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Amended Criteria for Testing the Asymptotic and Oscillatory Behavior of Solutions of Higher-Order Functional Differential Equations

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Abstract: Our interest in this article is to develop oscillation conditions for solutions of higher order differential equations and to extend recent results in the literature to differential equations of several delays. We obtain new asymptotic properties of a class from the positive solutions of an even higher order neutral delay differential equation. Then we use these properties to create more effective criteria for studying oscillation. Finally, we present some special cases of the studied equation and apply the new results to them.

Keywords: oscillatory; nonoscillatory; even-order; neutral; delay; differential equation

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1. Introduction

When modeling the length of time required to accomplish some hidden activities, the concept of delay in systems is considered as playing a crucial role. When the predator birth rate is influenced by historical levels of predators or prey rather than only present levels, the predator-prey model exhibits a delay. Sending measured signals to the remote control center has been much easier because to the quick development of communication technologies. The primary challenge for engineers, nevertheless, is the inescapable lag between the measurement and the signal received by the controller. To minimize the possibility of experimental instability and potential harm, this lag must be taken into account at the design stage. Delay differential equations (DDE) appear when modeling such phenomena, and others, see [1,2].

Many biological, chemical, and physical phenomena have mathematical models that use differential equations of the fourth-order delay. Examples of these applications include soil settlement and elastic issues. The oscillatory traction of a muscle, which takes place when the muscle is subjected to an inertial force, is one model that can be modeled by a fourth-order oscillatory equation with delay, see [3]. Heterogeneity in the Fisher-KPP reaction term is a research topic of interest. Palencia et al. [4] studied the existence of solutions, uniqueness, and travelling wave oscillatory properties.

Over the past few years, research has consistently focused on identifying necessary conditions for the oscillatory and non-oscillatory features of fourth and higher-order differential equations; see for example [5–9].

Below, we review in more detail some of the works that contributed to the development of the oscillation theory of higher order DDEs.

In 1998, Zafer [10] presented an oscillation criterion for the neutral differential equation (NDE)

$$(x(\ell)+p(\ell)x(\vartheta(\ell)))^{(n)}+G(\ell,x(\ell),x(h(\ell)))=0, \tag{1}$$

where $G(\ell,u,v)\in([0,\infty)\times\mathbb{R}\times\mathbb{R})$ and $vG(\ell,u,v)>0$ for $uv>0$.

Li et al. [11] and Zhang et al. [12] created and developed criteria for oscillation of the NDE

$$(x(\ell)+p(\ell)x(\vartheta(\ell)))^{(n)}+q(\ell)H(x(h(\ell)))=0, \tag{2}$$

The results obtained are an improvement and generalization of the results [10].

It is known that studies of the oscillatory behavior of solutions of differential equations are classified into two types, depending on the convergence or divergence of the integration $\int_{\ell_1}^{\ell} r^{-1/\alpha}(\mathfrak{a})\mathrm{d}\mathfrak{a}$ as $\ell\to\infty$. This is a result of the effect of this influence on the behavior of the positive solutions of the equation. In the case of equations with even orders, we find that the divergence of this integration means that there are no positive decreasing solutions.

Baculikova and Dzurina [13] studied the asymptotic and oscillation behavior of the solutions of the higher order delay differential equations

$$\left(r(\ell)\left(x'(\ell)\right)^{\alpha}\right)^{(n-1)}+q(\ell)x^{\alpha}(\vartheta(\ell))=0, \tag{3}$$

They set some oscillation conditions for (3) under the canonical condition

$$\int_{\ell_1}^{\ell} r^{-1/\alpha}(\mathfrak{a})\mathrm{d}\mathfrak{a}\to\infty \text{ as } \ell\to\infty. \tag{4}$$

where α is the ratio of two positive odd integers.

Sun et al. [14] studied the oscillation of NDE

$$(r(\ell)(x(\ell)+p(\ell)x(\vartheta(\ell))))^{(n)}+q(\ell)f(x(h(\ell)))=0, \tag{5}$$

under both the canonical condition (4) and non-canonical condition

$$\int_{\ell_0}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a}<\infty, \tag{6}$$

where $f(u)/u\geq k>0$.

Moaaz et al. [15] investigated the oscillatory properties of NDE

$$\left(r(\ell)\left((x(\ell)+p(\ell)x(\vartheta(\ell)))^{(n-1)}\right)^{\alpha}\right)'+q(\ell)x^{\alpha}(h(\ell))=0, \tag{7}$$

in the noncanonical case. They derived criteria for improving conditions that exclude the decreasing positive solutions of the considered equation.

In this study, we consider the more general neutral differential equation (NDE) of higher order and with several delays,

$$\frac{\mathrm{d}}{\mathrm{d}\ell}\left(r(\ell)\left(\frac{\mathrm{d}^{n-1}}{\mathrm{d}\ell^{n-1}}[x(\ell)+p(\ell)x(\vartheta(\ell))]\right)^{\alpha}\right)+\sum_{i=1}^{J}q_i(\ell)x^{\alpha}(h_i(\ell))=0,\ \ell\geq\ell_0, \tag{8}$$

which includes many of the previous equations as special cases. We deal with the oscillatory behavior of the solutions of Equation (8), so that we introduce new criteria that guarantee the oscillation of all solutions of this equation in the non-canonical condition. For this, we assume the following for n and α:

(H_1) $n\in\mathbb{N}, n\geq 4$, and $\alpha\in Q^{+}_{odd}:=\{a/b: a,b\in\mathbb{Z}^{+}$ and a,b are odd$\}$.

Moreover, r, p and q_i are continuous real functions on $[\ell_0,\infty)$, and r is differentiable, which satisfy the conditions:

(H_2) $r(\ell)>0, r'(\ell)\geq 0, 0\leq p(\ell)<1$ and $q_i(\ell)\geq 0$ for $i=1,2,\ldots,J$.

Furthermore, ϑ and h_i are continuous delay functions on $[\ell_0, \infty)$ and h_i is differentiable, which satisfy the conditions:

(H$_3$) $\vartheta(\ell) \leq \ell$, $h_i(\ell) \leq \ell$, $h_i'(\ell) > 0$. and $\lim_{\ell\to\infty} \vartheta(\ell) = \lim_{\ell\to\infty} h_i(\ell) = \infty$ for $i = 1, 2, \ldots, J$.

For convenience, we define the corresponding function $\mathcal{B} := x + p \cdot (x \circ \vartheta)$. A solution to Equation (8) is defined as a real differentiable function on $[\ell_x, \infty)$, $\ell_x \geq \ell_0$, which satisfies the properties $\mathcal{B} \in C^{(n-1)}([\ell_x, \infty))$, $r(\mathcal{B}^{(n-1)})^{\alpha} \in C^1([\ell_x, \infty))$ and x satisfies (8) on $[\ell_x, \infty)$. We will consider the eventually non-zero solutions, that is, $\sup\{|x(\ell)| : \ell \geqslant \ell_*\} > 0$, for $\ell_* \geq \ell_x$. A solution of (8) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is said to be nonoscillatory.

This article aims to extend recent previous results from the literature (see for example [16–19]) to differential equations with even-order and several delays, and to develop oscillation criteria for solutions of even order differential equations. For a class of positive solutions of NDE (8), we derive new asymptotic properties. Then, we construct better criteria for evaluating oscillation using these properties. We then apply the new results to a some particular cases of the equation under study.

2. Previous Results

In this part, we review some results from the literature.

Below, we review the most important results of paper [10], which studies the oscillatory behavior of solutions to Equation (1).

Theorem 1 ([10])**.** *Assume that $\psi(\ell) \in C([\ell_0, \infty), [0, \infty))$ and that $F \in C^1([\ell_0, \infty), [0, \infty))$ such that $F' \geq 0$,*

$$|G(\ell, u, v)| \geq \psi(\ell) F\left(\frac{|v|}{(1 - p(h(\ell)))h^{n-1}(\ell)}\right),$$

and

$$\int_{\ell_0}^{\zeta} \frac{1}{F(\mathfrak{a})} \mathrm{d}\mathfrak{a} < \infty \textit{ for all } \zeta > 0.$$

Then, all solutions of Equation (1) are oscillatory if

$$\int_{\ell_0}^{\infty} \psi(\mathfrak{a}) \mathrm{d}\mathfrak{a} = \infty,$$

In the following theorem we give the oscillation condition of Equation (2).

Theorem 2 ([12])**.** *Suppose that $|H(u)| \geq |u|$, for all $|u| \geq u_0 > 0$. Then, all solutions of Equation (2) are oscillatory if there is a $\lambda \in (0, 1)$ such that the first-order DDE*

$$Y'(\ell) + \frac{\lambda}{(n-1)!} q(\ell) h^{n-1}(\ell)(1 - p(h(\ell))) Y(h(\ell)) = 0,$$

is oscillatory.

Now, we present one of the results of the oscillation of the Equation (3).

Theorem 3 ([13])**.** *All solutions of Equation (3) are oscillatory if the first-order DDE*

$$Y'(\ell) + \frac{\alpha^{\alpha} \lambda^{\alpha}}{(n-2)!(n-2+\alpha)^{\alpha}} \frac{q(\ell) \vartheta^{n-2+\alpha}(\ell)}{r(\vartheta(\ell))} Y(\vartheta(\ell)) = 0,$$

is oscillatory, for some $\lambda \in (0, 1)$.

In the following two theorems, Sun et al. [14] provide two different criteria for the volatility of the Equation (5).

Theorem 4 ([14]). *Suppose that (4) holds and*

$$h(\ell) \leq \vartheta(\ell),\ p(\ell) \leq p_0,\ \vartheta'(\ell) \geq \vartheta_0 > 0 \text{ and } \vartheta \circ h = h \circ \vartheta. \tag{9}$$

Then, all solutions of Equation (5) are oscillatory if

$$\liminf_{\ell \to \infty} \int_{\vartheta^{-1}(h(\ell))}^{\ell} \frac{h^{n-1}(\mathfrak{a})}{r(h(\mathfrak{a}))} Q(\mathfrak{a}) \mathrm{d}\mathfrak{a} > (n-1)! \frac{(p_0 + \vartheta_0)}{k\vartheta_0 \mathrm{e}}, \tag{10}$$

where $Q(\ell) = \min\{q(\ell), q(\vartheta(\ell))\}$.

Theorem 5 ([14]). *Suppose that (6) and (9) hold. Then, all solutions of Equation (5) are oscillatory if (10) and*

$$\limsup_{\ell \to \infty} \int_{\ell_0}^{\ell} \left(\frac{\lambda}{(n-2)!} \xi(\mathfrak{a}) Q(\mathfrak{a}) h^{n-2}(\mathfrak{a}) - \frac{1 + p_0/\vartheta_0}{4} \frac{1}{r(\mathfrak{a})\xi(\mathfrak{a})} \right) \mathrm{d}\mathfrak{a} = \infty,$$

for $\lambda \in (0,1)$, *where* $\xi(\ell) := \int_{\ell}^{\infty} r^{-1/\alpha}(\mathfrak{a}) \mathrm{d}\mathfrak{a}$.

Finally, we present one of the results that guarantees the oscillation of Equation (7) in the non-canonical case.

Theorem 6 ([15]). *Suppose that*

$$\limsup_{\ell \to \infty} \int_{\ell_0}^{\ell} \left[q(\mathfrak{a}) \left(1 - p(h(\mathfrak{a})) R_0(\mathfrak{a}) \frac{\lambda h^{n-2}(\mathfrak{a})}{(n-2)!} \right)^{\alpha} - \frac{\frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}}{r^{1/\alpha}(\mathfrak{a}) R_0(\mathfrak{a})} \right] \mathrm{d}\mathfrak{a} = \infty,$$

holds for some constant $\lambda \in (0,1)$ *and*

$$\limsup_{\ell \to \infty} \left(R_{n-2}^{\alpha}(\ell) \int_{\ell_1}^{\ell} q(\mathfrak{a}) \widetilde{p}^{\alpha}(h(\mathfrak{a})) \mathrm{d}\mathfrak{a} \right) > 1.$$

Then all solutions of (7) are oscillatory, where

$$R_0(\ell) := \int_{\ell}^{\infty} \frac{1}{r^{1/\alpha}(\mathfrak{a})} \mathrm{d}\mathfrak{a},\ R_{n-2}(\ell) := \int_{\ell}^{\infty} R_{n-3}(\mathfrak{a}) \mathrm{d}\mathfrak{a},$$

and

$$\widetilde{p}(\ell) = 1 - p(\ell) \frac{R_{n-2}(\vartheta(\ell))}{R_{n-2}(\ell)} > 0.$$

In the next part, we review some lemmas from the literature that we will need in the proof of our results.

Lemma 1 ([20]). *Suppose that* $Y(\ell) \in C^m([\ell_0, \infty), \mathbb{R}^+)$, $Y^{(m)}(\ell)$ *is of constant sign and not identically zero on* $[\ell_0, \infty)$. *Assume also that* $Y^{(m-1)}(\ell) Y^{(m)}(\ell) \leq 0$, *eventually, and* $\lim_{u \to \infty} Y(\ell) \neq 0$. *Then, eventually,*

$$Y(\ell) \geq \frac{\lambda}{(m-1)!} u^{m-1} \left| Y^{(m-1)}(\ell) \right|, \text{ for } \lambda \in (0,1).$$

Lemma 2 ([21]). *Assume that* ϱ_1 *and* ϱ_2 *are real numbers,* $\varrho_1 > 0$, *then,*

$$\varrho_1 H^{(\alpha+1)/\alpha} - \varrho_2 H \geq -\frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}} \frac{\varrho_2^{\alpha+1}}{\varrho_1^{\alpha}}. \tag{11}$$

The following lemma classifies the positive solutions depending on the sign of their derivatives, which is a modification of Lemma 1.1 in [22] for the studied equation.

Lemma 3. *Suppose that $x \in C([\ell_0, \infty), \mathbb{R}^+)$ is a solution to (8). Then, $\mathcal{B}$ is positive, $r \cdot \left(\mathcal{B}^{(n-1)}\right)^{\alpha}$ is decreasing, and $\mathcal{B}$ satisfies one of the following cases:*

$(\mathbf{N}_1)$ $\mathcal{B}^{(r)}(\ell) > 0$ *for* $r = 1, 2, \ldots, n-1$ *and* $\mathcal{B}^{(n)}(\ell) < 0$;
$(\mathbf{N}_2)$ $\mathcal{B}^{(r)}(\ell) > 0$ *for* $r = 1, 2, \ldots, n-2$ *and* $\mathcal{B}^{(n-1)}(\ell) < 0$;
$(\mathbf{N}_3)$ $(-1)^r \mathcal{B}^{(r)}(\ell) > 0$ *for* $r = 0, 1, 2, \ldots, n-1$,

eventually.

3. Auxiliary Results

Next, we provide the following notations to help us display the results easily:

$$h(\ell) := \min\{h_i(\ell),\ i = 1, \ldots, J\},$$

$$R_0(\ell) := \int_{\ell}^{\infty} \frac{1}{r^{1/\alpha}(\mathfrak{a})} \mathrm{d}\mathfrak{a},$$

$$R_m(\ell) := \int_{\ell}^{\infty} R_{m-1}(\mathfrak{a}) \mathrm{d}\mathfrak{a},\ m = 1, 2, \ldots, n-2,$$

$$Q(\ell) := \sum_{i=1}^{J} q_i(\ell)(1 - p(h_i(\ell)))^{\alpha}$$

and

$$Q^*(\ell) := \sum_{i=1}^{J} q_i(\ell)\left(1 - p(h_i(\ell))\frac{R_{n-2}(\vartheta(h_i(\ell)))}{R_{n-2}(h_i(\ell))}\right)^{\alpha}.$$

Further, we denote the set of all eventually positive solutions of (8) which $\mathcal{B}(\ell)$ satisfies $\mathbf{N}_2$ by Ω.

Lemma 4. *Assume that $x \in \Omega$, then,*

$$\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)' \le -Q(\ell)\mathcal{B}^{\alpha}(h(\ell)).$$

Proof. Assume that $x \in \Omega$, we find $\mathcal{B}'(\ell) > 0$. Since $\vartheta(\ell) \le \ell$,then we have $x(\vartheta(\ell)) \le \mathcal{B}(\vartheta(\ell)) \le \mathcal{B}(\ell)$, therefore, we get

$$\begin{aligned} x(\ell) &= \mathcal{B}(\ell) - p(\ell)x(\vartheta(\ell)) \ge \mathcal{B}(\ell) - p(\ell)\mathcal{B}(\vartheta(\ell)) \\ &\ge (1 - p(\ell))\mathcal{B}(\ell). \end{aligned} \tag{12}$$

From (8) and (12), we have

$$\begin{aligned} \left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)' &= -\sum_{i=1}^{J} q_i(\ell)x^{\alpha}(h_i(\ell)) \le -\sum_{i=1}^{J} q_i(\ell)(1 - p(h_i(\ell)))^{\alpha}\mathcal{B}^{\alpha}(h_i(\ell)) \\ &\le -\mathcal{B}^{\alpha}(h(\ell))\sum_{i=1}^{J} q_i(\ell)(1 - p(h_i(\ell)))^{\alpha} \le -\mathcal{B}^{\alpha}(h(\ell))Q(\ell). \end{aligned} \tag{13}$$

The proof of the lemma is complete. □

Lemma 5. *Assume that $x \in \Omega$, then, $\mathcal{B}^{(n-2)}(\ell)/R_0(\ell)$ is increasing.*

Proof. Assume that $x \in \Omega$. From (8) we find that $r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}$ is decreasing.

Now, since

$$\mathcal{B}^{(n-2)}(\ell) \geq -\int_{\ell}^{\infty} \frac{r^{1/\alpha}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})} \mathcal{B}^{(n-1)}(\mathfrak{a}) d\mathfrak{a} \geq -R_0(\ell) r^{1/\alpha}(\ell) \mathcal{B}^{(n-1)}(\ell), \tag{14}$$

and so

$$\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)} \right)' = \frac{1}{r^{1/\alpha}(\ell) R_0^2(\ell)} \left(R_0(\ell) r^{1/\alpha}(\ell) \mathcal{B}^{(n-1)}(\ell) + \mathcal{B}^{(n-2)}(\ell) \right) \geq 0. \tag{15}$$

The proof of the lemma is complete. □

Lemma 6. *Assume that $x \in \Omega$, and there are $\gamma > 0$ and $\ell_1 \geq \ell_0$ such that*

$$\frac{1}{\alpha} r^{1/\alpha}(\ell) R_0^{1+\alpha}(\ell) \left(h^{n-2}(\ell) \right)^{\alpha} Q(\ell) \geq ((n-2)!)^{\alpha} \gamma, \tag{16}$$

then

$$\lim_{\ell \to \infty} \mathcal{B}^{(n-2)}(\ell) = 0,$$

where $\beta_0 = \mu_0 \gamma^{1/\alpha}$.

Proof. Assume that $x \in \Omega$, using Lemma 1 with $f = \mathcal{B}$ and $m = n-1$, we have

$$\mathcal{B}(\ell) \geq \frac{\mu_0}{(n-2)!} \ell^{n-2} \mathcal{B}^{(n-2)}(\ell), \tag{17}$$

for all $\mu_0 \in (0,1)$. Now, since $\mathcal{B}^{(n-2)}(\ell)$ is a positive decreasing function, we conclude that $\lim_{\ell \to \infty} \mathcal{B}^{(n-2)}(\ell) = c_1 \geq 0$. We claim that $c_1 = 0$. If not, then $\mathcal{B}^{(n-2)}(\ell) \geq c_1 > 0$ eventually, which with (17) gives

$$\mathcal{B}(\ell) \geq \frac{\mu_0}{(n-2)!} \ell^{n-2} \mathcal{B}^{(n-2)}(\ell) \geq \frac{\mu_0 c_1}{(n-2)!} \ell^{n-2},$$

for all $\mu_0 \in (0,1)$. Therefore, from (13), we have

$$\begin{aligned} \left(r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} \right)' &\leq -Q(\ell) \mathcal{B}^{\alpha}(h(\ell)) \leq -\left(\frac{\mu_0 c_1}{(n-2)!} h^{n-2}(\ell) \right)^{\alpha} Q(\ell) \\ &\leq -\mu_0^{\alpha} c_1^{\alpha} \frac{(h^{n-2}(\ell))^{\alpha}}{((n-2)!)^{\alpha}} Q(\ell), \end{aligned}$$

which with (16) becomes

$$\left(r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} \right)' \leq -\alpha c_1^{\alpha} \mu_0^{\alpha} \gamma \frac{1}{r^{1/\alpha}(\ell) R_0^{1+\alpha}(\ell)} \leq -\alpha c_1^{\alpha} \beta_0^{\alpha} \frac{1}{r^{1/\alpha}(\ell) R_0^{1+\alpha}(\ell)}. \tag{18}$$

Integrating (18) from ℓ_2 to ℓ, we have

$$\begin{aligned} r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} &\leq r(\ell_2) \left(\mathcal{B}^{(n-1)}(\ell_2) \right)^{\alpha} - \alpha c_1^{\alpha} \beta_0^{\alpha} \int_{\ell_2}^{\ell} \frac{1}{r^{1/\alpha}(\mathfrak{a}) R_0^{1+\alpha}(\mathfrak{a})} d\mathfrak{a} \\ &\leq \beta_0^{\alpha} c_1^{\alpha} \left(\frac{1}{R_0^{\alpha}(\ell_2)} - \frac{1}{R_0^{\alpha}(\ell)} \right). \end{aligned} \tag{19}$$

Since $R_0^{-\alpha}(\ell) \to \infty$ as $\ell \to \infty$, there is a $\ell_3 \geq \ell_2$ such that $R_0^{-\alpha}(\ell) - R_0^{-\alpha}(\ell_2) \geq \epsilon R_0^{-\alpha}(\ell)$ for all $\epsilon \in (0,1)$. Therefore, (19) becomes

$$\mathcal{B}^{(n-1)}(\ell) \leq -c_1 \epsilon^{1/\alpha} \beta_0 \frac{1}{r^{1/\alpha}(\ell) R_0(\ell)}, \tag{20}$$

for all $\ell \geq \ell_3$. Integrating (20) from ℓ_3 to ℓ, we have

$$\begin{aligned} \mathcal{B}^{(n-2)}(\ell) &\leq \mathcal{B}^{(n-2)}(\ell_3) - c_1 \epsilon^{1/\alpha} \beta_0 \int_{\ell_3}^{\ell} \frac{1}{r^{1/\alpha}(\mathfrak{a}) R_0(\mathfrak{a})} \mathrm{d}\mathfrak{a} \\ &\leq \mathcal{B}^{(n-2)}(\ell_3) - c_1 \epsilon^{1/\alpha} \beta_0 \ln \frac{R_0(\ell_3)}{R_0(\ell)} \to -\infty \text{ as } \ell \to \infty, \end{aligned}$$

which is a contradiction. Then, $c_1 = 0$. The proof of the lemma is complete. □

Lemma 7. *Assume that $x \in \Omega$, and (16) holds, then*

$$\mathcal{B}^{(n-2)}(\ell) / R_0^{\beta_0}(\ell) \text{ is decreasing} \tag{21}$$

and

$$\mathcal{B}^{(n-2)}(\ell) / R_0^{1-\beta_0}(\ell) \text{ is increasing} \tag{22}$$

for $\ell \geq \ell_0$, where $\beta_0 = \mu_0 \gamma^{1/\alpha}$, $\mu_0 \in (0,1)$ and $\alpha \leq 1$.

Proof. Assume that $x \in \Omega$, from (13), (16) and (17), we obtain

$$\left(r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} \right)' \leq -\frac{\alpha \beta_0^{\alpha}}{r^{1/\alpha}(\ell) R_0^{1+\alpha}(\ell)} \left(\mathcal{B}^{(n-2)}(h(\ell)) \right)^{\alpha}. \tag{23}$$

By integrating (23) from ℓ_1 to ℓ and using the fact $\mathcal{B}^{(n-1)}(\ell) < 0$, we have

$$\begin{aligned} r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} &\leq r(\ell_1) \left(\mathcal{B}^{(n-1)}(\ell_1) \right)^{\alpha} - \alpha \beta_0^{\alpha} \int_{\ell_1}^{\ell} \frac{1}{r^{1/\alpha}(\mathfrak{a}) R_0^{1+\alpha}(\mathfrak{a})} \left(\mathcal{B}^{(n-2)}(h(\mathfrak{a})) \right)^{\alpha} \mathrm{d}\mathfrak{a} \\ &\leq r(\ell_1) \left(\mathcal{B}^{(n-1)}(\ell_1) \right)^{\alpha} - \alpha \beta_0^{\alpha} \left(\mathcal{B}^{(n-2)}(\ell) \right)^{\alpha} \int_{\ell_1}^{\ell} \frac{1}{r^{1/\alpha}(\mathfrak{a}) R_0^{1+\alpha}(\mathfrak{a})} \mathrm{d}\mathfrak{a} \\ &\leq r(\ell_1) \left(\mathcal{B}^{(n-1)}(\ell_1) \right)^{\alpha} + \frac{\beta_0^{\alpha}}{R_0^{\alpha}(\ell_1)} \left(\mathcal{B}^{(n-2)}(\ell) \right)^{\alpha} - \frac{\beta_0^{\alpha}}{R_0^{\alpha}(\ell)} \left(\mathcal{B}^{(n-2)}(\ell) \right)^{\alpha}. \end{aligned}$$

Since $\mathcal{B}^{(n-2)}(\ell) \to 0$ as $\ell \to \infty$ there is a $\ell_2 \geq \ell_1$ such that

$$r(\ell_1) \left(\mathcal{B}^{(n-1)}(\ell_1) \right)^{\alpha} + \frac{\beta_0^{\alpha}}{R_0^{\alpha}(\ell_1)} \left(\mathcal{B}^{(n-2)}(\ell) \right)^{\alpha} \leq 0,$$

for $\ell \geq \ell_2$. Therefore, we get

$$r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} \leq -\frac{\beta_0^{\alpha}}{R_0^{\alpha}(\ell)} \left(\mathcal{B}^{(n-2)}(\ell) \right)^{\alpha},$$

and so

$$r^{1/\alpha}(\ell) \mathcal{B}^{(n-1)}(\ell) R_0(\ell) + \beta_0 \mathcal{B}^{(n-2)}(\ell) \leq 0, \tag{24}$$

then

$$\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)} \right)' = \frac{R_0(\ell) r^{1/\alpha}(\ell) \mathcal{B}^{(n-1)}(\ell) + \beta_0 \mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell) R_0^{1+\beta_0}(\ell)} \leq 0.$$

Now, from (13), (16), (17) and (24), we obtain

$$\begin{aligned}\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)' &\leq -\left(\frac{\mu_0}{(n-2)!}h^{n-2}(\ell)\right)^{\alpha}Q(\ell)\left(\mathcal{B}^{(n-2)}(h(\ell))\right)^{\alpha} \\ &\leq -\alpha\beta_0^{\alpha}\frac{1}{r^{1/\alpha}(\ell)R_0^{1+\alpha}(\ell)}\left(\mathcal{B}^{(n-2)}(h(\ell))\right)^{\alpha} \end{aligned} \quad (25)$$

and

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell) \leq -\beta_0\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)},$$

and so

$$\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)^{1-\alpha} \geq \left(\beta_0\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha}, \quad (26)$$

Now, we find

$$\begin{aligned}&\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)+\mathcal{B}^{(n-2)}(\ell)\right)' \\ &= \left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)'R_0(\ell)-\mathcal{B}^{(n-1)}(\ell)+\mathcal{B}^{(n-1)}(\ell) \\ &= \left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)'R_0(\ell) \\ &= \frac{1}{\alpha}\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)'\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)^{1-\alpha}R_0(\ell),\end{aligned}$$

from (25) and (26), we get

$$\begin{aligned}\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)+\mathcal{B}^{(n-2)}(\ell)\right)' &\leq -\beta_0^{\alpha}\frac{\left(\mathcal{B}^{(n-2)}(h(\ell))\right)^{\alpha}}{r^{1/\alpha}(\ell)R_0^{1+\alpha}(\ell)}\left(\beta_0\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha}R_0(\ell) \\ &\leq -\beta_0^{\alpha}\frac{\left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha}}{r^{1/\alpha}(\ell)R_0^{\alpha}(\ell)}\left(\beta_0\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha} \\ &\leq \frac{-\beta_0}{r^{1/\alpha}(\ell)R_0(\ell)}\mathcal{B}^{(n-2)}(\ell).\end{aligned}$$

Integrating the last inequality from ℓ to ∞ and using (14), we obtain

$$-r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)-\mathcal{B}^{(n-2)}(\ell) \leq -\beta_0\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})R_0(\mathfrak{a})}\mathcal{B}^{(n-2)}(\mathfrak{a})\mathrm{d}\mathfrak{a},$$

and so

$$\begin{aligned}r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)+\mathcal{B}^{(n-2)}(\ell) &\geq \beta_0\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})R_0(\mathfrak{a})}\mathcal{B}^{(n-2)}(\mathfrak{a})\mathrm{d}\mathfrak{a} \\ &\geq \beta_0\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a} \\ &\geq \beta_0\mathcal{B}^{(n-2)}(\ell),\end{aligned}$$

which means that

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)+(1-\beta_0)\mathcal{B}^{(n-2)}(\ell) \geq 0.$$

Then

$$\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{1-\beta_0}(\ell)}\right)' = \frac{R_0(\ell)r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)+(1-\beta_0)\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{2-\beta_0}(\ell)} \geq 0. \quad (27)$$

The proof of the lemma is complete. □

Lemma 8. *Assume that $x \in \Omega$, and (16) holds, then*

$$\lim_{\ell\to\infty} \mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell) = 0.$$

Proof. Since $\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell)$ is a positive decreasing function, $\lim_{\ell\to\infty} \mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell) = c_2 \geq 0$. We claim that $c_2 = 0$. If not, then $\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell) \geq c_2 > 0$ eventually. Now, we introduce the function

$$w(\ell) = \frac{\mathcal{B}^{(n-2)}(\ell) + R_0(\ell)r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)}.$$

From (16), we note that $w(\ell) > 0$ and

$$\begin{aligned} w'(\ell) &= \frac{\mathcal{B}^{(n-1)}(\ell) + R_0(\ell)\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)' - \mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)} \\ &\quad + \beta_0 \frac{\mathcal{B}^{(n-2)}(\ell) + R_0(\ell)r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} \\ &= \frac{\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)'}{R_0^{\beta_0-1}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)} \\ &= \frac{1}{\alpha} \frac{\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)' \left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)^{1-\alpha}}{R_0^{\beta_0-1}(\ell)} \\ &\quad + \beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)}. \end{aligned}$$

using (25) and (26), we have

$$\begin{aligned} w'(\ell) &\leq -\frac{\beta_0^{\alpha}}{R_0^{\beta_0-1}(\ell)} \frac{1}{r^{1/\alpha}(\ell)R_0^{1+\alpha}(\ell)} \left(\mathcal{B}^{(n-2)}(h(\ell))\right)^{\alpha} \left(\beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha} \\ &\quad + \beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)}. \end{aligned}$$

Since $\mathcal{B}^{(n-1)}(\ell) < 0$, $h(\ell) \leq \ell$, we find $\mathcal{B}^{(n-2)}(h(\ell)) \geq \mathcal{B}^{(n-2)}(\ell)$, and then

$$\begin{aligned} w'(\ell) &\leq -\frac{\beta_0^{\alpha}}{R_0^{\beta_0-1}(\ell)} \frac{1}{r^{1/\alpha}(\ell)R_0^{1+\alpha}(\ell)} \left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha} \left(\beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha} \\ &\quad + \beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)} \\ &\leq -\beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_0}(\ell)} + \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)} \\ &\leq \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)}. \end{aligned}$$

Since $\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell) \geq c_2$, and (24) holds, we obtain

$$w'(\ell) \leq \beta_0 \frac{\mathcal{B}^{(n-1)}(\ell)}{R_0^{\beta_0}(\ell)} \leq -\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)} \frac{\beta_0^2}{r^{1/\alpha}(\ell) R_0(\ell)}$$
$$\leq \frac{-c_2 \beta_0^2}{r^{1/\alpha}(\ell) R_0(\ell)} < 0. \tag{28}$$

The function $w(\ell)$ converges to a non-negative constant because it is a positive decreasing function. Integrating (28) from ℓ_3 to ∞, we have

$$-w(\ell_3) \leq -\beta_0^2 c_2 \lim_{\ell \to \infty} \ln \frac{R_0(\ell_3)}{R_0(\ell)},$$

and so

$$w(\ell_3) \geq \beta_0^2 c_2 \lim_{\ell \to \infty} \ln \frac{R_0(\ell_3)}{R_0(\ell)} \to \infty,$$

which is a contradiction and we get that $c_2 = 0$. The proof of the lemma is complete. □

If $\beta_0 \leq 1/2$, we can improve the properties in Lemma 7, as in the following lemma.

Lemma 9. *Assume that $x \in \Omega$, and (16) holds. If*

$$\lim_{\ell \to \infty} \frac{R_0(h(\ell))}{R_0(\ell)} = \delta < \infty, \tag{29}$$

and there exists an increasing sequence $\{\beta_r\}_{r=1}^{m}$ defined as

$$\beta_r := \beta_0 \frac{\delta^{\beta_{r-1}}}{(1 - \beta_{r-1})^{1/\alpha}},$$

with $\alpha \leq 1$, $\beta_0 = \mu_0 \gamma^{1/\alpha}$, $\beta_{m-1} \leq 1/2$ and β_m, $\mu_0 \in (0,1)$, then,

$$\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_m}(\ell) \text{ is decreasing.} \tag{30}$$

Proof. Since $x \in \Omega$, from Lemma 7, we have that (21) and (22) hold.
Now, assume that $\beta_0 \leq 1/2$ and

$$\beta_1 := \beta_0 \frac{\delta^{\beta_0}}{(1 - \beta_0)^{1/\alpha}}.$$

Next, we will prove (30) at $m = 1$. As in the proof of Lemma 7 we find

$$\left(r(\ell) \left(\mathcal{B}^{(n-1)}(\ell) \right)^{\alpha} \right)' \leq -\alpha \beta_0^{\alpha} \frac{1}{r^{1/\alpha}(\ell) R_0^{1+\alpha}(\ell)} \left(\mathcal{B}^{(n-2)}(h(\ell)) \right)^{\alpha}. \tag{31}$$

Integrating (31) from ℓ_1 to ℓ, and using (21) and (29), we have

$$
\begin{aligned}
& r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha} \\
\leq & r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}-\alpha\beta_0^{\alpha}\int_{\ell_1}^{\ell}\frac{\left(\mathcal{B}^{(n-2)}(h(\mathfrak{a}))\right)^{\alpha}}{r^{1/\alpha}(\mathfrak{a})R_0^{1+\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a} \\
\leq & r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}-\alpha\beta_0^{\alpha}\int_{\ell_1}^{\ell}\frac{R_0^{\alpha\beta_0}(h(\mathfrak{a}))}{r^{1/\alpha}(\mathfrak{a})R_0^{1+\alpha}(\mathfrak{a})}\left(\frac{\mathcal{B}^{(n-2)}(\mathfrak{a})}{R_0^{\beta_0}(\mathfrak{a})}\right)^{\alpha}\mathrm{d}\mathfrak{a} \\
\leq & r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}-\alpha\beta_0^{\alpha}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)}\right)^{\alpha}\int_{\ell_1}^{\ell}\frac{R_0^{-1-\alpha+\alpha\beta_0}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})}\frac{R_0^{\alpha\beta_0}(h(\mathfrak{a}))}{R_0^{\alpha\beta_0}(\mathfrak{a})}\mathrm{d}\mathfrak{a} \\
\leq & r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}-\alpha\beta_0^{\alpha}\delta^{\alpha\beta_0}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)}\right)^{\alpha}\int_{\ell_1}^{\ell}\frac{R_0^{-1-\alpha+\alpha\beta_0}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a} \\
\leq & r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}-\frac{\beta_0^{\alpha}\delta^{\alpha\beta_0}}{1-\beta_0}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)}\right)^{\alpha}\left(\frac{1}{R_0^{\alpha(1-\beta_0)}(\ell)}-\frac{1}{R_0^{\alpha(1-\beta_0)}(\ell_1)}\right) \\
\leq & r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}+\beta_1^{\alpha}\frac{1}{R_0^{\alpha(1-\beta_0)}(\ell_1)}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)}\right)^{\alpha}-\beta_1^{\alpha}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{\alpha}.
\end{aligned}
$$

Since $\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell)\to 0$ as $\ell\to\infty$, we get

$$r(\ell_1)\left(\mathcal{B}^{(n-1)}(\ell_1)\right)^{\alpha}+\beta_1^{\alpha}\frac{1}{R_0^{\alpha(1-\beta_0)}(\ell_1)}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_0}(\ell)}\right)^{\alpha}\leq 0.$$

Hence, we have

$$r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\leq-\beta_1^{\alpha}\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{\alpha},$$

and so

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)+\beta_1\mathcal{B}^{(n-2)}(\ell)\leq 0,$$

then

$$\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{\beta_1}(\ell)}\right)'=\frac{R_0(\ell)r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)+\beta_1\mathcal{B}^{(n-2)}(\ell)}{r^{1/\alpha}(\ell)R_0^{1+\beta_1}(\ell)}\leq 0.$$

By repeating the same approach used previously, we can prove that

$$\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0^{1-\beta_1}(\ell)}\right)'\geq 0.$$

Similarly, if $\beta_{k-1}<\beta_k\leq 1/2$, then we can prove

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell)+\beta_k\mathcal{B}^{(n-2)}(\ell)\leq 0, \tag{32}$$

for $k=2,3,\ldots,m$. The proof of the lemma is complete. □

Lemma 10. *Assume that x is a positive solution of (8) and $\mathcal{B}$ satisfies $\mathbf{N}_3$. Then*

$$\left(\frac{\mathcal{B}(\ell)}{R_{n-2}(\ell)}\right)'\geq 0. \tag{33}$$

Proof. Assume that x is a positive solution of (8) and $\mathcal{B}$ satisfies $\mathbf{N}_3$. From (8), we find $r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}$ is decreasing, and so

$$\begin{aligned} r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a} &\geq \int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})}r^{1/\alpha}(\mathfrak{a})\mathcal{B}^{(n-1)}(\mathfrak{a})\mathrm{d}\mathfrak{a} \\ &= \lim_{\ell\to\infty}\mathcal{B}^{(n-2)}(\ell) - \mathcal{B}^{(n-2)}(\ell). \end{aligned} \tag{34}$$

Since $\mathcal{B}^{(n-2)}(\ell)$ is a positive decreasing function, we have that $\mathcal{B}^{(n-2)}(\ell)$ converges to a nonnegative constant when $\ell \to \infty$. Thus, (34) becomes

$$-\mathcal{B}^{(n-2)}(\ell) \leq r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell), \tag{35}$$

from (35), we get

$$\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)' = \frac{\left(r^{1/\alpha}(\ell)R_0(\ell)\mathcal{B}^{(n-1)}(\ell) + \mathcal{B}^{(n-2)}(\ell)\right)}{r^{1/\alpha}(\ell)R_0^2(\ell)} \geq 0,$$

which leads to

$$\begin{aligned} -\mathcal{B}^{(n-3)}(\ell) &\geq \int_{\ell}^{\infty}\frac{\mathcal{B}^{(n-2)}(\mathfrak{a})}{R_0(\mathfrak{a})}R_0(\mathfrak{a})\mathrm{d}\mathfrak{a} \geq \frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\int_{\ell}^{\infty}R_0(\mathfrak{a})\mathrm{d}\mathfrak{a} \\ &= \frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}R_1(\mathfrak{a}). \end{aligned}$$

This implies

$$\left(\frac{\mathcal{B}^{(n-3)}(\ell)}{R_1(\ell)}\right)' = \frac{R_1(\ell)\mathcal{B}^{(n-2)}(\ell) + \mathcal{B}^{(n-3)}(\ell)R_0(\ell)}{R_1^2(\ell)} \leq 0.$$

Similarly, we repeat the same previous process $(n-4)$ times, we have

$$\left(\frac{\mathcal{B}'(\ell)}{R_{n-3}(\ell)}\right)' \leq 0.$$

Now,

$$\begin{aligned} -\mathcal{B}(\ell) &\leq \int_{\ell}^{\infty}\frac{\mathcal{B}'(\mathfrak{a})}{R_{n-3}(\mathfrak{a})}R_{n-3}(\mathfrak{a})\mathrm{d}\mathfrak{a} \leq \frac{\mathcal{B}'(\ell)}{R_{n-3}(\ell)}\int_{\ell}^{\infty}R_{n-3}(\mathfrak{a})\mathrm{d}\mathfrak{a} \\ &= \frac{\mathcal{B}'(\ell)}{R_{n-3}(\ell)}R_{n-2}(\ell). \end{aligned}$$

This implies

$$\left(\frac{\mathcal{B}(\ell)}{R_{n-2}(\ell)}\right)' = \frac{R_{n-2}(\ell)\mathcal{B}'(\ell) + \mathcal{B}(\ell)R_{n-3}(\ell)}{R_{n-2}^2(\ell)} \geq 0.$$

The proof of the lemma is complete. □

4. Main Results

In the following theorems, we prove that there are no positive solutions that satisfy case $\mathbf{N}_2$.

Theorem 7. *Assume that (16) holds. If*

$$\beta_0 > 1/2, \tag{36}$$

for some $\mu_0 \in (0,1)$, then the class Ω is empty, where β_0 is defined as in Lemma 7.

Proof. Assume the contrary that $x \in \Omega$. From Lemma 7, we have that the functions $\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_0}(\ell)$ and $\mathcal{B}^{(n-2)}(\ell)/R_0^{1-\beta_0}(\ell)$ are decreasing and increasing for $\ell \geq \ell_1$, respectively. In another meaning, we have

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) + \beta_0\mathcal{B}^{(n-2)}(\ell) \leq 0 \tag{37}$$

and

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) + (1-\beta_0)\mathcal{B}^{(n-2)}(\ell) \geq 0. \tag{38}$$

from (37) and (38), we get

$$\begin{aligned} 0 &\leq r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) + (1-\beta_0)\mathcal{B}^{(n-2)}(\ell) \\ &= r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) + \beta_0\mathcal{B}^{(n-2)}(\ell) + \mathcal{B}^{(n-2)}(\ell) - 2\beta_0\mathcal{B}^{(n-2)}(\ell) \\ &\leq (1-2\beta_0)\mathcal{B}^{(n-2)}(\ell). \end{aligned}$$

Since $\mathcal{B}^{(n-2)}(\ell) > 0$, must be $1 - 2\beta_0 \geq 0$, which measn that

$$\beta_0 \leq 1/2,$$

a contradiction. The proof of the theorem is complete. □

Theorem 8. *Assume that (16) and (29) hold. If there exists a positive integer number m such that*

$$w'(\ell) + \frac{1}{\alpha}\frac{\mu_0^\alpha\beta_m^{1-\alpha}}{((n-2)!)^\alpha(1-\beta_m)}\frac{R_0(\ell)}{R_0^{1-\alpha}(h(\ell))}\left(h^{n-2}(\ell)\right)^\alpha Q(\ell)w(h(\ell)) = 0, \tag{39}$$

then the class Ω is empty, where $\alpha \leq 1$ and β_m is defined as in Lemma 9.

Proof. Assume the contrary, that $x \in \Omega$. From Lemma 9, we have that (30) holds.

Now, we define the function

$$w(\ell) = r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) + \mathcal{B}^{(n-2)}(\ell).$$

It follows from (14) that $w(\ell) > 0$ for $\ell \geq \ell_1$. From (30), we obtain

$$r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) \leq -\beta_m\mathcal{B}^{(n-2)}(\ell).$$

Then, from the definition of $w(\ell)$, we find

$$\begin{aligned} w(\ell) &= r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell) + \beta_m\mathcal{B}^{(n-2)}(\ell) - \beta_m\mathcal{B}^{(n-2)}(\ell) + \mathcal{B}^{(n-2)}(\ell) \\ &\leq (1-\beta_m)\mathcal{B}^{(n-2)}(\ell). \end{aligned} \tag{40}$$

From (17) and (13), we get

$$\begin{aligned} w'(\ell) &= \left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)'R_0(\ell) \leq \frac{1}{\alpha}\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^\alpha\right)'\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)^{1-\alpha}R_0(\ell) \\ &\leq -\frac{1}{\alpha}Q(\ell)\mathcal{B}^\alpha(h(\ell))\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right)^{1-\alpha}R_0(\ell) \\ &\leq -\frac{1}{\alpha}Q(\ell)\mathcal{B}^\alpha(h(\ell))\left(\beta_m\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha}R_0(\ell) \\ &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}Q(\ell)R_0(\ell)\mathcal{B}^\alpha(h(\ell))\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha} \\ &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}Q(\ell)R_0(\ell)\left(\frac{\mu_0}{(n-2)!}h^{n-2}(\ell)\right)^\alpha\left(\mathcal{B}^{(n-2)}(h(\ell))\right)^\alpha\left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha}, \end{aligned}$$

from (15), we note that $\mathcal{B}^{(n-2)}(\ell)/R_0(\ell)$ is increasing, then

$$\frac{\mathcal{B}^{(n-2)}(h(\ell))}{R_0(h(\ell))} \leq \frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}$$

and

$$\left(\frac{\mathcal{B}^{(n-2)}(h(\ell))}{R_0(h(\ell))}\right)^{1-\alpha} \leq \left(\frac{\mathcal{B}^{(n-2)}(\ell)}{R_0(\ell)}\right)^{1-\alpha},$$

then, we have

$$\begin{aligned} w'(\ell) &\leq -\frac{1}{\alpha}\beta_m^{1-\alpha}Q(\ell)R_0(\ell)\left(\frac{\mu_0}{(n-2)!}h^{n-2}(\ell)\right)^{\alpha}\left(\mathcal{B}^{(n-2)}(h(\ell))\right)^{\alpha}\left(\frac{\mathcal{B}^{(n-2)}(h(\ell))}{R_0(h(\ell))}\right)^{1-\alpha} \\ &\leq -\frac{1}{\alpha}\frac{\beta_m^{1-\alpha}\mu_0^{\alpha}}{((n-2)!)^{\alpha}}Q(\ell)\frac{R_0(\ell)}{R_0^{1-\alpha}(h(\ell))}\left(h^{n-2}(\ell)\right)^{\alpha}\mathcal{B}^{(n-2)}(h(\ell)), \end{aligned}$$

which, from (40), gives

$$w'(\ell) + \frac{1}{\alpha}\frac{\mu_0^{\alpha}\beta_m^{1-\alpha}}{((n-2)!)^{\alpha}(1-\beta_m)}\frac{R_0(\ell)}{R_0^{1-\alpha}(h(\ell))}\left(h^{n-2}(\ell)\right)^{\alpha}Q(\ell)w(h(\ell)) \leq 0. \tag{41}$$

Hence, $w(\ell)$ is a positive solution of (41). Using [[23], Corollary 1], we see that (39) also has a positive solution, a contradiction. This contradiction completes the proof of the theorem. □

Corollary 1. *Assume that (16) and (29) hold. If*

$$\liminf_{\ell\to\infty}\int_{h(\ell)}^{\ell}\frac{1}{\alpha}\frac{R_0(\mathfrak{a})\left(h^{n-2}(\mathfrak{a})\right)^{\alpha}Q(\mathfrak{a})}{R_0^{1-\alpha}(h(\mathfrak{a}))}\mathrm{d}\mathfrak{a} > \frac{\beta_m^{\alpha-1}(1-\beta_m)((n-2)!)^{\alpha}}{\mathrm{e}}, \tag{42}$$

holds, then the class Ω is empty.

Theorem 9. *Assume that (16) and (29) hold. If*

$$\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left[\left(\frac{\lambda h^{n-2}(\mathfrak{a})}{(n-2)!}\right)^{\alpha}\frac{R_0^{\alpha\beta_m}(h(\mathfrak{a}))}{R_0^{-\alpha(1-\beta_m)}(\mathfrak{a})}Q(\mathfrak{a}) - \frac{\alpha^{\alpha+1}}{(1+\alpha)^{1+\alpha}}\frac{1}{R_0(\mathfrak{a})r^{1/\alpha}(\mathfrak{a})}\right]\mathrm{d}\mathfrak{a} = \infty, \tag{43}$$

holds for some constant $\lambda \in (0,1)$, then the class Ω is empty.

Proof. Assume the contrary that $x \in \Omega$. Define the function w by

$$w(\ell) = \frac{r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}}{\left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha}},\ \ell \geq \ell_1. \tag{44}$$

Then $w(\ell) < 0$ for $\ell \geq \ell_1$. Since $r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}$ is decreasing, we have

$$r^{1/\alpha}(\mathfrak{a})\mathcal{B}^{(n-1)}(\mathfrak{a}) \leq r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell),$$

for $\mathfrak{a} \geq \ell \geq \ell_1$. By dividing the last inequality by $r^{1/\alpha}(\mathfrak{a})$ and integrating it from ℓ to ∞, we obtain

$$0 \leq \mathcal{B}^{(n-2)}(\ell) + r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\int_{\ell}^{l}\frac{1}{r^{1/\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a},$$

and so

$$0 \leq \mathcal{B}^{(n-2)}(\ell) + r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell),$$

which produces

$$-\frac{r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)}{\mathcal{B}^{(n-2)}(\ell)}R_0(\ell) \leq 1.$$

Hence, from (44), we find

$$-w(\ell)R_0^{\alpha}(\ell) \leq 1. \tag{45}$$

From (44), we have

$$\begin{aligned} w'(\ell) &= \frac{\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)'}{\left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha}} - \alpha\frac{r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha+1}}{\left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha+1}} \\ &\leq -\frac{Q(\ell)\mathcal{B}^{\alpha}(h(\ell))}{\left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha}} - \alpha\frac{w^{(\alpha+1)/\alpha}}{r^{1/\alpha}(\ell)}. \end{aligned}$$

Using Lemma 1, we get

$$\mathcal{B}(h(\ell)) \geq \frac{\lambda}{(n-2)!}h^{n-2}(\ell)\mathcal{B}^{(n-2)}(h(\ell)),$$

for every $\lambda \in (0,1)$ and for all sufficiently large ℓ. Then,

$$w'(\ell) \leq -Q(\ell)\left(\frac{\lambda}{(n-2)!}h^{n-2}(\ell)\right)^{\alpha}\frac{\left(\mathcal{B}^{(n-2)}(h(\ell))\right)^{\alpha}}{\left(\mathcal{B}^{(n-2)}(\ell)\right)^{\alpha}} - \alpha\frac{w^{(\alpha+1)/\alpha}(\ell)}{r^{1/\alpha}(\ell)}.$$

Since $\mathcal{B}^{(n-2)}(\ell)/R_0^{\beta_m}(\ell)$ is decreasing, then

$$\mathcal{B}^{(n-2)}(\ell) \leq \frac{\mathcal{B}^{(n-2)}(h(\ell))}{R_0^{\beta_m}(h(\ell))}R_0^{\beta_m}(\ell), \tag{46}$$

for $h(\ell) \leq \ell$, thus

$$w'(\ell) \leq -Q(\ell)\frac{R_0^{\alpha\beta_m}(h(\ell))}{R_0^{\alpha\beta_m}(\ell)}\left(\frac{\lambda}{(n-2)!}h^{n-2}(\ell)\right)^{\alpha} - \alpha\frac{w^{(\alpha+1)/\alpha}(\ell)}{r^{1/\alpha}(\ell)}. \tag{47}$$

Multiplying (47) by $R_0^{\alpha}(\ell)$ and integrating it from ℓ_1 to ℓ, we obtain

$$\begin{aligned} &R_0^{\alpha}(\ell)w(\ell) - R_0^{\alpha}(\ell_1)w(\ell_1) + \alpha\int_{\ell_1}^{\ell}\frac{R_0^{\alpha-1}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})}w(\mathfrak{a})\mathrm{d}\mathfrak{a} \\ &+ \int_{\ell_1}^{\ell}Q(\mathfrak{a})\frac{R_0^{\alpha\beta_m}(h(\mathfrak{a}))}{R_0^{-\alpha(1-\beta_m)}(\mathfrak{a})}\left(\frac{\lambda h^{n-2}(\mathfrak{a})}{(n-2)!}\right)^{\alpha}\mathrm{d}\mathfrak{a} + \alpha\int_{\ell_1}^{\ell}\frac{w^{(\alpha+1)/\alpha}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})}R_0^{\alpha}(\mathfrak{a})\mathrm{d}\mathfrak{a} \leq 0. \end{aligned}$$

Using (11) with

$$\varrho_1 := \frac{R_0^{\alpha}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})},\ \varrho_2 := \frac{R_0^{\alpha-1}(\mathfrak{a})}{r^{1/\alpha}(\mathfrak{a})} \text{ and } u := -w(\mathfrak{a}),$$

we have

$$\begin{aligned} \int_{\ell_1}^{\ell}\left[\left(\frac{\lambda h^{n-2}(\mathfrak{a})}{(n-2)!}\right)^{\alpha}\frac{R_0^{\alpha\beta_m}(h(\mathfrak{a}))}{R_0^{-\alpha(1-\beta_m)}(\mathfrak{a})}Q(\mathfrak{a}) - \frac{\alpha^{\alpha+1}}{(1+\alpha)^{1+\alpha}}\frac{1}{R_0(\mathfrak{a})r^{1/\alpha}(\mathfrak{a})}\right]\mathrm{d}\mathfrak{a} \\ \leq R_0^{\alpha}(\ell_1)w(\ell_1) + 1, \end{aligned}$$

due to (45), which contradicts (43). This completes the proof of the theorem. □

In the following theorems, we establish new oscillation criteria for (8).

Theorem 10. *Let (16) and (29) hold. Assume that*

$$\liminf_{\ell\to\infty}\int_{h(\ell)}^{\ell} Q(\mathfrak{a})\frac{\left(h^{n-1}(\mathfrak{a})\right)^{\alpha}}{r(h(\mathfrak{a}))}\mathrm{d}\mathfrak{a} > \frac{((n-1)!)^{\alpha}}{\mathrm{e}}, \tag{48}$$

(43) and

$$\limsup_{\ell\to\infty}\int_{\ell_1}^{\ell}\left[Q^{*}(\mathfrak{a})R_{n-2}^{\alpha}(\mathfrak{a}) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{R_{n-3}(\mathfrak{a})}{R_{n-2}(\mathfrak{a})}\right]\mathrm{d}\mathfrak{a} = \infty, \tag{49}$$

hold for some constant $\lambda \in (0,1)$, then, every solution of (8) is oscillatory.

Proof. Assume that Equation (8) has a non-oscillatory solution x. Without loss of generality, we may assume that x is eventually positive. It follows from Equation (8) that there exist three possible cases as in Lemma 3.

Assume that $\mathbf{N}_1$ holds. Using Lemma 1, we have

$$\mathcal{B}(\ell) \geq \frac{\lambda\ell^{n-1}}{(n-1)!r^{1/\alpha}(\ell)}\left(r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\right), \tag{50}$$

for every $\lambda \in (0,1)$ and for all sufficiently large ℓ. Using (8) and (50), we obtain

$$\begin{aligned}\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)' &= -\sum_{i=1}^{J} q_i(\ell)x^{\alpha}(h_i(\ell)) \\ &\leq -Q(\ell)\mathcal{B}^{\alpha}(h(\ell)) \\ &\leq -Q(\ell)\frac{\lambda^{\alpha}\left(h^{n-1}(\ell)\right)^{\alpha}}{((n-1)!)^{\alpha}r(h(\ell))}r(h(\ell))\left(\mathcal{B}^{(n-1)}(h(\ell))\right)^{\alpha}.\end{aligned}$$

Letting $w(\ell) := r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}$, we find

$$w'(\ell) + Q(\ell)\frac{\lambda^{\alpha}\left(h^{n-1}(\ell)\right)^{\alpha}}{((n-1)!)^{\alpha}r(h(\ell))}w(h(\ell)) \leq 0. \tag{51}$$

This is a contradiction because condition (48) guarantees that (51) has no positive solution according to Theorem 2.1.1 in [24].

Assume that case $\mathbf{N}_2$ holds. The proof of the $\mathbf{N}_2$ is the same as that of Theorem 9.

Assume that $\mathbf{N}_3$ holds. Since $r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}$ is decreasing, we have

$$r^{1/\alpha}(\mathfrak{a})\mathcal{B}^{(n-1)}(\mathfrak{a}) \leq r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell),$$

for $\mathfrak{a} \geq \ell \geq \ell_1$. By dividing the last inequality by $r^{1/\alpha}(\mathfrak{a})$ and integrating it from ℓ to ∞, we have

$$0 \leq \mathcal{B}^{(n-2)}(\ell) + r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\int_{\ell}^{\infty}\frac{1}{r^{1/\alpha}(\mathfrak{a})}\mathrm{d}\mathfrak{a},$$

and so

$$0 \leq \mathcal{B}^{(n-2)}(\ell) + r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell),$$

which leads to

$$\mathcal{B}^{(n-2)}(\ell) \geq -r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_0(\ell). \tag{52}$$

Integrating (52) from ℓ to ∞ yields

$$\begin{aligned} -\mathcal{B}^{(n-3)}(\ell) &\geq -\int_{\ell}^{\infty} r^{1/\alpha}(\mathfrak{a})\mathcal{B}^{(n-1)}(\mathfrak{a})R_0(\mathfrak{a})\mathrm{d}\mathfrak{a} \geq -r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)\int_{\ell}^{\infty} R_0(\mathfrak{a})\mathrm{d}\mathfrak{a} \\ &\geq -r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_1(\ell). \end{aligned} \tag{53}$$

Similarly, Integrating (53) from ℓ to ∞ a total of $(n-4)$ times, we have

$$-\mathcal{B}'(\ell) \geq -r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_{n-3}(\ell). \tag{54}$$

Integrating (54) from ℓ to ∞ provides

$$\mathcal{B}(\ell) \geq -r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)R_{n-2}(\ell). \tag{55}$$

Now, define the function w by

$$w(\ell) = \frac{r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}}{\mathcal{B}^{\alpha}(\ell)},\ \ell \geq \ell_1. \tag{56}$$

Then $w(\ell) < 0$ for $\ell \leq \ell_1$. Differentiating (56), we obtain

$$w'(\ell) = \frac{\left(r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\right)'}{\mathcal{B}^{\alpha}(\ell)} - \alpha\frac{r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}\mathcal{B}'(\ell)}{\mathcal{B}^{\alpha+1}(\ell)}.$$

It follows from (8) and (56) that

$$w'(\ell) \leq -\frac{\sum_{i=1}^{J} q_i(\ell)x^{\alpha}(h_i(\ell))}{\mathcal{B}^{\alpha}(\ell)} - \alpha\frac{r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^{\alpha}}{\mathcal{B}^{\alpha}(\ell)}\frac{r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)}{\mathcal{B}(\ell)}R_{n-3}(\ell). \tag{57}$$

Since

$$x(\ell) = \mathcal{B}(\ell) - p(\ell)x(\vartheta(\ell)) \geq \mathcal{B}(\ell) - p(\ell)\mathcal{B}(\vartheta(\ell)), \tag{58}$$

from (33), we see that $\mathcal{B}(\ell)/R_{n-2}(\ell)$ is increasing, consequently

$$\frac{\mathcal{B}(\ell)}{R_{n-2}(\ell)} \geq \frac{\mathcal{B}(\vartheta(\ell))}{R_{n-2}(\vartheta(\ell))},$$

for $\vartheta(\ell) \leq \ell$. From (58), we have

$$x(\ell) \geq \left(1 - p(\ell)\frac{R_{n-2}(\vartheta(\ell))}{R_{n-2}(\ell)}\right)\mathcal{B}(\ell),$$

and

$$x(h_i(\ell)) \geq \left(1 - p(h_i(\ell))\frac{R_{n-2}(\vartheta(h_i(\ell)))}{R_{n-2}(h_i(\ell))}\right)\mathcal{B}(h_i(\ell))$$

also

$$\begin{aligned} \sum_{i=1}^{J} q_i(\ell)x^{\alpha}(h_i(\ell)) &\geq \sum_{i=1}^{J} q_i(\ell)\left(1 - p(h_i(\ell))\frac{R_{n-2}(\vartheta(h_i(\ell)))}{R_{n-2}(h_i(\ell))}\right)^{\alpha}\mathcal{B}^{\alpha}(h_i(\ell)) \\ &\geq \mathcal{B}^{\alpha}(h(\ell))\sum_{i=1}^{J} q_i(\ell)\left(1 - p(h_i(\ell))\frac{R_{n-2}(\vartheta(h_i(\ell)))}{R_{n-2}(h_i(\ell))}\right)^{\alpha} \\ &= Q^{*}(\ell)\mathcal{B}^{\alpha}(h(\ell)). \end{aligned}$$

Now, we see that (57) becomes

$$w'(\ell) \leq -Q^*(\ell)\frac{\mathcal{B}^\alpha(h(\ell))}{\mathcal{B}^\alpha(\ell)} - \alpha\frac{r(\ell)\left(\mathcal{B}^{(n-1)}(\ell)\right)^\alpha}{\mathcal{B}^\alpha(\ell)}\frac{r^{1/\alpha}(\ell)\mathcal{B}^{(n-1)}(\ell)}{\mathcal{B}(\ell)}R_{n-3}(\ell). \tag{59}$$

Multiplying (59) by $R^\alpha_{n-2}(\ell)$ and integrating it from ℓ_1 to ℓ, we have

$$R^\alpha_{n-2}(\ell)w(\ell) - R^\alpha_{n-2}(\ell_1)w(\ell_1) + \alpha\int_{\ell_1}^{\ell} R^{\alpha-1}_{n-2}(\mathfrak{a})R_{n-3}(\mathfrak{a})w(\mathfrak{a})\mathrm{d}\mathfrak{a}$$
$$+\int_{\ell_1}^{\ell} Q^*(\mathfrak{a})R^\alpha_{n-2}(\mathfrak{a})\mathrm{d}\mathfrak{a} + \alpha\int_{\ell_1}^{\ell} R_{n-3}(\mathfrak{a})R^\alpha_{n-2}(\mathfrak{a})w^{(\alpha+1)/\alpha}(\mathfrak{a})\mathrm{d}\mathfrak{a} \leq 0.$$

Using (11) with

$$\varrho_1 := R_{n-3}(\mathfrak{a})R^\alpha_{n-2}(\mathfrak{a}),\ \varrho_2 := R^{\alpha-1}_{n-2}(\mathfrak{a})R_{n-3}(\mathfrak{a}) \text{ and } u := -w(\mathfrak{a}),$$

we get

$$\int_{\ell_1}^{\ell}\left[Q^*(\mathfrak{a})R^\alpha_{n-2}(\mathfrak{a}) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}}\frac{R_{n-3}(\mathfrak{a})}{R_{n-2}(\mathfrak{a})}\right]\mathrm{d}\mathfrak{a} \leq R^\alpha_{n-2}(\ell_1)w(\ell_1) + 1,$$

due to (55), which contradicts (49). Therefore, every solution of (8) is oscillatory. □

Theorem 11. *Let (16) and (29) hold. Assume that (42), (48) and (49) hold for some constant $\lambda \in (0,1)$, then, every solution of (8) is oscillatory.*

Example 1. *Consider the NDE*

$$\left(\ell^{4\alpha}\left((x(\ell) + p_0x(\vartheta_0\ell))'''\right)^\alpha\right)' + \sum_{i=1}^{J} q_0\ell^{\alpha-1}x^\alpha(h_i\ell) = 0,\ \ell \geq 1, \tag{60}$$

where $0 \leq p_0 < 1$, $\vartheta_0, h_0 \in (0,1)$ and $q_0 > 0$. By comparing (8) and (60) we see that $n = 4$, $r(\ell) = \ell^{4\alpha}$, $q_i(\ell) = q_0\ell^{\alpha-1}$, $p(\ell) = p_0$, $\vartheta(\ell) = \vartheta_0\ell$, $h_i(\ell) = h_i\ell$. It is easy to find that

$$R_0(\ell) = \frac{1}{3\ell^3},\ R_1(\ell) = \frac{1}{6\ell^2},\ R_2(\ell) = \frac{1}{6\ell}$$

and

$$Q(\ell) = Jq_0\ell^{\alpha-1}(1-p_0)^\alpha.$$

For (16), we set

$$\gamma = \frac{J}{\alpha}\frac{h_0^{2\alpha}q_0}{2^\alpha 3^{\alpha+1}}(1-p_0)^\alpha,$$

where $h_0\ell = \min\{h_i\ell, i = 1,\ldots,J\}$. From (29), we get

$$\delta = \frac{1}{h_0^3}.$$

Now, we define the sequence $\{\beta_r\}_{r=1}^m$ as

$$\beta_r = \beta_0\frac{1}{(1-\beta_{r-1})^{1/\alpha}}\left(\frac{1}{h_0}\right)^{3\beta_{r-1}},$$

with

$$\beta_0 = \frac{J^{1/\alpha}\mu_0 q_0^{1/\alpha}}{6\alpha^{1/\alpha}3^{1/\alpha}}h_0^2(1-p_0).$$

Then, condition (36) reduces to

$$q_0 > \frac{3^{\alpha+1}\alpha}{\left(J\mu_0 h_0^2(1-p_0)\right)^{\alpha}}, \tag{61}$$

and condition (42) becomes

$$\begin{aligned}
&\liminf_{\ell\to\infty}\int_{h(\ell)}^{\ell}\frac{1}{\alpha}\frac{R_0(\ell)\left(h^{n-2}(\mathfrak{a})\right)^{\alpha}Q(\mathfrak{a})}{R_0^{1-\alpha}(h(\ell))}\mathrm{d}\mathfrak{a}\\
&=\liminf_{\ell\to\infty}\int_{h_0\ell}^{\ell}\frac{1}{\alpha}\frac{1}{3\mathfrak{a}^3}h_0^{2\alpha}\mathfrak{a}^{2\alpha}3^{1-\alpha}\mathfrak{a}^{3-3\alpha}h_0^{3-3\alpha}Jq_0\mathfrak{a}^{\alpha-1}(1-p_0)^{\alpha}\mathrm{d}\mathfrak{a}\\
&=\frac{1}{\alpha}\frac{J}{3^{\alpha}}h_0^{3-\alpha}q_0(1-p_0)^{\alpha}\ln\frac{1}{h_0},
\end{aligned}$$

which leads to

$$\frac{1}{\alpha}\frac{J}{6^{\alpha}}h_0^{3-\alpha}q_0(1-p_0)^{\alpha}\ln\frac{1}{h_0} > \frac{\beta_m^{\alpha-1}(1-\beta_m)}{\mathrm{e}}, \tag{62}$$

while condition (43) becomes

$$\begin{aligned}
&\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left[\left(\frac{\lambda h^{n-2}(\mathfrak{a})}{(n-2)!}\right)^{\alpha}\frac{R_0^{\alpha\beta_m}(h(\mathfrak{a}))}{R_0^{-\alpha(1-\beta_m)}(\mathfrak{a})}Q(\mathfrak{a})-\frac{\alpha^{\alpha+1}}{(1+\alpha)^{1+\alpha}}\frac{1}{R_0(\mathfrak{a})r^{1/\alpha}(\mathfrak{a})}\right]\mathrm{d}\mathfrak{a}\\
&=\limsup_{\ell\to\infty}\int_{\ell_0}^{\ell}\left[\frac{\lambda^{\alpha}}{2^{\alpha}}h_0^{2\alpha}\mathfrak{a}^{2\alpha}\frac{1}{3^{\alpha}\mathfrak{a}^{3\alpha}}\frac{1}{h_0^{3\alpha\beta_m}}Jq_0\mathfrak{a}^{\alpha-1}(1-p_0)^{\alpha}-\frac{\alpha^{\alpha+1}}{(1+\alpha)^{1+\alpha}}3\mathfrak{a}^3\frac{1}{\mathfrak{a}^4}\right]\mathrm{d}\mathfrak{a}\\
&=\left[\frac{\lambda^{\alpha}}{6^{\alpha}}\frac{J}{h_0^{3\alpha\beta_m-2\alpha}}q_0(1-p_0)^{\alpha}-\frac{3\alpha^{\alpha+1}}{(1+\alpha)^{1+\alpha}}\right]\limsup_{\ell\to\infty}\ln\frac{\ell}{\ell_0}=\infty,
\end{aligned}$$

which is achieved if

$$\frac{\lambda^{\alpha}}{6^{\alpha}}\frac{J}{h_0^{3\alpha\beta_m-2\alpha}}q_0(1-p_0)^{\alpha} > \frac{3\alpha^{\alpha+1}}{(1+\alpha)^{1+\alpha}}. \tag{63}$$

Using Theorem 7, Corollary 1 and Theorem 9, we note that the class Ω is empty if either (61), (62) or (63) holds, respectively.

Example 2. *Consider the NDE (60) where $\alpha = 1$, $p_0 = 1/2$, $2\vartheta_0 > 1$ and $J = 3$, then (60) becomes*

$$\left(\ell^4\left(\left(x(\ell)+\frac{1}{2}x(\vartheta_0\ell)\right)'''\right)\right)' + q_0(x(h_1u)+x(h_2u)+x(h_3u)) = 0,\ \ell \geq 1. \tag{64}$$

Clearly

$$h(\ell) = \min\{h_i(\ell),\ i = 1,2,3\} = h_0\ell$$

and

$$Q(\ell) = \frac{3}{2}q_0.$$

For (16), we set

$$\gamma = \frac{1}{12}h_0^2 q_0.$$

Form (29), we have $\delta = 1/h_0^3$. Now, we define the sequence $\{\beta_r\}_{r=1}^{m}$ as

$$\beta_r = \beta_0\frac{1}{(1-\beta_{r-1})^{1/\alpha}}\left(\frac{1}{h_0}\right)^{3\beta_{r-1}},$$

with

$$\beta_0 = \frac{1}{12}\mu_0 h_0^2 q_0.$$

Then, condition (36) reduces to

$$q_0 > \frac{6}{\mu_0 h_0^2}, \tag{65}$$

and condition (48) becomes

$$\begin{aligned} \liminf_{\ell\to\infty} \frac{1}{(n-1)!} \int_{h(\ell)}^{\ell} Q(\mathfrak{a}) \frac{h^{n-1}(\mathfrak{a})}{r(h(\mathfrak{a}))} \mathrm{d}\mathfrak{a} &= \liminf_{\ell\to\infty} \frac{1}{6} \int_{h_0\ell}^{\ell} \frac{3}{2} q_0 \frac{h_0^3 \mathfrak{a}^3}{h_0^4 \mathfrak{a}^4} \mathrm{d}\mathfrak{a} \\ &= \frac{1}{4} \frac{q_0}{h_0} \ln \frac{1}{h_0}, \end{aligned}$$

which leads to

$$\frac{1}{4} \frac{q_0}{h_0} \ln \frac{1}{h_0} > \frac{1}{\mathrm{e}}, \tag{66}$$

while condition (49) is abbreviated to

$$\begin{aligned} &\limsup_{\ell\to\infty} \int_{\ell_1}^{\ell} \left[Q^*(\mathfrak{a}) R_{n-2}^{\alpha}(\mathfrak{a}) - \frac{\alpha^{\alpha+1}}{(\alpha+1)^{\alpha+1}} \frac{R_{n-3}(\mathfrak{a})}{R_{n-2}(\mathfrak{a})} \right] \mathrm{d}\mathfrak{a} \\ &= \limsup_{\ell\to\infty} \int_{\ell_1}^{\ell} \left[3q_0 \left(1 - \frac{1}{2} \frac{1}{\vartheta_0}\right) \frac{1}{6} - \frac{1}{4} \right] \frac{1}{\mathfrak{a}} \mathrm{d}\mathfrak{a} \\ &= \left[\frac{1}{2} q_0 \left(1 - \frac{1}{2} \frac{1}{\vartheta_0}\right) - \frac{1}{4} \right] \limsup_{\ell\to\infty} \ln \frac{\ell}{\ell_1} = \infty, \end{aligned}$$

which is achieved when

$$q_0 \left(1 - \frac{1}{2} \frac{1}{\vartheta_0}\right) > \frac{1}{2}. \tag{67}$$

From Theorem 10 we see that every solution of (64) is oscillatory if (65), (66) and (67) holds.

5. Conclusions

In this paper, we have investigated the asymptotic properties of positive solutions of even-order neutral differential equations in the non-canonical case. We introduced several auxiliaries and important results on which our results depend. We used different techniques, including the Recati technique, and the comparison method to create the oscillation criteria for the studied equation. Finally, we provided some examples as special cases of the studied equation to illustrate the possibility of applying the results we obtained. Our obtained theorems not only generalize the existing results in the literature but also can be used to plan future research papers in a variety of directions. For example:

(1) One can consider Equation (8) with

$$\mathcal{B} := x + p_1 \cdot (x \circ \vartheta) + p_2 \cdot (x \circ \tau)$$

where $\tau(\ell) \leq \ell$.

(2) It would be of interest to extend the results of this paper for higher order equations of type (8), where $n \geq 3$ is an odd natural number.

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