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Projection Uniformity of Asymmetric Fractional Factorials

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Abstract: The objective of this paper is to study the issue of the projection uniformity of asymmetric fractional factorials. On the basis of level permutation and mixture discrepancy, the average projection mixture discrepancy to measure the uniformity for low-dimensional projection designs is defined, the uniformity pattern and minimum projection uniformity criterion are presented for evaluating and comparing any asymmetric factorials. Moreover, lower bounds to uniformity pattern have been obtained, and some illustrative examples are also provided.

Keywords: mixture discrepancy; generalized minimum aberration; uniformity pattern; lower bound

MSC: 62K15; 62K05; 62K99

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1. Introduction

Many criteria were proposed for comparing U-type designs, but none of these criteria can directly distinguish non-isomorphic saturated designs. A special criterion can measure all these subdesigns, and the related values are called its projection pattern. We can use the distribution or the vector of these projection values as a tool to distinguish the underlying designs. Ref. [1] firstly defined the projection discrepancy pattern and proposed the minimum projection uniformity (MPU) criterion, which is equivalent to generalized minimum aberration criterion (GMA [2]). Ref. [3] studied the projection discrepancies of two-level fractional factorials in terms of the centered L_2-discrepancy (CD [4]). Subsequently, ref. [5] discussed the relationships among criteria of MPU proposed in [1] and minimum generalized aberration [6]. Following this projection discrepancy, [7] studied the projection properties of two-level factorials in view of geometry and proposed the uniformity pattern and MPU criterion to assess and compare two-level factorials. The relations between MPU and minimum aberration, and GMA and orthogonality are clarified; this close relationship raises the hope of improving the connection between uniform design theory and factorial design theory.

Following the uniform pattern and MPU, projection uniformity of asymmetric design based on CD and wrap-around L_2-discrepancy (WD [8]) has been studied, respectively. As a measure of uniformity, CD does not have fewer cursed dimensions and WD is not sensitive to a shift for one or more dimensions, Mixture discrepancy (MD [9]) retains the good properties of CD and WD and overcomes the shortcomings of both. Aided by the level permutation technique in [10,11], ref. [12] obtained the relationship between the mean of mixture discrepancies and the generalized word–length pattern for multi-level designs. Ref. [13] defined the MPU criterion for two- and three-level factorials under MD. Refs. [14,15] generalize the findings in [13] to q-level and mixed two- and three-level factorials, respectively. Moreover, ref. [16] proposed the uniform projection design that have the smallest average CD values of all two-dimensional projections and are shown to have good-filling properties over all sub-spaces in terms of the distance, uniformity, and orthogonality. Based on the findings of [16], many applications and studies on uniform projection designs have emerged [17–22].

While the work of [13–15] discussed the projection uniformity for two-level, three-level, q-level, and mixed two- and three-level designs under MD, respectively, the present paper aims at obtaining further results. We extend the findings in [13–15] to general asymmetrical factorials. First, the uniformity pattern and MPU criterion are proposed for selecting asymmetrical designs. Second, we build some analytic linkages between uniformity pattern, orthogonality, and generalized word–length pattern. Third, we integrate two lower bound methods in [23], which can be served as a benchmark for searching MPU designs. Finally, the results of [13–15] can be used as our special cases, and some numerical examples are provided to illustrate our theoretical results.

This paper is organized as follows: Section 2 describes some notations and basic concepts such as distance distribution and generalized word–length pattern, which are useful throughout in this paper. Section 3 defines the average projection mixture discrepancy and related uniformity pattern, presents a statistical justification of MPU criterion, and establishes a connection between MPU and GMA. Section 4 provides a lower bound of the uniformity pattern. Some illustrative examples to verify our theoretical results are presented in Section 5.

2. Notations and Preliminaries

Consider a class of U-type designs, denoted by $\mathcal{U}(n;{q_1}^{s_1}\times {q_2}^{s_2})$, of mixed q_1- and q_2-level factorials in n runs and $s(=s_1+s_2)$ factors, where each factor of the first s_1 factors takes values from a set of $\{0,1,\ldots,q_1-1\}$ equally often and each factor of the last s_2 factors takes values from a set of $\{0,1,\ldots,q_2-1\}$ equally often. For any design $d\in\mathcal{U}(n;{q_1}^{s_1}\times {q_2}^{s_2})$, a typical treatment combination (or run) of design d is defined by $w=(w^{(1)},w^{(2)})$, where, for $i=1,2$, $w^{(i)}=(w_1^{(i)},\ldots,w_{s_i}^{(i)})$, $w_j^{(1)}\in\{0,1,\ldots,q_1-1\}$ and $w_j^{(2)}\in\{0,1,\ldots,q_2-1\}$. Denote $d=(d^{(1)},d^{(2)})$, where $w^{(i)}\in d^{(i)}$, $i=1,2$. If all the possible $q_1^{t_1}\times q_2^{t_2}$ level combinations corresponding to any $t(=t_1+t_2)$ columns of design d appear equally often, $0\le t_1\le s_1$, $0\le t_2\le s_2$, design d is called to be an orthogonal array of strength t and denoted by $OA(n;{q_1}^{s_1}\times {q_2}^{s_2},t)$.

For any design $d\in\mathcal{U}(n;{q_1}^{s_1}\times {q_2}^{s_2})$, its distance distribution is defined by

$$E_{j_1j_2}(d)=\frac{1}{n}\left|\{(i,k):H^{(1)}_{i_1k_1}=j_1,H^{(2)}_{i_2k_2}=j_2\}\right|,$$

where $|u|$ is the cardinality of the set $|u|$, H^t_{ik} is the Hamming distance between two runs i and k of design $d^{(t)}$, $t=1,2$, $0\le j_1\le s_1$, $0\le j_2\le s_2$.

The MacWilliams transforms of the $\{E_{j_1j_2}(d)\}$ of any design $d\in\mathcal{U}(n;{q_1}^{s_1}\times {q_2}^{s_2})$ are defined as

$$E'_{i_1i_2}(d)=\frac{1}{n}\sum_{j_1=0}^{s_1}\sum_{j_2=0}^{s_2}P_{i_1}(j_1;s_1,q_1)P_{i_2}(j_2;s_2,q_2)E_{j_1j_2}(d),\ i_1=0,\ldots,s_1,i_2=0,\ldots,s_2,$$

where $P_i(j;s,q)=\sum_{r=0}^{i}(-1)^r(q-1)^{i-r}\binom{j}{r}\binom{s-j}{i-r}$ is the Krawtchouk polynomial, $\binom{m}{k}=m(m-1)\cdots(m-k+1)/k!$ and $\binom{m}{k}=0$ for $m<k$.

Ref. [2] showed that the generalized word–length pattern is the MacWilliams transform of the distance distribution, that is,

$$A_i(d)=\sum_{i_1+i_2=i}E'_{i_1i_2}(d), \tag{1}$$

where the vector $(A_1(d),\ldots,A_s(d))$ is called the generalized word–length pattern. For any two designs d_1 and d_2 in $\mathcal{U}(n;{q_1}^{s_1}\times {q_2}^{s_2})$, d_1 is said to have less aberration than d_2 if there exists a positive integer $t\le s$, such that $A_t(d_1)<A_t(d_2)$ and $A_i(d_1)=A_i(d_2)$ for $i=1,\ldots,t-1$. The design d_1 has generalized minimum aberration if there is no other design with less aberration than d_1.

For any positive integer $g \leq s$, defined $C_g = \{(g_1, g_2) : g_1 = 0, \ldots, s_1, g_2 = 0, \ldots, s_2, g_1 + g_2 = g\}$, and for any $(g_1, g_2) \in C_g$, let $S_{g_1 g_2}$ be the set of all nonempty subsets of $\{1, \ldots, s\}$ with the first g_1 elements from $\{1, 2, \ldots, s_1\}$ and the next g_2 elements from $\{s_1 + 1, \ldots, s_1 + s_2\}$. For any g, $1 \leq g \leq s$, let S_g be the set of all nonempty subsets of $\{1, 2, \ldots, s\}$ with cardinality g, it is to be noted that $S_g = \bigcup_{(g_1, g_2) \in C_g} S_{g_1 g_2}$.

For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, define the nonempty set $u = u_1 \cup u_2 = \{u_{11}, \ldots, u_{1g_1}\} \cup \{u_{21}, \ldots, u_{2g_2}\} \in S_{g_1 g_2}$ and $g = g_1 + g_2$, let d_u be the corresponding projection design of d onto factors with indexes from u. A typical treatment combination of d_u is represented as $w_u = (w_u^{(1)}, w_u^{(2)})$, where $w_u^{(i)} = (w_{u_{i1}}^{(i)}, \ldots, w_{u_{ig_i}}^{(i)})$, $w_{u_{ig_i}}^{(i)} \in \{0, 1, \ldots, q_i - 1\}$, $i = 1, 2$. Let H_{ik}^u be the Hamming distance between two runs i^u and k^u of the projection design d_u, denote $\delta_{ik}^u = g - H_{ik}^u$ as the coincide number between two runs i^u and k^u, where $i^u = (i_1^u, i_2^u)$ and $k^u = (k_1^u, k_2^u)$.

3. Projection Uniformity of $\mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$

For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, $g (= g_1 + g_2) \leq s$ and $u \in S_g$, let $MD_u(d)$ be the mixture discrepancy value of the corresponding projection design d_u; following [9], we can derive the below formula for $MD_u(d)$,

$$[MD_u(d)]^2 = \left(\frac{7}{12}\right)^g - \frac{2}{n} \sum_{i=1}^{n} \prod_{j \in u} f_1(x_{ij}) + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{k=1}^{n} \prod_{j \in u} f(x_{ij}, x_{kj}), \tag{2}$$

where $f_1(x_{ij}) = \frac{2}{3} - \frac{1}{4}|x_{ij} - \frac{1}{2}| - \frac{1}{4}|x_{ij} - \frac{1}{2}|^2$, $f(x_{ij}, x_{kj}) = \frac{7}{8} - \frac{1}{4}|x_{ij} - \frac{1}{2}| - \frac{1}{4}|x_{kj} - \frac{1}{2}| - \frac{3}{4}|x_{ij} - x_{kj}| + \frac{1}{2}|x_{ij} - x_{kj}|^2$, $i, k = 1, \ldots, n$.

When considering all $q_1! \times q_2!$ possible level permutations for every factor of $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, there are $(q_1!)^{s_1} \times (q_2!)^{s_2}$ combinatorially isomorphic designs of d that can be obtained, and denote the set of these designs as $\mathcal{P}(d)$. Similarly, for any positive integer $g (= g_1 + g_2) \leq s$ and $u \in S_g$, we can obtain $(q_1!)^{g_1} \times (q_2!)^{g_2}$ combinatorially isomorphic designs of d_u; the corresponding set of these combinatorially isomorphic designs d'_u is denoted by $\mathcal{P}(d_u)$. The mean of projection mixture discrepancies of all the designs in $\mathcal{P}(d_u)$ is denoted by $AMD_u(d)$, that is,

$$AMD_u(d) = \frac{1}{(q_1!)^{g_1} (q_2!)^{g_2}} \sum_{d'_u \in \mathcal{P}(d_u)} [MD_u(d')]^2. \tag{3}$$

The following lemma, which can be proved similarly as [14,15], gives the expression for $AMD_u(d)$.

Lemma 1. *For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, $u \in S_g$ and $1 \leq g \leq s$,*
(i) when both q_1 and q_2 are even,

$$\begin{aligned} AMD_u(d) = & \left(\frac{7}{12}\right)^g - 2\left(\frac{28{q_1}^2 + 1}{48{q_1}^2}\right)^{g_1} \left(\frac{28{q_2}^2 + 1}{48{q_2}^2}\right)^{g_2} \\ & + \frac{1}{n} \left(\frac{3}{4}\right)^{g_1} \left(\frac{3}{4}\right)^{g_2} \sum_{i_1=0}^{g_1} \sum_{i_2=0}^{g_2} \left(\frac{7q_1 - 2}{9q_1}\right)^{i_1} \left(\frac{7q_2 - 2}{9q_2}\right)^{i_2} E_{i_1 i_2}(d_u); \end{aligned}$$

(ii) when both q_1 and q_2 are odd,

$$\begin{aligned} AMD_u(d) = & \left(\frac{7}{12}\right)^g - 2\left(\frac{7{q_1}^2 + 1}{12{q_1}^2}\right)^{g_1} \left(\frac{7{q_2}^2 + 1}{12{q_2}^2}\right)^{g_2} + \frac{1}{n} \left(\frac{6{q_1}^2 + 1}{8{q_1}^2}\right)^{g_1} \left(\frac{6{q_2}^2 + 1}{8{q_2}^2}\right)^{g_2} \\ & \times \sum_{i_1=0}^{g_1} \sum_{i_2=0}^{g_2} \left(\frac{14{q_1}^2 - 4q_1 + 3}{18{q_1}^2 + 3}\right)^{i_1} \left(\frac{14{q_2}^2 - 4q_2 + 3}{18{q_2}^2 + 3}\right)^{i_2} E_{i_1 i_2}(d_u); \end{aligned}$$

(*iii*) *when* q_1 *is even and* q_2 *is odd,*

$$AMD_u(d) = \left(\frac{7}{12}\right)^g - 2\left(\frac{28{q_1}^2+1}{48{q_1}^2}\right)^{g_1}\left(\frac{7{q_2}^2+1}{12{q_2}^2}\right)^{g_2} + \frac{1}{n}\left(\frac{3}{4}\right)^{g_1}\left(\frac{6{q_2}^2+1}{8{q_2}^2}\right)^{g_2} \times \sum_{i_1=0}^{g_1}\sum_{i_2=0}^{g_2}\left(\frac{7q_1-2}{9q_1}\right)^{i_1}\left(\frac{14{q_2}^2-4q_2+3}{18{q_2}^2+3}\right)^{i_2} E_{i_1 i_2}(d_u). \quad (4)$$

We can obtain the following lemma when the design d is an orthogonal array $OA(n;{q_1}^{s_1} \times {q_2}^{s_2}, t)$.

Lemma 2. *Suppose design* d *is an orthogonal array* $OA(n;{q_1}^{s_1} \times {q_2}^{s_2}, t)$*, then*

$$AMD_u(d) = \Phi_u,$$

where $|u| = g_1 + g_2$, $1 \le g_1 + g_2 \le t$, Φ_u *is a constant only depending on* q_1, q_2, g_1 *and* g_2*. In particular,*

(*i*) *when both* q_1 *and* q_2 *are even,*

$$\Phi_u = \left(\frac{7}{12}\right)^g - 2\left(\frac{28{q_1}^2+1}{48{q_1}^2}\right)^{g_1}\left(\frac{28{q_2}^2+1}{48{q_2}^2}\right)^{g_2} + \left(\frac{7{q_1}^2+2}{12{q_1}^2}\right)^{g_1}\left(\frac{7{q_2}^2+2}{12{q_2}^2}\right)^{g_2};$$

(*ii*) *when both* q_1 *and* q_2 *are odd,*

$$\Phi_u = \left(\frac{7}{12}\right)^g - 2\left(\frac{7{q_1}^2+1}{12{q_1}^2}\right)^{g_1}\left(\frac{7{q_2}^2+1}{12{q_2}^2}\right)^{g_2} + \left(\frac{14{q_1}^2+7}{24{q_1}^2}\right)^{g_1}\left(\frac{14{q_2}^2+7}{24{q_2}^2}\right)^{g_2};$$

(*iii*) *when* q_1 *is even and* q_2 *is odd,*

$$\Phi_u = \left(\frac{7}{12}\right)^g - 2\left(\frac{28{q_1}^2+1}{48{q_1}^2}\right)^{g_1}\left(\frac{7{q_2}^2+1}{12{q_2}^2}\right)^{g_2} + \left(\frac{7{q_1}^2+2}{12{q_1}^2}\right)^{g_1}\left(\frac{14{q_2}^2+7}{24{q_2}^2}\right)^{g_2}.$$

It is well known that strength is an important measure of orthogonality. For comparing the difference between design $d \in \mathcal{U}(n;{q_1}^{s_1} \times {q_2}^{s_2})$ and orthogonal array $OA(n;{q_1}^{s_1} \times {q_2}^{s_2}, t)$ of strength t, the definition of uniformity pattern of design d is given as follows, which provides a measure of the projection uniformity of d onto different dimensions.

Definition 1. *For any design* $d \in \mathcal{U}(n;{q_1}^{s_1} \times {q_2}^{s_2})$*, any positive integer* $g(= g_1 + g_2) \le s$ *and* $u \in S_g$*, define*

$$MI_g(d) = \sum_{|u|=g} [AMD_u(d) - \Phi_u],$$

where Φ_u *is shown in Lemma 2. The vector* $(MI_1(d), \ldots, MI_s(d))$ *is called the uniformity pattern of design* d*.*

We now state the above discussion as the following theorem, which gives a relationship between the uniformity pattern $(MI_1(d), \ldots, MI_s(d))$ of design d and the strength t of orthogonal array $OA(n;{q_1}^{s_1} \times {q_2}^{s_2}, t)$.

Theorem 1. *For any design* $d \in \mathcal{U}(n;{q_1}^{s_1} \times {q_2}^{s_2})$*, design* d *is an orthogonal array* $OA(n;{q_1}^{s_1} \times {q_2}^{s_2}, t)$ *if and only if* $MI_k(d) = 0$ *for* $k = 1, \ldots, t$ *and* $MI_{t+1}(d) \neq 0$*.*

Theorem 1 indicates that there is a close relationship between $MI_t(d)$ and strength t for a design $d \in \mathcal{U}(n;{q_1}^{s_1} \times {q_2}^{s_2})$, that is, the smaller the value of $MI_t(d)$, the design d will be closer to an orthogonal array of strength t. Based on Theorem 1, $\{MI_k(d)\}$ may be used as a measure for evaluating designs; it suggests to define some similar criteria, such as MPU.

Definition 2. *For two designs $d_1, d_2 \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, there is an integer t such that $MI_t(d_1) \neq MI_t(d_2)$ and $MI_k(d_1) = MI_k(d_2)$ for $k = 1, \ldots, t-1$; then, d_1 is said to have less MPU than d_2. If there is no other design in $\mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ that has less MPU than d_1, then d_1 is said to have MPU, or d_1 is an MPU design.*

Here, we mainly establish the connections between projection uniformity and orthogonality, and some relationships between criteria of MPU and GMA will also be included.

Theorem 2. *For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, any positive integer $g(= g_1 + g_2) \leq s$ and $u \in S_g$, we have*

$$MI_g(d) = \sum_{|u|=g} \alpha_{g_1 g_2} \sum_{(r_1, r_2) \in \mathcal{R}} \beta_{r_1 r_2} \binom{s_1 - r_1}{s_1 - g_1} \binom{s_2 - r_2}{s_2 - g_2} A_{r_1 + r_2}(d),$$

where $\mathcal{R} = \{(r_1, r_2) : r_1 = 0, \ldots, g_1, r_2 = 0, \ldots, g_2, (r_1, r_2) \neq (0, 0)\}$, and

(i) when both q_1 and q_2 are even,

$$\alpha_{g_1 g_2} = \left(\frac{7{q_1}^2 + 2}{12{q_1}^2}\right)^{g_1} \left(\frac{7{q_2}^2 + 2}{12{q_2}^2}\right)^{g_2}, \beta_{r_1 r_2} = \left(\frac{2q_1 + 2}{7{q_1}^2 + 2}\right)^{r_1} \left(\frac{2q_2 + 2}{7{q_2}^2 + 2}\right)^{r_2};$$

(ii) when both q_1 and q_2 are odd,

$$\alpha_{g_1 g_2} = \left(\frac{14{q_1}^2 + 7}{24{q_1}^2}\right)^{g_1} \left(\frac{14{q_2}^2 + 7}{24{q_2}^2}\right)^{g_2}, \beta_{r_1 r_2} = \left(\frac{4q_1 + 4}{14{q_1}^2 + 7}\right)^{r_1} \left(\frac{4q_2 + 4}{14{q_2}^2 + 7}\right)^{r_2};$$

(iii) when q_1 is even and q_2 is odd,

$$\alpha_{g_1 g_2} = \left(\frac{7{q_1}^2 + 2}{12{q_1}^2}\right)^{g_1} \left(\frac{14{q_2}^2 + 7}{24{q_2}^2}\right)^{g_2}, \beta_{r_1 r_2} = \left(\frac{2q_1 + 2}{7{q_1}^2 + 2}\right)^{r_1} \left(\frac{4q_2 + 4}{14{q_2}^2 + 7}\right)^{r_2}.$$

4. A Lower Bound of Uniformity Pattern

This section provides a lower bound of uniformity pattern defined in Definition 1. It is very important that the lower bounds of uniformity pattern can be served as a benchmark not only in searching for uniform designs with minimum projection uniformity but also in helping to validate that some good designs are in fact uniform.

Define $\Delta_u = q_1 \mathrm{e}^{v_1} + p_1 \mathrm{e}^{v_3} + q_2(\mathrm{e}^{v_2} - \mathrm{e}^{v_3})$ when $p_1 > q_2$, and $\Delta_u = p_2 \mathrm{e}^{v_1} + q_2 \mathrm{e}^{v_4} + p_1(\mathrm{e}^{v_2} - \mathrm{e}^{v_4})$ when $p_1 \leq q_2$.

Theorem 3. *For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ and positive integer $g(= g_1 + g_2) \leq s$, we have*

$$MI_g(d) \geq LMI_g'(d),$$

(i) when both q_1 and q_2 are even,

$$LMI_g'(d) = \sum_{|u|=g} \Psi_{g_1 g_2} + \frac{1}{n^2} \left(\frac{7q_1 - 2}{12q_1}\right)^{g_1} \left(\frac{7q_2 - 2}{12q_2}\right)^{g_2} \sum_{u \in S_g} \Delta_u,$$

where $\Psi_{g_1 g_2} = \frac{1}{n}(\frac{3}{4})^{g_1}(\frac{3}{4})^{g_2} - (\frac{7{q_1}^2+2}{12{q_1}^2})^{g_1}(\frac{7{q_2}^2+2}{12{q_2}^2})^{g_2}$;

(ii) when both q_1 and q_2 are odd,

$$LMI'_g(d) = \sum_{|u|=g} \Psi_{g_1g_2} + \frac{1}{n^2}\left(\frac{14{q_1}^2-4q_1+3}{24{q_1}^2}\right)^{g_1}\left(\frac{14{q_2}^2-4q_2+3}{24{q_2}^2}\right)^{g_2}\sum_{u\in S_g}\Delta_u,$$

where $\Psi_{g_1g_2} = \frac{1}{n}(\frac{6{q_1}^2+1}{8{q_1}^2})^{g_1}(\frac{6{q_2}^2+1}{8{q_2}^2})^{g_2} - (\frac{14{q_1}^2+7}{24{q_1}^2})^{g_1}(\frac{14{q_2}^2+7}{24{q_2}^2})^{g_2}$;

(iii) when q_1 is even and q_2 is odd,

$$LMI'_g(d) = \sum_{|u|=g} \Psi_{g_1g_2} + \frac{1}{n^2}\left(\frac{7q_1-2}{12q_1}\right)^{g_1}\left(\frac{14{q_2}^2-4q_2+3}{24{q_2}^2}\right)^{g_2}\sum_{u\in S_g}\Delta_u,$$

where $\Psi_{g_1g_2} = \frac{1}{n}(\frac{3}{4})^{g_1}(\frac{6{q_2}^2+1}{8{q_2}^2})^{g_2} - (\frac{7{q_1}^2+2}{12{q_1}^2})^{g_1}(\frac{14{q_2}^2+7}{24{q_2}^2})^{g_2}$.

Theorem 4. *For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ and positive integer $g(= g_1 + g_2) \leq s$,*

$$MI_g(d) \geq LMI''_g(d),$$

(i) when both q_1 and q_2 are even,

$$LMI''_g(d) = \sum_{|u|=g}\left[\frac{1}{n^2}\left(\frac{7q_1-2}{12q_1}\right)^{g_1}\left(\frac{7q_2-2}{12q_2}\right)^{g_2}\binom{g_1}{i_1}\binom{g_2}{i_2}\right.$$
$$\left.\times\left(\frac{2q_1+2}{7q_1-2}\right)^{i_1}\left(\frac{2q_2+2}{7q_2-2}\right)^{i_2}\theta_{i_1i_2} - \left(\frac{7q_1+2}{12q_1}\right)^{g_1}\left(\frac{7q_2+2}{12q_2}\right)^{g_2}\right];$$

(ii) when both q_1 and q_2 are odd,

$$LMI''_g(d) = \sum_{|u|=g}\left[\frac{1}{n^2}\left(\frac{14q_1^2-4q_1+3}{24q_1}\right)^{g_1}\left(\frac{14q_2^2-4q_2+3}{24q_2}\right)^{g_2}\binom{g_1}{i_1}\binom{g_2}{i_2}\right.$$
$$\left.\times\left(\frac{4q_1^2+4q_1}{14q_1^2-4q_1+3}\right)^{i_1}\left(\frac{4q_2^2+4q_2}{14q_2^2-4q_2+3}\right)^{i_2}\theta_{i_1i_2} - \left(\frac{14q_1^2+7}{24q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\right];$$

(iii) when q_1 is even and q_2 is odd,

$$LMI''_g(d) = \sum_{|u|=g}\left[\frac{1}{n^2}\left(\frac{7q_1-2}{12q_1}\right)^{g_1}\left(\frac{14q_2^2-4q_2+3}{24q_2}\right)^{g_2}\binom{g_1}{i_1}\binom{g_2}{i_2}\right.$$
$$\left.\times\left(\frac{2q_1+2}{7q_1-2}\right)^{i_1}\left(\frac{4q_2^2+4q_2}{14q_2^2-4q_2+3}\right)^{i_2}\theta_{i_1i_2} - \left(\frac{7q_1+2}{12q_1}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\right],$$

where $\theta_{i_1i_2} = n\lambda_{i_1i_2} + \mu_{i_1i_2}(1+\lambda_{i_1i_2})$, $\mu_{i_1i_2} = n - q_1^{i_1}q_2^{i_2}\lambda_{i_1i_2}$, $\lambda_{i_1i_2}$ *be the largest integer contained in* $n/(q_1^{i_1}q_2^{i_2})$.

Note that Theorem 3 is based on Hamming distances between any two runs of d, but Theorem 4 comes from the quadratic form $y_d^T D y_d$ in Appendix A Equation (A1). Some numerical examples show that these two lower bounds are not tight simultaneously. Therefore, we give another lower bound of uniformity pattern as the following theorem:

Theorem 5. *For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ and positive integer $g(= g_1 + g_2) \leq s$, we have*

$$MI_g(d) \geq LMI^*_g(d),$$

where $LMI_g^*(d) = \max\{LMI_g'(d), LMI_g''(d)\}$.

5. Illustrative Examples

In this section, some numerical examples are provided to illustrate our theoretical results.

Example 1. *Consider a design* $d_1 \in \mathcal{U}(4; 2^3 \times 4^3)$, *which are given below:*

$$d_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 3 & 2 \\ 1 & 0 & 1 & 2 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 3 & 1 & 3 \end{bmatrix}.$$

The number of columns in design d_1 is greater than the number of rows, its uniformity pattern in Definition 1, and its lower bound values in Theorems 3–5 are listed in Table 1.

Table 1. Numerical results of designs d_1.

g	**1**	**2**	**3**	**4**	**5**	**6**
$MI_g(d_1)$	0	0.0830	0.2193	0.2170	0.0954	0.0157
$LMI_g'(d_1)$	0	0.0830	0.2193	0.2170	0.0954	0.0157
$LMI_g''(d_1)$	0	0.0146	0.0397	0.0429	0.0318	0.0157
$LMI_g^*(d_1)$	0	0.0830	0.2193	0.2170	0.0954	0.0157

It is clear that d_1 is an orthogonal array of strength 1 and attains the lower bounds in Theorem 3.

Example 2. *Consider design* $d_2 \in \mathcal{U}(20; 2^3 \times 5)$ *and* $d_3 \in \mathcal{U}(48; 2^5 \times 3)$, *which are given below,*

$$d_2 = \begin{bmatrix} 0&0&1&1&1&1&0&0&0&0&1&1&1&1&0&0&0&0&1&1 \\ 0&1&0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1 \\ 0&1&1&0&1&0&0&1&0&1&1&0&1&0&0&1&0&1&1&0 \\ 0&0&0&0&1&1&1&1&2&2&2&2&3&3&3&3&4&4&4&4 \end{bmatrix}^T,$$

$$d_3 = \begin{bmatrix} 1111111100000000 & 0000000011111111 & 1111111100000000 \\ 1111000011110000 & 0000111100001111 & 1111000011110000 \\ 1100110011001100 & 0011001100110011 & 1100110011001100 \\ 1010101010101010 & 0101010101010101 & 1010101010101010 \\ 1001011001101001 & 0110100110010110 & 1001011001101001 \\ 0000000000000000 & 1111111111111111 & 2222222222222222 \end{bmatrix}^T.$$

The number of rows in designs d_2 and d_3 are greater than the number of columns, and the numerical results of both are shown in Table 2.

As can be seen from Table 2, designs d_2 and d_3 are an orthogonal array with strengths of 2 and 4, respectively, and both reach the lower bound in Theorem 4.

Table 2. Numerical results of designs d_2 and d_3.

g	1	2	3	4	5	6
$MI_g(d_2)$	0	0	7.8125×10^{-5}	1.2148×10^{-4}		
$LMI'_g(d_2)$	0	-0.0450	-0.0282	-0.0040		
$LMI''_g(d_2)$	0	0	7.8125×10^{-5}	1.2148×10^{-4}		
$LMI^*_g(d_2)$	0	0	7.8125×10^{-5}	1.2148×10^{-4}		
$MI_g(d_3)$	0	0	0	0	3.3908×10^{-6}	4.0973×10^{-6}
$LMI'_g(d_3)$	0	-0.1837	-0.1742	-0.1300	-0.0365	-0.0044
$LMI''_g(d_3)$	0	0	0	0	3.3908×10^{-6}	4.0973×10^{-6}
$LMI^*_g(d_3)$	0	0	0	0	3.3908×10^{-6}	4.0973×10^{-6}

It can be seen from Tables 1 and 2 that the lower bounds of uniformity pattern of designs d_1, d_2, and d_3 are achieved, so d_1, d_2, and d_3 are all MPU designs. We can also see that $LMI''_g(d)$ is better than $LMI'_g(d)$ for large n and smaller s. Similar to the findings of Fang et al. (2018) [24], none of the lower bounds in Theorems 3 and 4 are absolutely dominant for all combinations of the number of runs n and of factors s. Therefore, we choose the maximum value of Theorems 3–5.

6. Conclusions

In this paper, the projection uniformity and related properties under mixture discrepancy of asymmetric factorials are explored. The relationship between uniformity pattern and generalized minimum aberration is established. A lower bound of uniformity pattern is also obtained, which can be served as a benchmark for searching minimum projection uniformity designs. These results provide a theoretical basis for searching optimal asymmetric designs with minimum projection uniformity measured by average projection mixture discrepancy. Overall, this paper extends the results of [13–15] to the asymmetric case, which makes the corresponding theory more flexible.

The results in this paper can be extended to any asymmetric designs $d \in \mathcal{U}(N; q_1^{s_1} \times \cdots \times q_n^{s_n})$. Taking the first t factors as even and the last $n-t$ factors as odd, and using some simple calculation of tired multiplication, similar definition and results of uniformity pattern and lower bounds can be obtained.

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Appendix A

Proof of Lemma 2. If a design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ is an orthogonal array $OA(n; {q_1}^{s_1} \times {q_2}^{s_2}, t)$ of strength t; then, for any nonnegative integer $g (= g_1 + g_2) \leq s$ and $u = u_1 \cup u_2 \in S_{g_1 g_2}$, all possible ${q_1}^{g_1} \times {q_2}^{g_2}$ level combinations among any g columns of projection design d_u appear equally often. Given row $i_u^0 = (i_{u_1}^0, i_{u_2}^0) \in d_u$, it is easy to obtain that $|\{(i_u^0, k_u) : H_{i^0k}^{u_1} = j_1, H_{i^0k}^{u_2} = j_2, k_u \in d\}| = \binom{g_1}{j_1}\binom{g_2}{j_2}\frac{n(q_1-1)^{j_1}(q_2-1)^{j_2}}{q_1^{g_1}q_2^{g_2}}$.

Therefore, the third term in the right side of Formula (4) can be expressed as

$$\begin{aligned}&\frac{1}{n}\left(\frac{3}{4}\right)^{g_1}\left(\frac{6q_2^2+1}{8q_2^2}\right)^{g_2}\sum_{i_1=0}^{g_1}\sum_{i_2=0}^{g_2}\left(\frac{7q_1-2}{9q_1}\right)^{i_1}\left(\frac{14q_2^2-4q_2+3}{18q_2^2+3}\right)^{i_2}E_{i_1i_2}(d_u)\\ =&\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2},\end{aligned}$$

which completes the proof. □

Proof of Theorem 2. From Formulas (1), (3), (4) and Definition 1, we have

$$\begin{aligned}&MI_g(d)\\ =&\sum_{|u|=g}\left[\frac{1}{n}\left(\frac{3}{4}\right)^{g_1}\left(\frac{6q_2^2+1}{8q_2^2}\right)^{g_2}\sum_{i_1=0}^{g_1}\sum_{i_2=0}^{g_2}\left(\frac{7q_1-2}{9q_1}\right)^{i_1}\left(\frac{14q_2^2-4q_2+3}{18q_2^2+3}\right)^{i_2}E_{i_1i_2}(d_u)\right.\\ &\left.-\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\right]\\ =&\sum_{|u|=g}\left[\left(\frac{3}{4q_1}\right)^{g_1}\left(\frac{6q_2^2+1}{8q_2^3}\right)^{g_2}\sum_{r_1=0}^{g_1}\sum_{r_2=0}^{g_2}\sum_{i_1=0}^{g_1}\sum_{i_2=0}^{g_2}\left(\frac{7q_1-2}{9q_1}\right)^{i_1}\left(\frac{14q_2^2-4q_2+3}{18q_2^2+3}\right)^{i_2}\right.\\ &\left.\times P_{i_1}(r_1;g_1,q_1)P_{i_2}(r_2;g_2,q_2)E'_{r_1r_2}(d_u)-\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\right]\\ =&\sum_{|u|=g}\left[\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\sum_{r_1=0}^{g_1}\sum_{r_2=0}^{g_2}\left(\frac{2q_1+2}{7q_1^2+2}\right)^{r_1}\left(\frac{4q_2+4}{14q_2^2+7}\right)^{r_2}E'_{r_1r_2}(d_u)\right.\\ &\left.-\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\right]\\ =&\sum_{|u|=g}\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\sum_{(r_1,r_2)\in\mathcal{R}}\left(\frac{2q_1+2}{7q_1^2+2}\right)^{r_1}\left(\frac{4q_2+4}{14q_2^2+7}\right)^{r_2}E'_{r_1r_2}(d_u)\\ =&\sum_{|u|=g}\left(\frac{7q_1^2+2}{12q_1^2}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\sum_{(r_1,r_2)\in\mathcal{R}}\left(\frac{2q_1+2}{7q_1^2+2}\right)^{r_1}\left(\frac{4q_2+4}{14q_2^2+7}\right)^{r_2}\\ &\times\binom{s_1-r_1}{s_1-g_1}\binom{s_2-r_2}{s_2-g_2}A_r(d),\end{aligned}$$

which completes the proof. □

In order to prove Theorem 3, we need to know Lemmas A1–A3, where Lemma A1 can be obtained from Lemma 1 and Definition 1.

Lemma A1. *For any design $d \in \mathcal{U}(n; q_1^{s_1} \times q_2^{s_2})$, positive integer $g(= g_1 + g_2) \le s$ and $u = u_1 \cup u_2 \in S_g$,*

(i) when both q_1 and q_2 are even,

$$MI_g(d) = \sum_{|u|=g}\Psi_{g_1g_2} + \frac{1}{n^2}\sum_{|u|=g}\left(\frac{7q_1-2}{12q_1}\right)^{g_1}\left(\frac{7q_2-2}{12q_2}\right)^{g_2}\sum_{i=1}^{n}\sum_{k(\neq i)=1}^{n}e^{\theta_{ik}^u},$$

where $\Psi_{g_1g_2}$ is shown in Theorem 3, $\theta_{ik}^u = \ln(\frac{9q_1}{7q_1-2})\cdot\delta_{ik}^{u_1} + \ln(\frac{9q_2}{7q_2-2})\cdot\delta_{ik}^{u_2}$;

(ii) when both q_1 and q_2 are odd,

$$MI_g(d) = \sum_{|u|=g} \Psi_{g_1 g_2} + \frac{1}{n^2} \sum_{|u|=g} \left(\frac{14{q_1}^2 - 4q_1 + 3}{24{q_1}^2} \right)^{g_1} \left(\frac{14{q_2}^2 - 4q_2 + 3}{24{q_2}^2} \right)^{g_2} \sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} \mathrm{e}^{\theta_{ik}^u},$$

where $\Psi_{g_1 g_2}$ is shown in Theorem 3, $\theta_{ik}^u = \ln(\frac{18{q_1}^2+3}{14{q_1}^2-4q_1+3}) \cdot \delta_{ik}^{u_1} + \ln(\frac{18{q_2}^2+3}{14{q_2}^2-4q_2+3}) \cdot \delta_{ik}^{u_2}$;
(iii) when q_1 is even and q_2 is odd,

$$MI_g(d) = \sum_{|u|=g} \Psi_{g_1 g_2} + \frac{1}{n^2} \sum_{|u|=g} \left(\frac{7q_1 - 2}{12q_1} \right)^{g_1} \left(\frac{14{q_2}^2 - 4q_2 + 3}{24{q_2}^2} \right)^{g_2} \sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} \mathrm{e}^{\theta_{ik}^u},$$

where $\Psi_{g_1 g_2}$ is shown in Theorem 3, $\theta_{ik}^u = \ln(\frac{9q_1}{7q_1-2}) \cdot \delta_{ik}^{u_1} + \ln(\frac{18{q_2}^2+3}{14{q_2}^2-4q_2+3}) \cdot \delta_{ik}^{u_2}$.

The proof of Lemma A1 is similar to [14], so it is omitted.

Lemma A2 ([25])**.** *For any design $d \in \mathcal{U}(n; q^s)$ and positive integer t, we have*

$$\sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} (\delta_{ik})^t = Pw^t + Q(w+1)^t.$$

where $w = \lfloor \frac{(n-q)s}{q(n-1)} \rfloor$, P and Q are integers such that $P + Q = n(n-1)$, and $\lfloor A \rfloor$ means the largest integer contained in A.

Lemma A3 ([26])**.** *For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$ and positive integer t, we have*

$$\sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} \theta_{ik} = \frac{\alpha_1 n(n-q_1) g_1}{q_1} + \frac{\alpha_2 n(n-q_2) g_2}{q_2}, \text{ and}$$

$$\sum_{i=1}^{n} \sum_{k(\neq i)=1}^{n} (\theta_{ik})^t \geq \begin{cases} Q_1 v_1^t + Q_2 v_2^t + (P_1 - Q_2) v_3^t, \text{ when } P_1 > Q_2; \\ P_2 v_1^t + P_1 v_2^t + (Q_2 - P_1) v_4^t, \text{ when } P_1 \leq Q_2. \end{cases}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ are weights, P_1 and Q_1 are integers such that $P_1 + Q_1 = n(n-1)$ and $P_1 w_1 + Q_1(w_1+1) = n(n-q_1)s_1/q_1$, P_2 and Q_2 are integers such that $P_2 + Q_2 = n(n-1)$ and $P_2 w_2 + Q_2(w_2+1) = n(n-q_2)s_2/q_2$. Let $v_1 = \alpha_1(w_1+1) + \alpha_2 w_2$, $v_2 = \alpha_1 w_1 + \alpha_2(w_2+1)$, $v_3 = \alpha_1 w_1 + \alpha_2 w_2$, $v_4 = \alpha_1(w_1+1) + \alpha_2(w_2+1)$, $w_1 = \lfloor \frac{(n-q_1)s_1}{q_1(n-1)} \rfloor$, $w_2 = \lfloor \frac{(n-q_2)s_2}{q_2(n-1)} \rfloor$.

Proof of Theorem 4. According to [23,24], let $\boldsymbol{I}_q$ and $\mathbf{1}_q$ respectively be the $q \times q$ identity matrix and the $q \times 1$ vector with all elements unity, define

$$\boldsymbol{L}(0) = \mathbf{1}_q^T,\ \boldsymbol{L}(1) = \boldsymbol{I}_q,\ \boldsymbol{J}_q = \mathbf{1}_q \mathbf{1}_q^T.$$

Let $D_{g_1}^{(1)}$ and $D_{g_2}^{(2)}$ be the g_1-fold and g_2-fold Kronecker products of $D_0^{(1)}$ and $D_0^{(2)}$, respectively. Let Ω be the set of all binary $(q_1 + q_2)$ tuples, $\Omega_{i_1 i_2}$ be the set of Ω consisting of those binary $(g1 + g2)$-tuples with exactly i_1 elements of x_1 unity and i_2 elements of x_2 unity, respectively, where $\Omega = \{x = (x^{(1)}, x^{(2)}) : x^{(1)} = (x_1^{(1)}, \ldots, x_{g_1}^{(1)}) \in \Omega^{(1)}, x^{(2)} = (x_1^{(2)}, \ldots, x_{g_2}^{(2)}) \in \Omega^{(2)}\}$.

$$D = D_{g_1}^{(1)} \bigotimes D_{g_2}^{(2)},\ D_{g_1}^{(1)} = \bigotimes_{i_1=1}^{g_1} D_0^{(1)},\ D_{g_2}^{(2)} = \bigotimes_{i_2=1}^{g_2} D_0^{(2)}.$$

For any design $d \in \mathcal{U}(n; {q_1}^{s_1} \times {q_2}^{s_2})$, Lemma A1 gives an expression between the uniformity pattern and the number of coincide. Based on this, we can obtain

(i) when both q_1 and q_2 are even,

$$D_0^{(1)} = \frac{q_1+1}{6q_1}\boldsymbol{I}_{q_1} + \frac{7q_1-2}{12q_1}\boldsymbol{J}_{q_1},\ D_0^{(2)} = \frac{q_2+1}{6q_2}\boldsymbol{I}_{q_2} + \frac{7q_2-2}{12q_2}\boldsymbol{J}_{q_2};$$

(ii) when both q_1 and q_2 are odd,

$$D_0^{(1)} = \frac{q_1+1}{6q_1}\boldsymbol{I}_{q_1} + \frac{14q_1^2-4q_1+3}{24q_1^2}\boldsymbol{J}_{q_1},\ D_0^{(2)} = \frac{q_2+1}{6q_2}\boldsymbol{I}_{q_2} + \frac{14q_2^2-4q_2+3}{24q_2^2}\boldsymbol{J}_{q_2};$$

(iii) when q_1 is even and q_2 is odd,

$$D_0^{(1)} = \frac{q_1+1}{6q_1}\boldsymbol{I}_{q_1} + \frac{7q_1-2}{12q_1}\boldsymbol{J}_{q_1},\ D_0^{(2)} = \frac{q_2+1}{6q_2}\boldsymbol{I}_{q_2} + \frac{14q_2^2-4q_2+3}{24q_2^2}\boldsymbol{J}_{q_2}.$$

Considering the case (iii) where q_1 is even and q_2 is odd, we have

$$MI_g(d) = \sum_{|u|=g}\left[\frac{1}{n^2}y_d^T D y_d - \left(\frac{7q_1+2}{12q_1}\right)^{g_1}\left(\frac{14q_2^2+7}{24q_2^2}\right)^{g_2}\right], \tag{A1}$$

where

$$D = \gamma_{g_1g_2}\sum_{x^{(1)}\in\Omega^{(1)}}\sum_{x^{(2)}\in\Omega^{(2)}}\left(\frac{2q_1+2}{7q_1-2}\right)^{\sum x_i^{(1)}}\left(\frac{4q_2{}^2+4q_2}{14q_2{}^2-4q_2+3}\right)^{\sum x_i^{(2)}} H(x)'H(x),$$

$$y_d' D y_d = \gamma_{g_1g_2}\sum_{i_1=0}^{g_1}\sum_{i_2=0}^{g_2}\left(\frac{2q_1+2}{7q_1-2}\right)^{i_1}\left(\frac{4q_2{}^2+4q_2}{14q_2{}^2-4q_2+3}\right)^{i_2}\sum_{x\in\Omega_{i_1i_2}} y_d' H(x)'H(x)y_d,$$

and $\gamma_{g_1g_2} = (\frac{q_1+1}{6q_1})^{g_1}(\frac{q_2+1}{6q_2})^{g_2}$.

Let $y_d(x)$ be the number of times the treatment combination x occurs in d and y_d be the $n \times 1$ vector with elements $y_d(x)$ arranged in the lexicographic order. For any $\sum_{x\in\Omega_{i_1i_2}}$, the elements of the $q_1^{i_1}q_2^{i_2} \times 1$ vector $H(x)y_d$ are nonnegative integers with sum n; then, by [24], we have

$$y_d' H(x)'H(x)y_d \le \lambda_{i_1i_2}^2(q_1^{i_1}q_2^{i_2} - \mu_{i_1i_2}) + (\lambda_{i_1i_2}+1)^2\mu_{i_1i_2} = n\lambda_{i_1i_2} + \mu_{i_1i_2}(\lambda_{i_1i_2}+1),$$

which completes the proof of Case (iii).

The proof of Case (i) and Case (ii) are similar to Case (iii). □

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