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Article

Certain New Class of Analytic Functions Defined by Using a Fractional Derivative and Mittag-Leffler Functions

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Abstract: Fractional calculus has a number of applications in the field of science, specially in mathematics. In this paper, we discuss some applications of fractional differential operators in the field of geometric function theory. Here, we combine the fractional differential operator and the Mittag-Leffler functions to formulate and arrange a new operator of fractional calculus. We define a new class of normalized analytic functions by means of a newly defined fractional operator and discuss some of its interesting geometric properties in open unit disk.

Keywords: analytic functions; fractional derivative operator; Mittag-Leffler function; convolution

MSC: 05A30; 30C45; 11B65; 47B38

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions η of the form

$$\eta(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the normalization condition

$$\eta(0) = \eta'(0) - 1 = 0.$$

Furthermore, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of functions of the form (1), which are also univalent in U.

For two functions $\eta, y \in \mathcal{A}$, we say that η subordinated to y, written as

$$\eta(z) \prec y(z),$$

or equivalently

$$\eta(z) = y(k(z)),$$

where, $k(z)$ is the Schwarz function in U along with the condition, (see [1])

$$k(0) = 0 \text{ and } |k(z)| < 1.$$

If y is univalent in U, then

$$\eta(z) \prec y(z) \Longleftrightarrow \eta(0) = y(0) \text{ and } \eta(U) \subset y(U).$$

The majorization of two analytic function $(\eta \ll y)$ if and only if

$$\eta(z) = k(z)y(z), \ \ z \in U,$$

and also the coefficient inequality is satisfied

$$|a_n| \leq |b_n|.$$

There exists a wide formation between the subordination and majorization [2] in U for established different classes including the the class of starlike functions $(\mathcal{S}^*)$:

$$\mathrm{Re}\left(\frac{z\eta'(z)}{\eta(z)}\right) > 0, \ \ z \in U$$

and convex functions $(\mathcal{C})$:

$$1 + \mathrm{Re}\left(\frac{z\eta''(z)}{\eta'(z)}\right) > 0, \ \ z \in U.$$

Related to classes $\mathcal{S}^*$ and $\mathcal{C}$, we define the class $\mathcal{P}$ of analytic functions $m \in \mathcal{P}$, which are normalized by

$$m(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

such that

$$\mathrm{Re}m(z) > 0 \text{ in } U \text{ and } \ m(0) = 1.$$

The convolution $(*)$ of η and y, defined by

$$(\eta * y)(z) = \sum_{n=0}^{\infty} a_n b_n z^n,$$

where,

$$y(z) = \sum_{n=0}^{\infty} b_n z^n, \qquad (z \in U).$$

Srivastava et al. [3] geometrically explored the class of complex fractional operators (differential and integral) and Ibrahim [4] provided the generality for a class of analytic functions into two-dimensional fractional parameters in U. Number of authors used these operators to illustrate various subclasses of analytic functions, fractional analytic functions and differential equations of complex variable [5–7].

Definition 1. *Pochhammer symbol $(\alpha)_n$ can be defined as:*

$$(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1) \ \ \textit{if} \ \ n \neq 0$$

and

$$(\alpha)_n = 1 \ \ \textit{if} \ \ n = 0.$$

Definition 2. *The $(\alpha)_n$ can be expressed in terms of the Gamma function as:*

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \ \ (n \in \mathbb{N}).$$

In [8], Mittag-Leffler introduced Mittag-Leffler functions $\mathcal{H}_{\alpha}(z)$ as:

$$\mathcal{H}_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+1)}z^{n},\ (\alpha\in\mathbb{C},\ \mathrm{Re}(\alpha))>0,$$

and its generalization $\mathcal{H}_{\alpha,\beta}(z)$ introduced by Wiman [9] as:

$$\mathcal{H}_{\alpha,\beta}(z)=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\alpha n+\beta)}z^{n},\ (\alpha,\ \beta\in\mathbb{C},\ \mathrm{Re}(\alpha),\ \mathrm{Re}(\beta))>0. \tag{2}$$

Now we define the normalization of Mittag-Leffler function $\mathcal{M}_{\alpha,\beta}(z)$ as follows:

$$\begin{aligned}\mathcal{M}_{\alpha,\beta}(z) &= z\Gamma(\beta)\mathcal{H}_{\alpha,\beta}(z)\\ \mathcal{M}_{\alpha,\beta}(z) &= z+\sum_{n=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)}z^{n},\end{aligned} \tag{3}$$

where, $z\in U$, $\mathrm{Re}\alpha>0$, $\beta\in\mathbb{C}\backslash\{0,-1,-2,\dots\})$.

A function $f\in\mathcal{A}$ is called bounded turning if it satisfies the condition

$$\mathrm{Re}\left(\eta^{'}(z)\right)>0.$$

For $0\leq v<1$, let $B(v)$ denote the class of functions η of the form (1), so that $\mathrm{Re}\left(\eta^{'}\right)>v$ in U. The functions in $B(v)$ are called functions of bounded turning (c.f. [1], Vol. II). Nashiro–Warschowski Theorem (see, e.g., [1], Vol. I) stated that the functions in $B(v)$ are univalent and also close-to-convex in U. Now recall the definition of class $\mathcal{R}$ of bounded turning functions and can be defined as:

$$\mathcal{R}=\left\{\eta\in\mathcal{A}:\eta^{'}(z)\prec\frac{1+z}{1-z},\quad z\in U\right\}.$$

In [3], Srivastava and Owa gave definitions for fractional derivative operator and fractional integral operator in the complex z-plane $\mathbb{C}$ as follows:

The fractional integral of order δ is defined for a function $\eta(z)$, by

$$I_{z}^{\delta}\eta(z)\equiv I_{z}^{-\delta}\eta(z)=\frac{1}{\Gamma(\delta)}\int_{0}^{z}(z-t)^{\delta-1}\eta(t)d(t),\ (\delta>0).$$

The fractional derivative operator D_z of order δ is defined by

$$\begin{aligned}D_{z}^{\delta}\eta(z) &= D_{z}I_{z}^{1-\delta}\eta(z)\\ &= \frac{1}{\Gamma(1-\delta)}D_{z}\int_{0}^{z}\frac{\eta(t)}{(z-t)^{\delta}}d(t),\quad (0\leq\delta<1).\end{aligned}$$

where, the function $\eta(z)$ is analytic in the simply-connected region of the complex z-plane $\mathbb{C}$ containing the origin, and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-t)$ to be real when $(z-t)>0$.

Let $\delta>0$ and m be the smallest integer, and the extended fractional derivative of $\eta(z)$ of order δ is defined as:

$$D_{z}^{\delta}\eta(z)=D_{z}^{m}I_{z}^{m-\delta}\eta(z),\ 0\leq\delta,\ n>-1, \tag{4}$$

provided that it exists. We find from (4) that is

$$D_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n-\delta}, \ (0 \leq \delta < 1, \ n > -1)$$

and

$$I_z^{\delta} z^n = \frac{\Gamma(n+1)}{\Gamma(n+1-\delta)} z^{n+\delta}, \ (0 < \delta, \ n > -1).$$

Owa and Srivastava [10], defined the differential integral operator $\Omega_z^{\delta} : \mathcal{A} \to \mathcal{A}$ in the term of series:

$$\begin{aligned} \Omega_z^{\delta} \eta(z) &= \frac{\Gamma(2-\delta)}{\Gamma(2)} z^{\delta} D_z^{\delta} \eta(z) \\ &= z + \sum_{n=2}^{\infty} \frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} a_n z^n, \end{aligned} \tag{5}$$

where,

$$(\delta < 2, \text{and } z \in U).$$

Here, $D_z^{\delta}\eta(z)$ represents the fractional integral of $\eta(z)$ of order δ when $-\infty < \delta < 0$ and a fractional derivative of $\eta(z)$ of order δ when $0 \leq \delta < 2$.

Now, by using the definition of convolution on (3) and (5), we define fractional differential integral operator $\mathfrak{D}_z^{\delta,\alpha,\beta} : \mathcal{A} \to \mathcal{A}$, associated with normalized Mittag-Leffler function $\mathcal{M}_{\alpha,\beta}(z)$ as follows:

$$\mathfrak{D}_z^{\delta,\alpha,\beta} \eta(z) = z + \sum_{n=2}^{\infty} \left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right) a_n z^n,$$

where,

$$(\delta < 2, \ \mathrm{Re}\alpha > 0, \ \beta \in \mathbb{C} \backslash \{0, -1, -2, \dots\}), \ z \in U.$$

It is noted that

$$\mathfrak{D}_z^{0,0,1} \eta(z) = \eta(z).$$

Again, by using fractional differential integral operator $\mathfrak{D}_z^{\delta,\alpha,\beta}$, we also define a linear multiplier fractional differential integral operator ${}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}$ as follows:

$${}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m} \eta(z) = {}_{\beta}^{\alpha} \Delta_{\lambda}^{\delta,1} \left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m-1} \eta(z) \right), \tag{6}$$

where,

$${}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,0} \eta(z) = \eta(z),$$

and

$${}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,1} \eta(z) = (1-\lambda) \mathfrak{D}_z^{\delta,\alpha,\beta} \eta(z) + \lambda z \mathcal{D} \left(\mathfrak{D}_z^{\delta,\alpha,\beta} \eta(z) \right).$$

It is seen from $\eta(z)$ given by (1) and from (6), we have

$${}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m} \eta(z) = z + \sum_{n=2}^{\infty} A(\lambda, \delta, \alpha, \beta, m, n) a_n z^n, \tag{7}$$

where,

$$A(\lambda, \delta, \alpha, \beta, m, n) = \left[\left(\frac{\Gamma(2-\delta)\Gamma(n+1)}{\Gamma(2)\Gamma(n+1-\delta)} \right) \left(\frac{\Gamma(\beta)}{\Gamma(\alpha(n-1)+\beta)} \right) (1-\lambda+n\lambda) \right]^m$$

and

$$(\delta < 2, \ m \in \mathbb{N}, \ \lambda \geq 0, \ \mathrm{Re}\alpha > 0, \ \beta \in \mathbb{C} \backslash \{0, -1, -2, \dots\}), \ z \in U.$$

Remark 1. *When, $\delta = 0$, $\alpha = 0$, and $\beta = 1$, in (7) then it is reduced to the operator given by Al-Oboudi [11].*

Remark 2. *For, $\delta = 0$, $\lambda = 1$, $\alpha = 0$, and $\beta = 1$ in (7) then it is reduced to the operator given by Salagean [12].*

Definition 3. *A function $\eta \in \mathcal{A}$, is in the class ${}^{\beta}_{\alpha}\mathcal{S}^{*\delta,m}_{\lambda}(\sigma)$ if and only if*

$$ {}^{\beta}_{\alpha}\mathcal{S}^{*\delta,m}_{\lambda}(\sigma) = \left\{ \eta \in \mathcal{A} : \frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)} \prec \sigma(z), \ \ \sigma(0) = 1 \right\}. $$

Definition 4. *A function $\eta \in \mathcal{A}$, is in the class ${}^{\alpha}_{\beta}J^{\delta,m}_{\lambda}(L, M, b)$ if and only if*

$$ {}^{\alpha}_{\beta}J^{\delta,m}_{\lambda}(L, M, b) = \left\{ \eta \in \mathcal{A} : 1 + \frac{1}{b}\left(\frac{2\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z) - {}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(-z)} \right) \prec \frac{1+Lz}{1+Mz} \right\}. $$

The following lemmas will be use to prove our main results.

Lemma 1 ([13])**.** *For $\varrho \in \mathbb{C}$ and a positive integer n, the class of analytic functions is given by*

$$ \mathcal{H}(\eta, n) = \left\{ \eta : \eta(z) = \varrho + \varrho_n z^n + \varrho_{n+1} z^{n+1} + \dots \right\}. $$

(i) Let $l \in \mathbb{R}$. Then

$$ Re\left(\eta(z) + lz\eta'(z)\right) > 0 \longrightarrow Re(\eta(z)) > 0. $$

Moreover, $l > 0$ and $\eta \in \mathcal{H}(1, n)$, then there is constant $\delta > 0$ and $k > 0$, such that

$$ k = k(l, \delta, n) $$

and

$$ \eta(z) + lz\eta'(z) \prec \left(\frac{1+z}{1-z}\right)^k \longrightarrow \eta(z) \prec \left(\frac{1+z}{1-z}\right)^\delta. $$

(ii) For $\eta \in \mathcal{H}(1, n)$, and for fixed real number $l > 0$ and let $c \in [0, 1)$, so that

$$ Re\left(\eta^2(z) + 2\eta(z)(zD\eta(z))\right) > c \longrightarrow Re(\eta(z)) > l. $$

(iii) Let $\eta \in \mathcal{H}(\eta, n)$, with $Re(\eta) > 0$, then

$$ Re\left(\eta(z) + z\eta'(z) + z^2\eta''(z)\right) > 0, $$

or for $\mathbb{N} : U \to \mathbb{R}$, such that

$$ Re\left(\eta(z) + \left(\frac{z\eta'(z)}{\eta(z)}\right)\mathbb{N}(z)\right) > 0. $$

Then

$$ Re(\eta(z)) > 0. $$

2. Main Results

To make use of Lemma 1, first of all, we illustrate differential integral operator ${}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)$ is also bounded turning function.

Theorem 1. *Let $\eta \in A$, and*

$$(i)\ {}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z) \text{ is of bounded turning function.}$$

$$(ii)\ \left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)' \prec \left(\frac{1+z}{1-z}\right)^{k},\ k > 0,\ z \in U.$$

$$(iii)\ Re\left(\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right)\right) > \frac{c}{2},\ c \in [0,1).$$

$$(iv)\ Re\left(z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'' - \left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)' + 2\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right)\right) > 0.$$

$$(v)\ Re\left(\left(z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)' / {}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right) + 2\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)/z\right)\right) > 1.$$

Then

$$\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right) \in \mathcal{P}(\lambda), \text{ for some } \lambda \in [0,1).$$

Proof. Define a function $m(z)$ as follows:

$$m(z) = \frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}, \quad z \in U. \tag{8}$$

Then computation implies that

$$zm'(z) + m(z) = \left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'.$$

From the first inequality (i), we have ${}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)$ is bounding turning function, and this give us

$$\text{Re}\left(zm'(z) + m(z)\right) > 0.$$

Thus, Lemma 1, part (i) implies that

$$\text{Re}(m(z)) > 0.$$

Hence (i) is proved. Accordingly, part (ii) is confirmed.

By the virtue of Lemma 1 and part (i), let $l > 0$, such that $k = k(l)$ and

$$\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z} \prec \left(\frac{1+z}{1-z}\right)^{l}.$$

This indicates that

$$\text{Re}\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right) > \lambda, \quad \lambda \in [0,1).$$

Suppose that

$$\begin{aligned} &\text{Re}\left(m^2(z) + 2m(z).zm'(z)\right) \\ = \quad &2\text{Re}\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\left(\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)' - \frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{2z}\right)\right) > c,\ c \in [0,1). \end{aligned} \tag{9}$$

From the Lemma 1 and part (ii), there exists a fixed real number $l > 0$ and satisfying the condition
$$\mathrm{Re}(m(z)) > l$$
and
$$m(z) = \frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z} \in \mathcal{P}(\lambda).$$
It follows from (9) that
$$\mathrm{Re}\Big(\Big({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\Big)\Big)' > 0.$$
Taking the derivative (8), we then obtain
$$\begin{aligned} &\mathrm{Re}\Big(m(z) + zm'(z) + z^2m''(z)\Big) \\ = \quad &\mathrm{Re}\left(z\Big({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\Big)'' - \Big({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\Big)' + 2\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right)\right) > 0. \end{aligned}$$
Hence, Lemma 1 (ii) implies that
$$\mathrm{Re}\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right) > 0.$$
The logarithmic differentiation of (8) yields
$$\begin{aligned} &\mathrm{Re}\left(m(z) + \frac{zm'(z)}{m(z)} + z^2m''(z)\right) \\ = \quad &\mathrm{Re}\left(\frac{z\Big({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\Big)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{q,\lambda}\eta(z)} + 2\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right) - 1\right) > 0. \end{aligned}$$
Hence, Lemma 1 (iii) implies, where $N(z) = 1$,
$$\mathrm{Re}\left(\frac{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}{z}\right) > 0.$$
□

Now we find the upper bounds of the operator ${}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)$ by using the exponential integral in U, which provided $\eta \in \Big({}^{\beta}_{\alpha}\mathcal{S}^{*\delta,m}_{\lambda}(\sigma)\Big)$.

Theorem 2. *Let $\eta \in \Big({}^{\beta}_{\alpha}\mathcal{S}^{*\delta,m}_{\lambda}(\sigma)\Big)$, where $\sigma(z)$ is convex in U. Then,*
$$ {}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z) \prec z\exp\int_0^z \frac{\sigma(\phi(w)) - 1}{w}dw, \tag{10}$$
where, $\phi(z)$ is analytic in U having condition
$$\phi(0) = 0 \text{ and } |\phi(z)| < 1.$$

Furthermore, for $|z| = \xi$, we have

$$\exp \int_0^1 \frac{\sigma(\phi(-\xi)) - 1}{w} d\xi \leq \left| \frac{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)}{z} \right| \leq \exp \int_0^1 \frac{\sigma(\phi(\xi)) - 1}{w} d\xi.$$

Proof. By the hypothesis we received the following conclusion:

$$\begin{aligned} \frac{z\left({}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)} &\prec \sigma(z) \\ \frac{z\left({}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)} &= \sigma(\phi(z)),\ z \in U, \end{aligned}$$

and

$$\frac{\left({}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)} - \frac{1}{z} = \frac{\sigma(\phi(z)) - 1}{z}. \tag{11}$$

Consequently, integrating (11), we obtain

$$\log\left(\frac{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)}{z}\right) = \int_0^z \frac{\sigma(\phi(w) - 1}{w} dw. \tag{12}$$

By the definition of subordination we attain

$${}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z) \prec z \exp \int_0^z \frac{\sigma(\Psi(w) - 1}{w} dw.$$

Hence (10) is proved.

Note that the function $\sigma(z)$ convex and symmetric with respect to real axis. That is

$$\sigma(-\zeta|z|) \leq \mathrm{Re}\{\sigma(\Psi(\xi z)\} \leq \sigma(\xi|z|) \quad (0 < \xi < 1,\ z \in U),$$

then we have the inequalities

$$\sigma(-\xi) \leq \sigma(-\xi|z|),\ \sigma(\xi|z|) \leq \sigma(\xi).$$

Consequently, we obtain

$$\int_0^1 \frac{\sigma(\Psi(-\xi|z|)) - 1}{\xi} d\xi \leq \mathrm{Re} \int_0^1 \frac{\sigma(\Psi(\xi)) - 1}{\xi} d\xi \leq \int_0^1 \frac{\sigma(\Psi(\xi|z|)) - 1}{\xi} d\xi.$$

In the sight of Equation (12), we obtain

$$\int_0^1 \frac{\sigma(\Psi(-\xi|z|)) - 1}{\xi} d\xi \leq \log\left|\frac{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)}{z}\right| \leq \int_0^1 \frac{\sigma(\Psi(\xi|z|)) - 1}{\xi} d\xi.$$

which implies that

$$\exp \int_0^1 \frac{\sigma(\Psi(-\xi|z|)) - 1}{\xi} d\xi \leq \left|\frac{{}^{\alpha}_{\beta}\Delta_{\lambda}^{\delta,m}\eta(z)}{z}\right| \leq \exp \int_0^1 \frac{\sigma(\Psi(\xi|z|)) - 1}{\xi} d\xi.$$

Hence, we have

$$\exp\int_0^1 \frac{\sigma(\Psi(-\xi))-1}{\xi}d\xi \le \left|\frac{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}{z}\right| \le \exp\int_0^1 \frac{\sigma(\Psi(\xi))-1}{\xi}d\xi.$$

□

Now we investigate the sufficient condition of η to be in the class ${}_{\beta}^{\alpha}\mathcal{S}_{\lambda}^{*,\delta,m}(\sigma)$, where σ is convex univalent satisfying $\sigma(0)=1$.

Theorem 3. *If $\eta \in A$, satisfies the inequality*

$$\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\left(2+\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)''}{\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}\right)-\left(\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\right)\prec\sigma(z),$$

then, $\eta \in {}_{\beta}^{\alpha}\,\mathcal{S}_{\lambda}^{*,\delta,m}(\sigma)$.

Proof. Let

$$m(z)=\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}$$

and $m(z)=1$ in the inequality

$$m(z)+m(z)\left(zm'(z)\right)\prec\sigma(z),$$

then, we obtain

$$\begin{aligned}&m(z)+m(z)\left(zm'(z)\right)\\=\;&\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\times\left(2+\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)''}{\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}-\left(\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\right)\right)\prec\sigma(z).\end{aligned}$$

This implies that

$$m(z)=\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\prec\sigma(z),$$

that is

$$\eta\in\left({}_{\beta}^{\alpha}\mathcal{S}_{\lambda}^{*,\delta,m}(\sigma)\right).$$

□

Corollary 1. *Let the assumption of Theorem 3. Then,*

$$\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\times\left(1+\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)''}{\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}\right)-\left(\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}\right)\ll\sigma'(z).$$

Proof. Let

$$m(z)=\frac{z\left({}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)\right)'}{{}_{\beta}^{\alpha}\Delta_{\lambda}^{\delta,m}\eta(z)}.$$

In the view of Theorem 3, we have

$$\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)} \prec \sigma(z),$$

where, $\sigma \in \mathbb{C}$. Then, by [2] (Theorem 3), we obtain

$$m'(z) \ll \sigma'(z)$$

for some $z \in U$, where

$$m'(z) = \frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}\left(1+\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)''}{\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}\right)-\left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}\right).$$

□

It is well known that the function $\sigma(z) = e^{\theta z}, 1 < |\theta| \leq \frac{\pi}{2}$ is not convex in U, where the domain $\sigma(U)$ is lima-bean (see [13], p. 123). Now, we can find the same result of Theorem 3 as follows:

Theorem 4. *If $\eta \in A$, it satisfies the inequality*

$$1+\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)''}{\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'} \prec e^{\theta z}.$$

Then,

$$\eta \in \left({}^{\alpha}_{\beta}\mathcal{S}^{*,\delta,m}_{\lambda}(e^{\theta z})\right).$$

Proof. Let

$$m(z) = \frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}.$$

After some simple computation implies that

$$\begin{aligned}
& m(z)+\frac{zm'(z)}{m(z)} \\
= \quad & \left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}\right)+\frac{\left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}\right)\left(1+\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)''}{\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}-\left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}\right)\right)}{\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}} \\
= \quad & \left(1+\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)''}{\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}\right) \prec e^{\theta z}.
\end{aligned}$$

This implies that (see [13], p. 123)

$$m(z) = \frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)} \prec e^{\theta z},$$

that is

$$\eta \in \left({}^{\alpha}_{\beta}\mathcal{S}^{*,\delta,m}_{\lambda}(e^{\theta z})\right).$$

□

Theorem 5. *If* $\eta \in \left({}^{\alpha}_{\beta}J^{\delta,m}_{\lambda}(L,M,b)\right)$*, then*

$$\mathcal{M}(z) = \frac{1}{2}[\eta(z) - \eta(-z)]$$

satisfies

$$1 + \frac{1}{b}\left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\mathcal{B}(z)\right)'}{\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\mathcal{B}(z)\right)}\right) \prec \frac{1+Lz}{1+Mz},$$

$$Re\left(\frac{z\mathcal{B}'(z)}{\mathcal{B}(z)}\right) \geq \frac{1-\vartheta^2}{1+\vartheta^2}, \quad |z| = \vartheta < 1.$$

Proof. Let $\eta \in \left({}^{\alpha}_{\beta}J^{\delta,m}_{\lambda}(L,M,b)\right)$, then there occurs a function $J(z)$ such that

$$\begin{aligned} b(J(z)-1) &= \frac{2z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z) - {}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(-z)}, \\ b(J(-z)-1) &= \frac{2z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(-z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{q,\lambda}\eta(-z) - {}^{\alpha}_{\beta}\Delta^{\delta,m}_{q,\lambda}\eta(z)}. \end{aligned}$$

This confirm that

$$1 + \frac{1}{b}\left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\mathcal{G}(z)\right)'}{\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\mathcal{G}(z)\right)} - 1\right) = \frac{J(z) + J(-z)}{2}.$$

However, J satisfies

$$J(z) \prec \frac{1+Lz}{1+Mz},$$

which is univalent, then we get

$$1 + \frac{1}{b}\left(\frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\mathcal{G}(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\mathcal{G}(z)} - 1\right) \prec \frac{1+Lz}{1+Mz}.$$

Additionally, $\mathcal{G}(z)$ is starlike in z, and which implies that

$$h(z) = \frac{z\mathcal{G}(z)'}{\mathcal{G}(z)} \prec \frac{1-z^2}{1+z^2}.$$

Hence, their exist a Schwarz function $w(z)$, such that $|w(z)| \leq |z| < 1$, $k(0) = 0$, we get

$$h(z) \prec \frac{1-w(z)^2}{1+w(z)^2},$$

which leads to

$$w(\zeta)^2 = \frac{1-h(\zeta)}{1+h(\zeta)}, \;\; \zeta \in z, \;\; |\zeta| = r < 1.$$

A simple calculation yields

$$\left|\frac{1-h(\zeta)}{1+h(\zeta)}\right| = |w(\zeta)|^2 \leq |\zeta|^2.$$

Therefore, we get the following inequalities:

$$\begin{aligned}\left|h(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\right|^2 &\leq \frac{4|\zeta|^4}{\left(1-|\zeta|^4\right)^2}, \\ \left|h(\zeta) - \frac{1+|\zeta|^4}{1-|\zeta|^4}\right| &\leq \frac{2|\zeta|^2}{\left(1-|\zeta|^4\right)}.\end{aligned}$$

Thus, we have

$$\mathrm{Re}\left(\frac{z\mathcal{G}'(z)}{\mathcal{G}(z)}\right) \geq \frac{1-\vartheta^2}{1+\vartheta^2}, \quad |\zeta| = \vartheta < 1.$$

This completes the proof of Theorem 5. □

Example 1. *Let*

$$\begin{aligned}\frac{z\eta'(z)}{\eta(z)} &= \frac{z\left({}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)\right)'}{{}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z)}, \\ {}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z) &= \frac{z}{(1-z)^2}, \ \eta \in \mathcal{A}.\end{aligned}$$

Then the solution of $\frac{z\eta'(z)}{\eta(z)} = \frac{1+z}{1-z}$ *is formulated as follows:*

$${}^{\alpha}_{\beta}\Delta^{\delta,m}_{\lambda}\eta(z) = \frac{z}{(1-z)^2}, \ \ \eta \in \mathcal{A}.$$

Moreover, the solution of the equation

$$\eta(z) + \frac{z\eta'(z)}{\eta(z)} = \frac{1+z}{1-z}$$

is approximated to

$$\eta(z) = \frac{z}{1-z}.$$

3. Conclusions

Many researchers have discussed some applications of fractional differential operator in different areas of mathematics. In this paper, we combined fractional differential operator and the Mittag-Leffler functions and formulated a new operator of fractional calculus for a class of normalized functions in the open unit disk. We considered this operator on the two classes of analytic functions and investigated some of its applications in the field of geometric function theory. The suggested operator can be utilized to define some more classes of analytic functions or to generalize other types of differential operators.

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