2569	1.0688	95.50
						$a = 0.50$ and $b = 0.75$					
100	λ	0.8091	0.2685	0.8024	0.2591	0.7887	0.2494	0.1823	1.5890	1.4068	95.70
	θ	0.3810	0.0445	0.3791	0.0435	0.3677	0.0400	0.0772	0.6911	0.6140	96.00
	a	0.5455	0.0066	0.5445	0.0065	0.5399	0.0062	0.4125	0.6696	0.2571	96.40
	b	0.7605	0.1464	0.7543	0.1407	0.7416	0.1380	0.2130	1.5316	1.3186	95.80
200	λ	0.7649	0.1888	0.7597	0.1828	0.7474	0.1750	0.2591	1.4660	1.2069	95.70
	θ	0.3754	0.0370	0.3737	0.0362	0.3628	0.0331	0.1190	0.6343	0.5153	96.30
	a	0.5617	0.0063	0.5611	0.0062	0.5585	0.0059	0.4738	0.6615	0.1877	97.90
	b	0.7692	0.0992	0.7644	0.0966	0.7531	0.0950	0.2360	1.3608	1.1249	95.40
						$a = 0.75$ and $b = 0.50$					
100	λ	0.4809	0.0563	0.4787	0.0553	0.4702	0.0543	0.1399	1.0081	0.8682	96.00
	θ	0.3096	0.0443	0.3082	0.0436	0.2989	0.0413	0.0009	0.6334	0.6325	95.50
	a	0.8275	0.0244	0.8249	0.0239	0.8182	0.0229	0.5893	1.0826	0.4934	97.20
	b	0.7806	0.2892	0.7751	0.2795	0.7640	0.2732	0.2631	1.4980	1.2349	96.60
200	λ	0.4979	0.0765	0.4954	0.0736	0.4877	0.0713	0.1726	1.0339	0.8613	96.30
	θ	0.3123	0.0361	0.3110	0.0355	0.3019	0.0336	0.0695	0.6814	0.6119	97.80
	a	0.8837	0.0270	0.8818	0.0265	0.8775	0.0256	0.7195	1.0529	0.3333	97.80
	b	0.7820	0.2022	0.7772	0.1966	0.7668	0.1894	0.2254	1.3674	1.1419	96.30
						$a = 0.75$ and $b = 0.75$					
100	λ	0.5980	0.1115	0.5944	0.1078	0.5832	0.1042	0.2021	1.2069	1.0048	96.70
	θ	0.3263	0.0389	0.3245	0.0380	0.3132	0.0348	0.0150	0.6615	0.6464	96.70
	a	0.8225	0.0163	0.8201	0.0158	0.8140	0.0149	0.6520	1.0320	0.3800	97.20
	b	0.9579	0.2684	0.9493	0.2557	0.9355	0.2512	0.2655	1.8190	1.5534	96.10
200	λ	0.5567	0.0635	0.5540	0.0618	0.5450	0.0594	0.2204	1.1121	0.8916	95.90
	θ	0.3086	0.0313	0.3071	0.0306	0.2969	0.0283	0.0629	0.6873	0.6244	98.50
	a	0.8658	0.0220	0.8642	0.0216	0.8602	0.0207	0.7201	1.0532	0.3332	99.50

Table 9. *Cont.*

n		**BE: SEL**		**BE:LINEX**		**BE: GE**		**HPD**			
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Lower**	**Upper**	**AIL**	**CP (%)**
	b	0.9864	0.2140	0.9785	0.2045	0.9663	0.1975	0.3605	1.8260	1.4655	96.40

Table 10. Avg. estimated values and MSEs of the BE using MCMC for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 0.50, \theta = 0.50$

n		**BE: SEL**		**BE:LINEX**		**BE: GE**		**HPD**			
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Lower**	**Upper**	**AIL**	**CP (%)**
						$a = 0.50$ and $b = 0.50$					
100	λ	0.6658	0.1536	0.6613	0.1484	0.6497	0.1426	0.1455	1.3650	1.2195	95.20
	θ	0.4548	0.0393	0.4523	0.0388	0.4397	0.0395	0.1388	0.7979	0.6591	95.80
	a	0.5672	0.0108	0.5660	0.0106	0.5607	0.0099	0.4224	0.7208	0.2983	97.30
	b	0.5675	0.1006	0.5641	0.0976	0.5539	0.0940	0.1314	1.1523	1.0209	95.20
200	λ	0.6356	0.0959	0.6320	0.0927	0.6214	0.0882	0.1976	1.1913	0.9937	95.60
	θ	0.4348	0.0303	0.4324	0.0298	0.4200	0.0308	0.1497	0.6999	0.5503	95.70
	a	0.5822	0.0102	0.5815	0.0101	0.5784	0.0096	0.4830	0.7011	0.2181	97.50
	b	0.5522	0.0614	0.5497	0.0600	0.5408	0.0581	0.1729	1.0280	0.8551	95.40
						$a = 0.50$ and $b = 0.75$					
100	λ	0.7804	0.2526	0.7745	0.2439	0.7618	0.2360	0.1986	1.5670	1.3684	96.00
	θ	0.4173	0.0448	0.4150	0.0444	0.4026	0.0459	0.1080	0.7519	0.6438	95.50
	a	0.5639	0.0106	0.5628	0.0105	0.5580	0.0099	0.4138	0.6983	0.2845	96.70
	b	0.7286	0.1998	0.7224	0.1833	0.7112	0.1881	0.1863	1.3979	1.2116	95.10
200	λ	0.7554	0.1803	0.7502	0.1744	0.7381	0.1670	0.2744	1.4806	1.2063	96.30
	θ	0.4145	0.0317	0.4126	0.0317	0.4015	0.0335	0.1361	0.6977	0.5617	96.20
	a	0.5769	0.0089	0.5763	0.0088	0.5735	0.0084	0.4798	0.6892	0.2094	98.20
	b	0.7129	0.0970	0.7082	0.0944	0.6966	0.0935	0.2340	1.3315	1.0975	95.60
						$a = 0.75$ and $b = 0.50$					
100	λ	0.5024	0.0801	0.4995	0.0779	0.4901	0.0750	0.1226	1.2241	1.1016	95.80
	θ	0.3734	0.0535	0.3715	0.0534	0.3602	0.0552	0.0416	0.7058	0.6642	95.80
	a	0.9352	0.0625	0.9315	0.0605	0.9238	0.0574	0.6001	1.2427	0.6427	96.70
	b	0.7618	0.2500	0.7562	0.2419	0.7442	0.2347	0.1930	1.8617	1.6687	95.80
200	λ	0.4132	0.0494	0.4113	0.0487	0.4033	0.0483	0.0947	0.8132	0.7185	96.00
	θ	0.3463	0.0653	0.3444	0.0650	0.3341	0.0665	0.0498	0.7401	0.6903	96.00
	a	0.9399	0.0505	0.9378	0.0497	0.9330	0.0482	0.7321	1.1718	0.4397	99.00
	b	0.8996	0.3778	0.8915	0.3640	0.8777	0.3520	0.2745	1.7283	1.4538	96.00
						$a = 0.75$ and $b = 0.75$					
100	λ	0.5441	0.0594	0.5412	0.0575	0.5311	0.0549	0.1428	0.9901	0.8473	95.40
	θ	0.3335	0.0440	0.3315	0.0426	0.3207	0.0393	0.0000	0.6359	0.6359	95.40
	a	0.8283	0.0204	0.8259	0.0198	0.8200	0.0188	0.6028	1.0679	0.4650	97.40
	b	1.0214	0.2856	1.0115	0.2702	0.9981	0.2618	0.3154	1.9135	1.5981	95.40
200	λ	0.5788	0.0725	0.5761	0.0708	0.5671	0.0682	0.2098	1.0824	0.8726	95.30
	θ	0.3227	0.0329	0.3210	0.0322	0.3096	0.0295	0.0157	0.6086	0.5929	96.30
	a	0.8720	0.0242	0.8704	0.0237	0.8664	0.0229	0.7249	1.0765	0.3516	99.50
	b	0.9800	0.2226	0.9724	0.2144	0.9599	0.2078	0.3522	1.8339	1.4817	96.70

Table 11. Avg. estimated values and MSEs of the BE using MCMC for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 1.50, \theta = 0.25$

n		BE: SEL		BE:LINEX		BE: GE		HPD			
		Avg.	**MSE**	**Avg.**	**MSE**	**Avg.**	**MSE**	**Lower**	**Upper**	**AIL**	**CP (%)**
						$a = 0.50$ and $b = 0.50$					
100	λ	1.5865	0.5295	1.5627	0.4923	1.5485	0.4999	0.4954	3.0986	2.6033	95.50
	θ	0.4091	0.0563	0.4070	0.0551	0.3949	0.0509	0.1118	0.7391	0.6273	96.30
	a	0.5625	0.0088	0.5614	0.0086	0.5566	0.0081	0.4457	0.7099	0.2642	98.50
	b	0.7110	0.1426	0.7060	0.1372	0.6944	0.1302	0.2133	1.2958	1.0825	95.50
200	λ	1.5255	0.3875	1.5053	0.3671	1.4903	0.3725	0.4860	2.6242	2.1381	95.20
	θ	0.4014	0.0446	0.3994	0.0437	0.3877	0.0399	0.1672	0.6924	0.5252	97.20
	a	0.5743	0.0081	0.5736	0.0080	0.5708	0.0076	0.4806	0.6745	0.1938	97.70
	b	0.7281	0.1339	0.7233	0.1291	0.7113	0.1219	0.2825	1.3011	1.0186	96.10
						$a = 0.50$ and $b = 0.75$					
100	λ	1.6533	0.5052	1.6292	0.4702	1.6155	0.4742	0.5608	3.0550	2.4942	95.40
	θ	0.3718	0.0430	0.3700	0.0421	0.3594	0.0389	0.0806	0.7218	0.6412	96.40
	a	0.5750	0.0096	0.5739	0.0094	0.5694	0.0088	0.4659	0.7065	0.2406	97.20
	b	1.0035	0.2142	0.9943	0.2033	0.9803	0.1944	0.3801	1.7651	1.3851	95.60
200	λ	1.7344	0.4795	1.7104	0.4493	1.6962	0.4504	0.7395	3.1014	2.3618	96.40
	θ	0.3463	0.0277	0.3448	0.0272	0.3342	0.0249	0.1245	0.5995	0.4750	96.70
	a	0.5889	0.0101	0.5882	0.0099	0.5857	0.0095	0.4970	0.6762	0.1792	96.30
	b	0.9472	0.1602	0.9395	0.1523	0.9264	0.1455	0.3824	1.5918	1.2094	95.70
						$a = 0.75$ and $b = 0.50$					
100	λ	1.2376	0.2899	1.2244	0.2897	1.2082	0.3005	0.4985	2.2758	1.7773	97.40
	θ	0.3196	0.0475	0.3179	0.0468	0.3071	0.0444	0.0025	0.7014	0.6989	95.70
	a	0.8419	0.0269	0.8391	0.0262	0.8323	0.0249	0.6153	1.0881	0.4728	97.40
	b	0.8761	0.3155	0.8659	0.2947	0.8507	0.2790	0.3109	1.5729	1.2620	95.70
200	λ	1.2803	0.3823	1.2648	0.3662	1.2518	0.3740	0.4053	2.2012	1.7959	95.20
	θ	0.3266	0.0370	0.3252	0.0364	0.3152	0.0345	0.0627	0.6774	0.6146	96.40
	a	0.8934	0.0295	0.8916	0.0289	0.8876	0.0278	0.7496	1.0709	0.3214	97.60
	b	0.8990	0.3388	0.8913	0.3264	0.8780	0.3145	0.3484	1.8689	1.5206	97.60
						$a = 0.75$ and $b = 0.75$					
100	λ	1.3833	0.2552	1.3668	0.2485	1.3504	0.2554	0.5322	2.2118	1.6796	96.70
	θ	0.2915	0.0361	0.2903	0.0357	0.2819	0.0337	0.0010	0.6494	0.6484	95.40
	a	0.8479	0.0229	0.8454	0.0222	0.8393	0.0209	0.6459	1.0501	0.4042	96.70
	b	1.1431	0.3846	1.1285	0.3538	1.1130	0.3411	0.4450	2.1598	1.7147	96.10
200	λ	1.3792	0.3270	1.3640	0.3136	1.3512	0.3200	0.5131	2.3935	1.8804	95.30
	θ	0.3530	0.0481	0.3513	0.0473	0.3406	0.0444	0.0075	0.6698	0.6623	95.30
	a	0.8574	0.0223	0.8559	0.0220	0.8518	0.0215	0.6812	1.0437	0.3625	97.70
	b	1.2064	0.4315	1.1932	0.4067	1.1793	0.3924	0.4454	2.1931	1.7477	96.10

Table 12. Avg. estimated values and MSEs of the BE using MCMC for the $BBE2$ distribution at different sample sizes n and different values of (a, b) when $\lambda = 1.50, \theta = 0.50$

n		BE: SEL		BE:LINEX		BE: GE		HPD			
		Avg.	MSE	Avg.	MSE	Avg.	MSE	Lower	Upper	AIL	CP (%)
					$a = 0.50$ and $b = 0.50$						
100	λ	1.5949	0.4241	1.5707	0.3963	1.5552	0.3990	0.4817	2.8664	2.3847	95.60
	θ	0.4526	0.0418	0.4500	0.0412	0.4370	0.0416	0.1209	0.7990	0.6781	95.20
	a	0.5628	0.0096	0.5616	0.0094	0.5566	0.0089	0.4147	0.7011	0.2864	96.40
	b	0.6333	0.0935	0.6293	0.0905	0.6176	0.0857	0.1976	1.1518	0.9541	95.20
200	λ	1.6267	0.4213	1.6063	0.3994	1.5925	0.4020	0.6017	2.8506	2.2489	96.10
	θ	0.4479	0.0295	0.4456	0.0292	0.4338	0.0302	0.1657	0.7233	0.5576	96.00
	a	0.5802	0.0100	0.5795	0.0099	0.5765	0.0095	0.4659	0.6950	0.2291	97.50
	b	0.6299	0.0917	0.6261	0.0882	0.6159	0.0835	0.2437	1.1356	0.8919	96.00
					$a = 0.50$ and $b = 0.75$						
100	λ	1.6810	0.5096	1.6553	0.4738	1.6399	0.4789	0.4820	3.0446	2.5626	95.00
	θ	0.3865	0.0405	0.3844	0.0402	0.3728	0.0419	0.1189	0.6796	0.5608	96.20
	a	0.5850	0.0118	0.5839	0.0115	0.5793	0.0107	0.4692	0.7145	0.2453	97.40
	b	0.8896	0.1460	0.8813	0.1381	0.8668	0.1312	0.3052	1.6220	1.3168	95.50
200	λ	1.7339	0.4436	1.7104	0.4134	1.6970	0.4136	0.7003	3.0303	2.3299	96.20
	θ	0.3927	0.0329	0.3908	0.0331	0.3794	0.0354	0.1379	0.6684	0.5305	96.50
	a	0.5989	0.0126	0.5982	0.0125	0.5954	0.0119	0.4832	0.6917	0.2084	95.60
	b	0.8693	0.1271	0.8624	0.1206	0.8498	0.1154	0.3559	1.5547	1.1988	96.00
					$a = 0.75$ and $b = 0.50$						
100	λ	1.2899	0.3308	1.2755	0.3247	1.2611	0.3314	0.5131	2.4233	1.9102	96.90
	θ	0.3458	0.0728	0.3439	0.0726	0.3329	0.0745	0.0192	0.7276	0.7085	96.90
	a	0.9204	0.0514	0.9167	0.0499	0.9084	0.0474	0.5942	1.1140	0.5198	97.90
	b	0.8444	0.3324	0.8382	0.3233	0.8258	0.3142	0.2734	1.7783	1.5050	95.80
200	λ	1.1301	0.3316	1.1212	0.3342	1.1080	0.3452	0.3093	1.9100	1.6007	95.50
	θ	0.3039	0.0734	0.3030	0.0735	0.2957	0.0762	0.0598	0.7096	0.6498	95.50
	a	0.9675	0.0570	0.9654	0.0561	0.9612	0.0543	0.7966	1.1437	0.3471	98.50
	b	0.9585	0.5883	0.9451	0.5285	0.9340	0.5299	0.3774	2.0226	1.6452	97.00
					$a = 0.75$ and $b = 0.75$						
100	λ	1.3484	0.2612	1.3342	0.2551	1.3196	0.2616	0.4972	2.0795	1.5823	95.10
	θ	0.3393	0.1481	0.3378	0.1469	0.3292	0.1488	0.0014	0.7717	0.7703	95.10
	a	0.9262	0.0872	0.9228	0.0852	0.9157	0.0831	0.6704	1.1831	0.5127	98.10
	b	1.0871	0.4269	1.0753	0.3921	1.0629	0.3931	0.3857	1.8090	1.4234	95.10
200	λ	1.1800	0.2372	1.1686	0.2372	1.1548	0.2447	0.5466	1.9582	1.4116	96.00
	θ	0.2716	0.0908	0.2704	0.0907	0.2625	0.0930	0.0099	0.6410	0.6311	96.00
	a	0.9434	0.0462	0.9416	0.0455	0.9377	0.0442	0.7688	1.1019	0.3330	98.70
	b	1.2299	0.4617	1.2179	0.4424	1.2035	0.4292	0.5190	2.2479	1.7289	96.00

8. Applications: Kevlar Data

A real-world data example is provided to demonstrate the flexibility of the $BBE2$ distribution and its sub-models for data modeling. The set of data in Table 13 comprised 101 observations referring to failure times data of Kevlar 49/epoxy strands at 90% pressure. The failure time data were originally given in [2,43] and further analyzed by [44]. The $BBE2$ distribution is compared with the Kumaraswamy log-logistic Rayleigh (KLLoGR), exponentiated log-logistic Weibull (ELLoGW), exponentiated log-logistic exponential (ELLoGE), exponentiated log-logistic Rayleigh (ELLoGR), log-logistic Weibull (LLoGW), log-logistic

Rayleigh (LLoGR), beta-modified Weibull (BMW) and generalized log-logistic Weibull (GLLoGW) distributions. All these competitive models are mentioned in [45].

Table 13. Failure time data of Stress-Rupture Life of Kevlar 49/Epoxy Strands with pressure at 90% Data.

0.01	0.01	0.02	0.02	0.02	0.03	0.03	0.04	0.05	0.06	0.07	0.07	0.08	0.09	0.09
0.10	0.10	0.11	0.11	0.12	0.13	0.18	0.19	0.20	0.23	0.24	0.24	0.29	0.34	0.35
0.36	0.38	0.40	0.42	0.43	0.52	0.54	0.56	0.60	0.60	0.63	0.65	0.67	0.68	0.72
0.72	0.72	0.73	0.79	0.79	0.80	0.80	0.83	0.85	0.90	0.92	0.95	0.99	1.00	1.01
1.02	1.03	1.05	1.10	1.10	1.11	1.15	1.18	1.20	1.29	1.31	1.33	1.34	1.40	1.43
1.45	1.50	1.51	1.52	1.53	1.54	1.54	1.55	1.58	1.60	1.63	1.64	1.8	1.8	1.81
2.02	2.05	2.14	2.17	2.33	3.03	3.03	3.34	4.20	4.69	7.89				

Some properties of the dataset were computed in Table 14.

Table 14. Some properties of Stress-Rupture Life of Kevlar 49/Epoxy Strands Data.

Q_1	Median	Q_3	Mean	Variance	Kurtosis	Skewness
0.2400	0.800	1.4500	1.0249	1.2530	14.4745	3.0472

The asymptotic covariance matrix of the MLEs for the $BBE2$ distribution $I_n^{-1}(\hat{\psi})$ is provided via:

$$I_0^{-1}(\hat{a},\hat{b},\hat{\theta},\hat{\lambda}) = \begin{pmatrix} 0.028 & 0.008376 & -0.034 & -0.312 \\ 0.008376 & 0.036 & 0.043 & -1.238 \\ -0.034 & 0.043 & 0.24 & -1.266 \\ -0.312 & -1.238 & -1.266 & 43.531 \end{pmatrix}$$

and the approximate 99% CIs for the parameters are $a \in [0.69, 0.782]$, $b \in [0.114, 0.219]$, $\theta \in [0.235, 0.506]$ and $\lambda \in [4.359, 8.003]$, respectively.

The following statistics given in Table 15 consists of Akaike information criterion (AIC), consistent Akaike information criterion (CAIC)and Bayesian information criterion (BIC). Plots of the fitted pdf, cdf and survival are QQ plots provided in Figures 3 and 4.

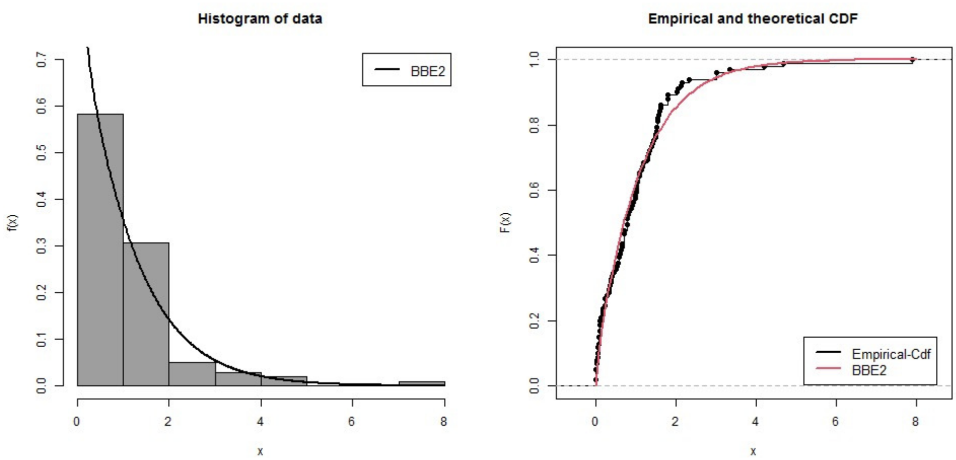

Figure 3. Plots of the epdf and ecdf for the $BBE2$ model.

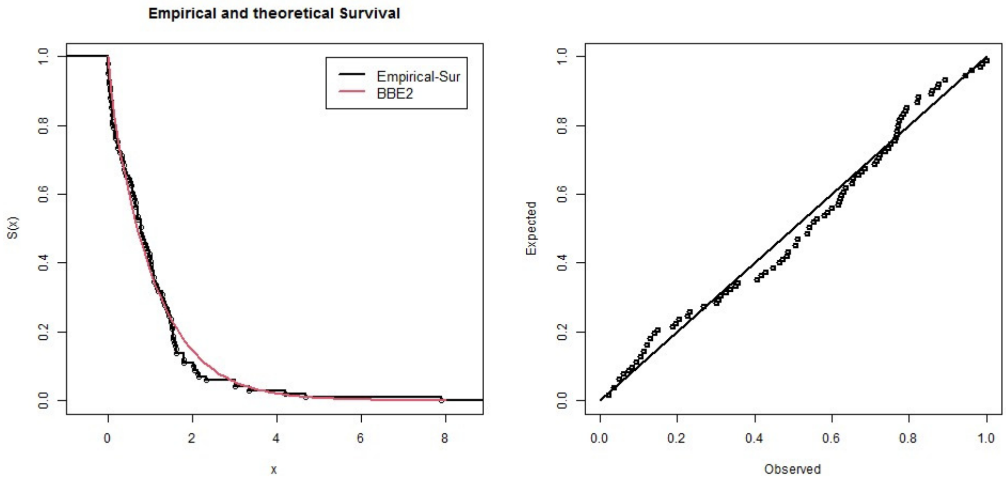

Figure 4. Plots of the estimated survival and QQ plot for the $BBE2$ model.

Table 15. Summary of fitted distributions corresponding to Stress-Rupture life of Kevlar 49/Epoxy Strands data.

Distribution	Estimates					Measures			
						$-2logL$	AIC	$CAIC$	BIC
$KLLoGR(a,b,\alpha,\beta)$	0.2734	0.5547	2.7501	0.06908	2.0	205.9	213.9	214.3	224.3
$ELLoGW(a,b,\alpha,\beta)$	1.7650	1.0	0.4598	0.7387	1.0910	204.9	212.9	213.3	223.3
$ELLoGE(a,b,\alpha,\beta)$	1.9878	1.0	0.4051	0.8586	1.0	205.0	211.0	211.2	218.8
$ELLoGR(a,b,\alpha,\beta)$	0.6750	1.0	1.3884	0.06580	2.0	210.1	216.1	216.4	224.0
$LLoGW(a,b,\alpha,\beta)$	1.0	1.0	2.1998	0.4113	0.5413	207.5	213.5	213.7	221.3
$LLoGR(a,b,\alpha,\beta)$	1.0	1.0	0.9114	0.1423	2.0	213.3	217.3	217.5	226
$BetaMW(a,b,\alpha,\gamma,\lambda)$	108.86	25.631	1.6632	0.0534	0.0343	207.3	217.3	217.9	230.38
$GLLoGW(c,\alpha,\beta,\delta,\theta)$	0.2365	0.2591	0.9648	4.3962	0.1396	204.1	214.01	214.6	227.1
$BBE2(a,b,\theta,\lambda)$	0.736	0.167	0.371	6.181		204.722	212.722	213.13	212.739

Now, we apply the formal goodness-of-fit tests in order to verify which distribution fits better to these data. We consider the Cramér–von Mises ($W*$) and Anderson–Darling ($A*$), which are presented in Table 16.

The $BBE2$ model has the smallest value of the numerical results, which are mentioned in Tables 15 and 16 for the measures $-2logL$, AIC, $CAIC$, BIC, A^* and W^*. The $BBE2$ model has the best fit for the proposed dataset.

Table 16. Goodness-of-fit tests corresponding to the Stress-Rupture life of Kevlar 49/Epoxy Strands data.

Distribution	Statistics	
	W^*	A^*
$KLLoGR(a, b, \alpha, \beta)$	0.1635	0.9753
$ELLoGW(a, b, \alpha, \beta)$	0.1319	0.8073
$ELLoGE(a, b, \alpha, \beta)$	0.1447	0.8635
$ELLoGR(a, b, \alpha, \beta)$	0.2610	1.4415
$LLoGW(a, b, \alpha, \beta)$	0.1070	0.7446
$LLoGR(a, b, \alpha, \beta)$	0.1776	1.1049
$BetaMW(a, b, \alpha, \gamma, \lambda)$	0.1955	1.1190
$GLLoGW(c, \alpha, \beta, \delta, \theta)$	0.1322	0.7996
$BBE2(a, b, \theta, \lambda)$	0.12446	0.77445

9. Conclusions

In this article, a new four-parameter lifetime distribution called the beta binomial exponential 2 ($BBE2$) distribution is proposed. Some mathematical features, including quantile function, moments, generating function and characteristic function, of the $BBE2$ distribution are computed. The acceptance sampling plans have been derived based on the $BBE2$ distribution when the life test is truncated at the median life of the proposed distribution. At different parameters of the proposed distribution and different levels of consumer risk, the minimum sample size was computed under multiple truncation times. Further, at the obtained sample sizes, the probability of acceptance was computed to ensure that it is less than or equal to the complement of the consumer's risk $(1 - \alpha)$. Some useful tables are provided and applied to establish acceptance sampling plans. Classical (ML and $MPSP$ estimation methods) and Bayesian estimation approaches are utilized to estimate the model parameters. The performance of the model parameters is examined through the simulation study by using the three different approaches of estimation. Subsequently, we examine real-world data applications to demonstrate the versatility and potential of the $BBE2$ model. A real-world application demonstrates that the new distribution can offer a better fit than other competitive lifetime models. Future work can be extended for double and group-acceptance sampling plans based on the $BBE2$ distribution.

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