

## Article

# Nearly Sasakian Manifolds of Constant Type

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**Abstract:** The article deals with nearly Sasakian manifolds of a constant type. It is proved that the almost Hermitian structure induced on the integral manifolds of the maximum dimension of the first fundamental distribution of the nearly Sasakian manifold is a nearly Kähler manifold. It is proved that the class of nearly Sasakian manifolds of the zero constant type coincides with the class of Sasakian manifolds. The concept of constancy of the type of an almost contact metric manifold is introduced through its Nijenhuis tensor, and the criterion of constancy of the type of an almost contact metric manifold is proved. The coincidence of both concepts of type constancy for the nearly Sasakian manifold is proved. It is proved that the almost Hermitian structure induced on the integral manifolds of the maximum dimension of the first fundamental distribution of the almost contact metric manifold of the zero constant type is the Hermitian structure.

**Keywords:** nearly Sasakian manifold; Sasakian manifold; constant-type manifold

**MSC:** 53C25; 53D15



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## 1. Introduction

The concept of constancy of the type of nearly Kähler manifolds was introduced by A. Gray [1] and proved to be very useful in the study of the geometry of nearly Kähler manifolds. An exhaustive description of nearly Kähler manifolds of a constant type was obtained by V.F. Kirichenko [2].

The contact analogue of the concept of type constancy was introduced in the papers [3,4] for generalized Kenmotsu manifolds and nearly cosymplectic manifolds. In the article [2] it was proved that the class of generalized Kenmotsu manifolds of the zero constant type coincides with the class of Kenmotsu manifolds, and the class of generalized manifolds of the non-zero constant type are transformed by the concircular transformation into almost contact metric manifolds locally equivalent to the product of the six-dimensional Kähler manifold by the real line. This gives us a complete characterization of the generalized Kenmotsu manifolds of the constant type.

For nearly cosymplectic manifolds of a constant type [4], it was proved that the point constancy of the type of a connected nearly cosymplectic manifold of dimension greater than 3 is equivalent to the global constancy of its type. The class of nearly cosymplectic manifolds of the zero constant type coincides with the class of eigen with nearly cosymplectic manifolds or with the class of cosymplectic manifolds. The class of nearly cosymplectic manifolds of a non-zero constant type coincides with the class of seven-dimensional eigens of nearly cosymplectic manifolds.

Naturally, there comes the question of studying nearly Sasakian manifolds of a constant type.

In this article, we examine the nearly Sasakian manifolds of a constant type, so the study is organized as follows. In Section 2, we provide the necessary information for further investigation. In particular, we give a complete group of structural equations on the space of the attached G-structure, some identities that structural tensors satisfy. It is proved that the almost-Hermitian structure induced on the integral manifolds of the maximum

dimension of the first fundamental distribution of the nearly Sasakian manifold is the nearly Kähler structure. In Section 3, we consider the attached Q-algebra of the nearly Sasakian structure and prove that it is anticommutative. In Section 4, we prove that the point constancy of a type of nearly Sasakian manifolds is equivalent to the global constancy of its type. We also prove that the class of nearly Sasakian manifolds of the zero-constant type coincides with the class of Sasakian manifolds. Section 5 introduces the concept of type permanence on the basis of the Nijenhuis tensor. The criterion of constancy of the type of almost contact metric manifold is obtained. It is shown that for nearly Sasakian manifolds, the introduced concept of type constancy coincides with the concept of type introduced in Section 4. It is proved that the almost-Hermitian structure induced on the integral manifolds of the maximum dimension of the first fundamental distribution of the almost contact metric manifold of the zero constant type is the Hermitian structure.

This article is a continuation of the study [4].

## 2. Preliminary Information

**Definition 1** ([5]). *The near-contact metric structure is called **nearly Sasakian** ( $\mathcal{NS}$ ) structure if its structural endomorphism  $\Phi$  satisfies the identity*

$$\nabla_X(\Phi)X = \langle X, X \rangle \xi - \eta(X)X; \quad X \in \mathcal{X}(M). \quad (1)$$

These structures were introduced by K. Yano, D. E. Blair and D. K. Showers in [6] and relate to Sasakian structures to the same extent as the nearly Kähler structures are related to the Kähler structures. An example of such a structure is an almost contact metric structure induced on a five-dimensional sphere  $S^5$ , embedded in a six-dimensional sphere  $S^6 \subset \mathbf{O}$  as a completely umbilic hypersurface [6]. This structure is not a Sasakian structure.

Z. Olszak in [7,8] investigated some properties of nearly Sasakian non-Sasakian manifolds. It was proved that locally symmetric, constant holomorphic sectional curvature and conformally flat nearly Sasakian non-Sasakian manifolds are constants of curvature and have five dimensions. The author also gave some equivalent conditions for nearly Sasakian manifolds that are not Sasakian manifolds to show that such manifolds are Einsteinian.

The result, which gives an exhaustive description of the structure of nearly Sasakian manifolds, was published in the paper of V. F. Kirichenko [9]. It was proven that any of the nearly Sasakian manifolds is either a Sasakian manifold or is locally equivalent to a five-dimensional sphere equipped with a canonical nearly Sasakian structure. This study also presented a complete classification of nearly Sasakian manifolds of the constant  $\Phi$ -holomorphic curvature of the section, significantly generalizing and refining the Tanno classification [10] of complete single-connected Sasaki manifolds of the constant  $\Phi$ -holomorphic curvature of the section.

The study in paper [11] continued the systematic study of nearly Sasakian manifolds. It was proved that any nearly Sasakian manifold allows for two types of integrable distributions with quite geodesic layers, which are respectively Sasakian and 5-dimensional nearly Sasakian manifolds. As a consequence, any nearly Sasakian manifold is a contact manifold. It was also proved that there is a one-to-one correspondence between the five-dimensional nearly Sasakian structures and a special class of nearly hypo-SU(2)-structures. By deforming such a SU(2)-structure, the Sasaki–Einstein structure was obtained.

The study [12] proved that any nearly Sasakian manifold of dimension greater than five is Sasakian. This gives a new criterion for the Sasakianity of an almost-contact-metric manifold. Thus, the study [13] provided a new, autonomous and more conceptual proof of the result that an almost-contact-metric manifold of dimension greater than five is Sasakian if and only if it is nearly Sasakian.

This review motivates the study of nearly Sasakian manifolds of a constant type.

The complete group of structural equations of nearly Sasakian structure has the following form [5]:

$$\begin{aligned}
 (1) \quad & d\omega = F_{ab}\omega^a \wedge \omega^b + F^{ab}\omega_a \wedge \omega_b - 2\sqrt{-1}\delta_a^b\omega^a \wedge \omega_b; \\
 (2) \quad & d\omega^a = -\theta_b^a \wedge \omega^b + C^{abc}\omega_b \wedge \omega_c + \frac{3}{2}F^{ab}\omega_b \wedge \omega + \sqrt{-1}\delta_a^b\omega^b \wedge \omega; \\
 (3) \quad & d\omega_a = \theta_a^b \wedge \omega_b + C_{abc}\omega^b \wedge \omega^c + \frac{3}{2}F_{ab}\omega^b \wedge \omega + \sqrt{-1}\delta_a^b\omega_b \wedge \omega, \\
 (4) \quad & d\theta_b^a = -\theta_c^a \wedge \theta_b^c + (A_{bc}^{ad} - 2C^{adh}C_{hbc} - \frac{3}{2}F^{ad}F_{bc} - \delta_{bc}^{ad})\omega^c \wedge \omega_d \\
 & \quad - \sqrt{-1}F_{a[c}\delta_{d]}^b\omega^c \wedge \omega^d - \sqrt{-1}F^a[c]\delta_b^{d]}\omega_c \wedge \omega_d; \\
 (5) \quad & dC^{abc} + C^{dbc}\theta_d^a + C^{adc}\theta_d^b + C^{abd}\theta_d^c = C^{abcd}\omega_d + 3\sqrt{-1}F^{[ab}\delta_c^{d]}\omega^d; \\
 (6) \quad & dC_{abc} - C_{dbc}\theta_d^a - C_{adc}\theta_d^b - C_{abd}\theta_d^c = C_{abcd}\omega^d - 3\sqrt{-1}F_{[ab}\delta_c^{d]}\omega^d; \\
 (7) \quad & dF^{ab} + F^{cb}\theta_c^a + F^{ac}\theta_c^b = 2\sqrt{-1}C^{abc}\omega_c + \sqrt{-1}F^{ab}\omega; \\
 (8) \quad & dF_{ab} - F_{cb}\theta_c^a - F_{ac}\theta_c^b = -2\sqrt{-1}C_{abc}\omega^c - \sqrt{-1}F_{ab}\omega.
 \end{aligned} \tag{2}$$

At the same time, there are the following equalities:

$$(1)C^{abd}F_{dc} = 0; \quad 2)C^{abch}F_{hd} = 0; \quad 3)C^{abh}C_{hcd} = F^{h[a}F_{h[c}\delta_{d]}^b] - \frac{1}{2}F^{ab}F_{cd}. \tag{3}$$

**Theorem 1.** *The almost Hermitian structure induced on the integral manifolds of the maximum dimension of the first fundamental distribution of the  $\mathcal{NS}$ -manifold is the nearly Kähler structure.*

**Proof.** Assume that  $M$  is a nearly Sasakian manifold with a fully integrable first fundamental distribution  $\mathcal{L}$ . The first group of structural equations of such a manifold has the following form:

$$\begin{aligned}
 d\omega &= F_{ab}\omega^a \wedge \omega^b + F^{ab}\omega_a \wedge \omega_b - 2\sqrt{-1}\delta_a^b\omega^a \wedge \omega_b; \\
 d\omega^a &= -\theta_b^a \wedge \omega^b + C^{abc}\omega_b \wedge \omega_c + \frac{3}{2}F^{ab}\omega_b \wedge \omega + \sqrt{-1}\delta_a^b\omega^b \wedge \omega; \\
 d\omega_a &= \theta_a^b \wedge \omega_b + C_{abc}\omega^b \wedge \omega^c + \frac{3}{2}F_{ab}\omega^b \wedge \omega - \sqrt{-1}\delta_a^b\omega_b \wedge \omega.
 \end{aligned}$$

Let  $N \in M$  be the integral manifold of the maximum dimension of the first fundamental distribution of the manifold  $M$ . Then, it canonically induces the almost-Hermitian structure  $\langle J, \tilde{g} \rangle$ , where  $J = \Phi|_{\mathcal{L}}$ ,  $\tilde{g} = g|_{\mathcal{L}}$ . Since the form  $\omega$  is the Pfaff form of the first fundamental distribution, the first group of structural equations of an almost-Hermitian structure on  $N$  has the following form:

$$\begin{aligned}
 d\omega^a &= -\theta_b^a \wedge \omega^b + B^{abc}\omega_b \wedge \omega_c; \\
 d\omega_a &= \theta_a^b \wedge \omega_b + B_{abc}\omega^b \wedge \omega^c; \\
 d\omega &= 0.
 \end{aligned} \tag{4}$$

Bearing in mind the Gray–Hervella classification of the almost-Hermitian structures, written in the form of a table ([5], p. 450), we obtain that the almost-Hermitian structure induced on the integral submanifolds of the manifolds of the manifold  $M$  is almost-Kähler.  $\square$

### 3. Q-Algebras of Nearly Sasakian Manifolds

The concepts of a generalized almost Hermitian manifold and the Q-algebra attached to it were formed in the 1980s and were studied in a number of articles by V. F. Kirichenko (see, for example [14–18]).

**Definition 2 ([15]).** *Q-algebra* is the triple  $\{V, \langle \langle \cdot, \cdot \rangle \rangle, *\}$ , where  $V$  is the modulus over the commutative associative ring  $K$  with (nontrivial) involution;  $\langle \langle \cdot, \cdot \rangle \rangle$ —non-degenerate Hermitian form on  $V$ ;  $*$ —a binary operation  $* : V \times V \rightarrow V$ , antilinear for each argument, for which the axiom Q-algebra  $\langle \langle X * Y, Z \rangle \rangle + \langle \langle Y, X * Z \rangle \rangle = 0$ ,  $X, Y, Z \in V$  is executed.

If  $K = \mathbb{C}$ , then  $V$  is complex Q-algebra.

**Definition 3** ([18]). *Q-algebra  $V$  is called the following:*

- *Abelian, or commutative Q-algebra, if  $X * Y = 0$ ,  $(X, Y \in V)$ ;*
- *K-algebra, or anti-commutative Q-algebra, if  $X * Y = -Y * X$ ,  $(X, Y \in V)$ ;*
- *A-algebra, or pseudocommutative Q-algebra, if  $\langle X * Y, Z \rangle + \langle Y * Z, X \rangle + \langle Z * X, Y \rangle = 0$ ,  $(X, Y, Z \in V)$ .*

Recall [16] that in module  $\mathcal{X}(M)$ , an almost-contact-metric manifold is naturally introduced by the structure of Q-algebra  $\mathfrak{R}$  over a ring of complex-valued smooth functions with the operation

$$X * Y = T(X, Y) = \frac{1}{4} \{ \Phi \nabla_{\Phi X}(\Phi) \Phi Y - \Phi \nabla_{\Phi^2 X}(\Phi) \Phi^2 Y \}; X, Y \in \mathcal{X}(M) \quad (5)$$

and metrics

$$\langle \langle X, Y \rangle \rangle = \langle X, Y \rangle + \sqrt{-1} \langle X, \Phi Y \rangle; X, Y \in \mathcal{X}(M). \quad (6)$$

This is **attached** Q-algebra.

Let  $M$  be a  $\mathcal{NS}$ -manifold. In  $C^\infty(M)$ -module  $\mathcal{X}(M)$  of smooth vector fields of the  $M$  manifold, we introduce the binary operation “ $*$ ” using the formula  $X * Y = T(X, Y) = \frac{1}{4} \{ \Phi \nabla_{\Phi X}(\Phi) \Phi Y - \Phi \nabla_{\Phi^2 X}(\Phi) \Phi^2 Y \}; X, Y \in \mathcal{X}(M)$ .

**Theorem 2** ([18]).  *$\mathcal{NS}$ -structure has anti-commutative attached Q-algebra.*

**Proof.** From Definition 1, it easily follows that

$$\Phi \nabla_{\Phi X}(\Phi) \Phi Y + \Phi \nabla_{\Phi Y}(\Phi) \Phi X = 0; X, Y \in \mathcal{X}(M),$$

that is

$$\Phi \nabla_{\Phi X}(\Phi) \Phi Y = -\Phi \nabla_{\Phi Y}(\Phi) \Phi X; X, Y \in \mathcal{X}(M).$$

It means that

$$\Phi \nabla_{\Phi^2 X}(\Phi) \Phi^2 Y = -\Phi \nabla_{\Phi^2 Y}(\Phi) \Phi^2 X; X, Y \in \mathcal{X}(M).$$

Then,

$$T(X, Y) = \frac{1}{4} \{ \Phi \nabla_{\Phi X}(\Phi) \Phi Y - \Phi \nabla_{\Phi^2 X}(\Phi) \Phi^2 Y \} = -\frac{1}{4} \{ \Phi \nabla_{\Phi Y}(\Phi) \Phi X - \Phi \nabla_{\Phi^2 Y}(\Phi) \Phi^2 X \} = -T(Y, X); X, Y \in \mathcal{X}(M).$$

That is, the attached Q-algebra with  $\mathcal{NS}$ -structure is anti-commutative Q-algebra.  $\square$

#### 4. Nearly Sasakian Manifolds of Constant Type

In this Section, we consider nearly Sasakian manifolds of a constant type. It is necessary to recall some definitions.

**Definition 4.** *Complex Q-algebra  $\mathfrak{R}$  is named as Q-algebra of a **constant type**, if  $\exists c \in \mathbb{C} \forall X, Y \in \mathcal{L} : \langle \langle X, Y \rangle \rangle = 0 \Rightarrow \|X * Y\|^2 = c \|X\|^2 \|Y\|^2$ , where  $\mathcal{L}$  – is the hyperplane in the reification of the  $c$ -module  $\mathfrak{R}$ .*

**Definition 5.** *The almost-contact-metric manifold  $M$  is called a manifold of a **point constant type** if its attached Q-algebra has a constant type at each point of  $M$ . The function  $c$ , if it exists, is called a constant type of the almost-contact-metric manifold. If also  $c = \text{const}$ , then  $M$  is called an almost-contact-metric manifold of a **globally constant type**.*

This definition is a contact analogue of the concept of constancy of the type of almost-Hermitian manifolds, introduced by V. F. Kirichenko [14].

**Theorem 3.** An almost-contact-metric manifold  $M$  is a point-constant-type manifold if and only if

$$\forall X, Y \in \mathcal{X}(M) \langle \langle X, Y \rangle \rangle = 0 \Rightarrow \|C(X, Y)\|^2 = c\|X\|^2\|Y\|^2. \quad (7)$$

The proof is conducted similarly to the proof of theorem 4 of [4].

Assume that  $M$ — $\mathcal{NS}$ -manifold—is of a point-constant type.

Let us introduce 4-form  $C(X, Y, Z, W) = \langle \langle X * Y, Z * W \rangle \rangle = \langle \langle C(X, Y), C(Z, W) \rangle \rangle$ . The following theorems are proved in the same way as theorems 4 and 5 of [4].

**Theorem 4.**  $\mathcal{NS}$ -manifold is a manifold of a point-constant type if and only if

$$C(X, Y, Z, W) = \langle \langle C(X, Y), C(Z, W) \rangle \rangle = c\{\langle \langle W, Y \rangle \rangle \langle \langle Z, X \rangle \rangle - \langle \langle W, X \rangle \rangle \langle \langle Z, Y \rangle \rangle\}.$$

**Theorem 5.** Let  $M$  be a  $\mathcal{NS}$ -manifold. Then the following statements are equivalent:

- (1)  $M$ — $\mathcal{NS}$ -manifold of point constant of  $c$ -type;
- (2) The first structural tensor of  $\mathcal{NS}$ -manifold satisfies identity

$$\langle \langle C(X, Y), C(Z, W) \rangle \rangle = c\{\langle \langle W, Y \rangle \rangle \langle \langle Z, X \rangle \rangle - \langle \langle W, X \rangle \rangle \langle \langle Z, Y \rangle \rangle\}.$$

- (3) In the space of the attached  $G$ -structure, the following ratio is true:

$$C^{abh}C_{hcd} = \frac{c}{2}\delta_{cd}^{ab}, \quad (8)$$

where  $\delta_{cd}^{ab} = \delta_c^a\delta_d^b - \delta_c^b\delta_d^a$ —second-order Kronecker delta.

Let us externally differentiate the identity (8)  $dC^{abh}C_{hcd} + C^{abh}dC_{hcd} = \frac{1}{2}dc\delta_{cd}^{ab}$ . Taking into account the structural equations of  $\mathcal{NS}$ -manifolds, we have

$$(-C^{gbh}\theta_g^a - C^{agh}\theta_g^b - C^{abg}\theta_g^h + C^{abhg}\omega_g + 3\sqrt{-1}F^{[ab}\delta_g^{h]}\omega_g)C_{hcd} + C^{abh}(C_{gcd}\theta_h^g + C_{hgd}\theta_c^g + C_{hcg}\theta_d^g + C_{hcdg}\omega^g - 3\sqrt{-1}F_{[cd}\delta_h^g]\omega_g) = \frac{1}{2}\delta_{cd}^{ab}dc.$$

By opening the parentheses and giving similar terms, we obtain

$$(C^{abh}C_{hcdg} + \sqrt{-1}F^{ab}C_{gcd})\omega_g + (C^{abhg}C_{hcd} - \sqrt{-1}C^{abg}F_{cd})\omega_g = \frac{1}{2}\delta_{cd}^{ab}(c_g\omega^g + c^g\omega_g + c_0\omega).$$

Hence, we have (in case  $\dim M > 3$ )

$$\begin{aligned} (1) \delta_{cd}^{ab}c_g &= C^{abh}C_{hcdg} + \sqrt{-1}F^{ab}C_{gcd}; \\ (2) \delta_{cd}^{ab}c^g &= C^{abhg}C_{hcd} - \sqrt{-1}C^{abg}F_{cd}; \quad (3) c_0 = 0. \end{aligned} \quad (9)$$

Folding the equality 9:(1) and 8:(2) with the object  $F^{df}$ , because of (3), we obtain

$$(\delta_c^a F^{bf} - \delta_c^b F^{af})c_g = 0. \quad (10)$$

Folding the obtained equality on the indices  $a$  and  $c$ , we obtain  $(n-1)F^{bf}c_g = 0$ , i.e., since  $\dim M > 3$ , then either  $F^{ab} = 0$ , or  $c_g = 0$ .

Similarly, from 9:(2), we obtain either  $F_{ab} = 0$ , or  $c^g = 0$ .

Suppose that  $F_{ab} = F^{ab} = 0$ . Then according to 3:(3) we have  $C^{abh}C_{hcd} = 0$ , i.e., taking into account (8), we have that  $c = 0$ . Folding this identity over the pairs of indices  $a$  and  $c$ ,  $b$  and  $d$ , taking into account the oblique symmetry of the objects  $C^{abh}$  and  $C_{hcd}$ , we obtain that  $\Sigma_{a,b,c} |C_{abc}|^2 = C^{abh}C_{hcd} = 0$ . Thus, the manifold is a Sasakian manifold of the zero constant type.

It follows from the condition  $dc = 0$ , that if  $M$  is coherent, then  $c = \text{const}$ . Thus, the following theorem is proved.

**Theorem 6.** *The point constancy of the type of a connected  $\mathcal{NS}$ -manifold of dimension greater than 3 is equivalent to the global constancy of its type.*

Let us make a complete convolution of identity (8):  $\Sigma_{a,b,c} |C_{abc}|^2 = C^{abh} C_{hcd} = \frac{c}{2} n(n-1)$ , where  $n$  is the complex dimension of the contact distribution  $\mathcal{L}$ . It follows that  $c \in \mathbb{R}$ ,  $c \geq 0$ , and  $c = 0$  if and only if  $C_{hcd} = C^{abh} = 0$ ,  $F_{ab} = F^{ab} = 0$ , i.e.,  $M$  is a Sasakian manifold. The opposite is obvious. Thus, it is proved.

**Theorem 7.** *The class of  $\mathcal{NS}$ -manifolds of the zero constant type coincides with the class of Sasakian manifolds.*

### 5. Nijenhuis Tensor of $\mathcal{NS}$ -Manifolds

The Nijenhuis tensor (or the operator) of the structural endomorphism  $\Phi$  of an almost-contact-metric structure is called the  $N_\Phi$  tensor of the type (2,1) defined by the formula

$$N_\Phi(X, Y) = \frac{1}{4} \{ \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] \}; \quad X, Y \in \mathcal{X}(M).$$

It is easy to see that this tensor has the following properties:

$$N_\Phi(X, Y) = \frac{1}{4} \{ \Phi^2[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] \}; \quad X, Y \in \mathcal{X}(M).$$

By directly counting, taking into account the identity of  $[X, Y] = \nabla_X Y - \nabla_Y X$ , expressing the absence of torsion of Riemannian connection, we write the last identity as

$$N_\Phi(X, Y) = \frac{1}{4} \{ \nabla_{\Phi X}(\Phi)Y - \nabla_{\Phi Y}(\Phi)X + \Phi \nabla_Y(\Phi)X - \Phi \nabla_X(\Phi)Y \}; \quad X, Y \in \mathcal{X}(M). \quad (11)$$

In the space of the attached G-structure, the components of the Nijenhuis tensor  $N_\Phi$  are defined by equality:

$$\begin{aligned} (1) \quad N_{\Phi_{ab}}^0 &= -\frac{\sqrt{-1}}{2} \Phi_{[a,b]}^0, \quad (2) \quad N_{\Phi_{\hat{a}\hat{b}}}^0 = -N_{\Phi_{b\hat{a}}}^0 = -\frac{\sqrt{-1}}{2} \Phi_{(\hat{a},b)}^0, \quad (3) \quad N_{\Phi_{\hat{a}\hat{b}}}^0 = -\frac{\sqrt{-1}}{2} \Phi_{[\hat{a},\hat{b}]}^0, \\ (4) \quad N_{\Phi_{\hat{b}0}}^a &= -N_{\Phi_{0\hat{b}}}^a = \frac{\sqrt{-1}}{4} \Phi_{\hat{b},0}^a - \frac{\sqrt{-1}}{2} \Phi_{0,\hat{b}}^a, \quad (5) \quad N_{\Phi_{\hat{b}\hat{c}}}^a = \sqrt{-1} \Phi_{[\hat{b},\hat{c}]}^a = 2B^{abc}, \\ (6) \quad N_{\Phi_{bc}}^{\hat{a}} &= -\sqrt{-1} \Phi_{[b,c]}^{\hat{a}} = 2B_{abc}, \quad (7) \quad N_{\Phi_{b0}}^{\hat{a}} = -N_{\Phi_{0b}}^{\hat{a}} = \frac{\sqrt{-1}}{2} \Phi_{0,b}^{\hat{a}} - \frac{\sqrt{-1}}{4} \Phi_{b,0}^{\hat{a}}. \end{aligned} \quad (12)$$

The remaining components of this tensor are identically zero.

**Definition 6.** *We name the almost-contact-metric manifold  $M$  a constant type  $c$  manifold if*

$$\forall X, Y \in \mathcal{L} : \langle \langle X, Y \rangle \rangle = 0 \Rightarrow \|N_\Phi(X, Y)\|^2 = c \|X\|^2 \|Y\|^2, \quad (13)$$

where  $\langle \langle X, Y \rangle \rangle = \langle X, Y \rangle + \sqrt{-1} \langle X, \Phi Y \rangle$ ;  $X, Y \in \mathcal{L}$ ,  $\mathcal{L}$  – is a contact distribution.

**Theorem 8.** *An almost-contact-metric manifold  $M$  is a constant type  $c$  manifold only if*

$$B^{abh} B_{hcd} = \frac{c}{2} \delta_{cd}^{ab}, \quad (14)$$

where  $\delta_{cd}^{ab} = \delta_c^a \delta_d^b - \delta_c^b \delta_d^a$  is the second-order Kronecker delta.

**Proof.** For  $X, Y \in \mathcal{L}$  we have

$$\langle X, Y \rangle = g_{ij} X^i Y^j = g_{00} X^0 Y^0 + g_{a\hat{b}} X^a Y^{\hat{b}} + g_{\hat{a}b} X^{\hat{a}} Y^b = X^a Y_a + X_a Y^a$$

and

$$\langle X, \Phi Y \rangle = g_{ij} X^i \Phi_k^j Y^k = g_{ab} X^a \Phi_c^b Y^c + g_{ab} X^a \Phi_c^b Y^c = -\sqrt{-1} X^a Y_a + \sqrt{-1} X_a Y^a.$$

Thus,  $\langle \langle X, Y \rangle \rangle = \langle X, Y \rangle + \sqrt{-1} \langle X, \Phi Y \rangle = 0$  is equivalent to the execution of relations on the space of the attached G-structure:  $X^a Y_a + X_a Y^a = 0$ ,  $X^a Y_a - X_a Y^a = 0$ , which are performed if and only if  $X^a Y_a = 0$ . Consider equality  $\|N_\Phi(X, Y)\|^2 = c\|X\|^2\|Y\|^2$  in the space of the attached G-structure. On the one hand,

$$\begin{aligned} \|N_\Phi(X, Y)\|^2 &= \langle N_\Phi(X, Y), N_\Phi(X, Y) \rangle = \\ &= N_\Phi^a(X, Y) N_{\Phi_a}(X, Y) + N_{\Phi_a}(X, Y) N_\Phi^a(X, Y) = 2N_\Phi^a(X, Y) N_{\Phi^a}(X, Y), \end{aligned}$$

and taking into account (12), we obtain  $\|N_\Phi(X, Y)\|^2 = 2 \cdot 2B^{abc} X_b Y_c \cdot 2B_{abc} X^b Y^c = 8B^{abc} B_{abc} X_b Y_c X^b Y^c$ .

On the other hand,  $c\|X\|^2\|Y\|^2 = c \cdot 2X^a X_a \cdot 2Y^b Y_b = 4c(X^a X_a)(Y^b Y_b)$ , thus,  $B^{abc} B_{adh} X_b Y_c X^d Y^h = \frac{c}{2}(X^a X_a)(Y^b Y_b)$ .

We introduce the object into consideration  $\delta_{cd}^{ab} = \delta_c^a \delta_d^b - \delta_c^b \delta_d^a$ , then

$$\begin{aligned} \delta_{cd}^{ab} X^c Y^d X_a Y_b &= (\delta_c^a \delta_d^b - \delta_c^b \delta_d^a) X^c Y^d X_a Y_b = \\ &= X^a Y^b X_a Y_b - X^b Y^a X_a Y_b = (X^a X_a)(Y^b Y_b) - (X^b Y_b)(Y^a X_a). \end{aligned} \quad (15)$$

Therefore, the condition (13) on the space of the attached G-structure can be rewritten as follows:

$$\forall X, Y \in \mathcal{L} : X^a Y_a = 0 \Rightarrow B^{adh} B_{hbc} X_a Y_d X^b Y^c = \frac{c}{2} \delta_{bc}^{ad} X_a Y_d X^b Y^c.$$

Indicate

$$B_{bc}^{ad} = B^{adh} B_{hbc} - \frac{c}{2} \delta_{bc}^{ad},$$

We reduce the definition of type constancy to the following:

$$\forall X, Y \in \mathcal{L} : X^a Y_a = 0 \Rightarrow B_{bc}^{ad} X_a Y_d X^b Y^c = 0.$$

Then the complex tensor  $B(X, Y, Z, W) = B_{bc}^{ad} X^b Y^c Z_a W_d$ ,  $X, Y, Z, W \in \mathcal{X}(M)$  has the following properties:

- (1)  $B(\Phi X, Y, Z, W) = B(X, \Phi Y, Z, W) = \sqrt{-1} B(X, Y, Z, W)$ ;
- (2)  $B(X, Y, \Phi Z, W) = B(X, Y, Z, \Phi W) = -\sqrt{-1} B(X, Y, Z, W)$ ;
- (3)  $\overline{B(X, Y, Z, W)} = B(Z, W, X, Y)$ ;
- (4)  $B(X, Y, Z, W) = -B(Y, X, Z, W)$ ;
- (5)  $B(X, Y, Z, W) = -B(X, Y, W, Z)$ ;
- (6)  $B(X, Y, X, Y) = 0$ , if  $X^a Y_a = 0$ .

Involutional polarization of the property (6) gives  $B(X, Y, Z, W) = 0$ ,  $\forall X, Y, Z, W \in \mathcal{X}(M)$ . It follows that the tensor  $B = 0$ , i.e.,  $B^{adh} B_{hbc} - \frac{c}{2} \delta_{bc}^{ad} = 0$ .  $\square$

**Theorem 9.** *The almost-Hermitian structure induced on the contact distribution of the almost-contact-metric manifold of the zero constant type is the Hermitian structure.*

**Proof.** Assume that  $M$  is an almost-contact-metric manifold of a zero constant type. Then,  $B^{adh} B_{hbc} = 0$ . A complete convolution of this equality gives  $B^{adh} = B_{hbc} = 0$ . Therefore, the first group of structural equations of the manifold  $M$  takes the following form:



$$\begin{aligned}d\omega &= C_{ab}\omega^a \wedge \omega^b + C^{ab}\omega_a \wedge \omega_b + C_a^b\omega^a \wedge \omega_b + C_b\omega \wedge \omega^b + C^b\omega \wedge \omega^b; \\d\omega^a &= -\theta_b^a \wedge \omega^b + B^{ab}_c\omega^c \wedge \omega_b + B^{ab}\omega \wedge \omega_b + B^a_b\omega \wedge \omega^b; \\d\omega_a &= \theta_a^b \wedge \omega_b + B_{ab}^c\omega_c \wedge \omega^b + B_{ab}\omega \wedge \omega_b + B_a^b\omega \wedge \omega^b.\end{aligned}$$

Then the first group of structural equations of an almost-Hermitian structure induced on contact distribution of an almost-contact-metric manifold of a zero constant type will take the form

$$\begin{aligned}d\omega &= 0; \\d\omega^a &= -\theta_b^a \wedge \omega^b + B^{ab}_c\omega^c \wedge \omega_b; \\d\omega_a &= \theta_a^b \wedge \omega_b + B_{ab}^c\omega_c \wedge \omega^b.\end{aligned}$$

According to the Gray–Hervella classification of almost-Hermitian structures, written in the form of a table ([5], p. 450), we obtain that the almost-Hermitian structure induced on the integral submanifolds of the manifold  $M$  is Hermitian.  $\square$

Assume that  $M$  now is a nearly Sasakian manifold. Then Equation (12) will take the following form:

$$\begin{aligned}(1) \ N_{\Phi_{ab}}^0 &= -N_{\Phi_{ba}}^0 = -\frac{\sqrt{-1}}{2}\delta_b^a; \quad (2) \ N_{\Phi_{bc}}^a = \sqrt{-1}\Phi_{[b,c]}^a = 2C^{abc}; \\(3) \ N_{\Phi_{bc}}^{\hat{a}} &= -\sqrt{-1}\Phi_{[b,c]}^{\hat{a}} = 2C_{abc}.\end{aligned}\tag{16}$$

Identity (14) will take the following form:  $C^{abh}C_{hcd} = \frac{c}{2}\delta_{cd}^{ab}$ . Thus, the concepts of type constancy defined by Definitions 5 and 6 for  $\mathcal{NS}$ -manifolds coincide.

## 6. Conclusions

The article presents some interesting identities (3), which are satisfied by the structural tensors of the  $\mathcal{NS}$ -structure. The concept of type constancy for  $\mathcal{NS}$ -manifold is introduced, and criteria for pointwise type constancy for  $\mathcal{NS}$ -manifold are proved. It is also proved that the pointwise constancy of the type of a connected  $\mathcal{NS}$ -manifold of dimension greater than three is equivalent to the global constancy of its type. The class of  $\mathcal{NS}$ -manifolds of zero constant type coincides with the class of Sasakian manifolds. It is proved that the almost-Hermitian structure induced on the integral manifolds of the maximum dimension of the first fundamental distribution of the  $\mathcal{NS}$ -manifold is the nearly Kähler structure. A new concept of constancy of the type of an almost-contact-metric structure is introduced on the basis of its Nijenhuis tensor. The presented method allows us to study the constancy of the type of a wide class of these varieties. In particular, nearly cosymplectic, nearly trans-Sasakian, generalized Kenmotsu manifolds, and others. It is proved that the almost-Hermitian structure induced on the contact distribution of the almost-contact-metric manifold of the zero constant type is a Hermitian structure.

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