

Article

Six-Dimensional Space with Symmetric Signature and Some Properties of Elementary Particles

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Abstract: The six-dimensional pseudo-Euclidean space $\mathbb{E}_{3,3}$ with signature (3,3) is proposed as a model of real physical space at the subparticle scale. The conserved quantum characteristics of elementary particles, such as spin, isospin, electric and baryon charges, and hypercharge, are expressed through the symmetries of this space. The symmetries are brought out by the various representation of the metric in $\mathbb{E}_{3,3}$ with the aid of spinors and hyperbolic complex numbers. The properties of the metric allow predicting the number of quarks equal to 18. The violation of strong conservation laws in weak interactions is treated through compactifying the three-dimensional temporal subspace at the subparticle scale into single-dimensional time at bigger scales, which reduces symmetry from the spherical to axial type.

Keywords: pseudo-Euclidean space; hidden groups of metric motion; spinors; hyperbolic complex numbers; hyperbolic unitary operators

MSC: 81P16; 81Q65; 81V05; 81V15



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1. Introduction

In the theory of grand unification based on the structural group $SU(3) \times SU(2) \times U(1)$, the mathematical number of space dimensions is equal to 11. In this case, four dimensions refer to real physical space–time, whereas the remaining seven correspond to some abstract space [1], within the framework of which one can introduce such quantum characteristics of elementary particles as isospin, hypercharge, colour, flavour, etc. The number of dimensions of that additional abstract space may increase as ever new conserved quantum characteristics of elementary particles are discovered. This scheme of development of the theory is rather simple; however, one cannot get rid of the feeling that it is ad hoc and strained.

There were several attempts to introduce additional dimensions: supermembranes [2,3], extending M-theory [4], F-theory [5,6], strings [7], 12D super Yang–Mills and supergravity theories for constant lightlike vectors [8,9]. An elaborate theory is proposed in [10] and then developed in respect to elementary particles in [11] and gravity [12]. The efforts of introducing additional dimensions in various ways were presented in the review [13].

From the literature, one can see that the standard theory itself, as well as most of alternative ways introduce many additional dimensions to form a mathematical structure, which is able to support necessary quantum characteristics. Even if the increase in dimensions is moderate as in [10], the proposed modifications seem arbitrary from the mathematical point of view and look more like a fitting to explain certain features.

Our key idea here is to recall the basic principle of relativity theory: the fundamental equality in treatment of space and time, which is reflected in the formula of the invariant interval:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 \quad (1)$$

in the system where $c \equiv 1$. So far, t was perceived as a single-dimensional time and x was a three-dimensional space, giving

$$\Delta s^2 = \Delta t^2 - \Delta x_1^2 - \Delta x_2^2 - \Delta x_3^2. \tag{2}$$

Despite the declared principle, time and space are no longer handled equally in (2). We propose to establish the equality truly, to obtain a really symmetric scheme, where each spatial dimension has its temporal counterpart:

$$\Delta s^2 = \sum_{i=1}^n \Delta t_i^2 - \sum_{i=1}^n \Delta x_i^2. \tag{3}$$

It turns out that $n = 3$ is sufficient to construct a space, the symmetry groups of which describe known quantum characteristics.

In [14], the six-dimensional manifold of symmetric signature (3,3) and the special metric is proposed to build a unified theory of gravity and electromagnetism. Here, we consider its typical tangential layer, which is pseudo-Euclidean space $\mathbb{E}_{3,3}$, as being the real physical space–time in the microcosm.

The article has the following structure. At first, $\mathbb{E}_{3,3}$ is considered as an image of spinor space, and its hidden groups of proper motions are examined. It is shown that these motions constitute the $SU(4)$ group. Taking this, the conserved quantum characteristics generated by $SU(4)$ are derived in a standard way. The simplest non-trivial representation of $SU(4)$ is built as an example. It is shown that starting from $\mathbb{E}_{3,3}$, one can obtain known conserved quantum characteristics and quarks in a mathematically rigorous way. Then, the representations of $\mathbb{E}_{3,3}$ through hyperbolic numbers are considered. The properties of the metric in this space allow predicting the number of quarks equal to 18. The appropriate scheme is presented. Finally, the question of a reduction from $\mathbb{E}_{3,3}$ space to our usual $\mathbb{E}_{1,3}$ space is examined. It is assumed that the world time flows along a certain axis in the three-dimensional temporal subspace of the six-dimensional space. For physical processes in very small time intervals, the temporal subspace is isotropic with fair accuracy. However, for a larger time duration, the presence of a certain time axis leads to the violation of spherical symmetry. Some symmetries are lost, which leads to the violation of conservation laws for a bigger time scale. This behaviour can be an explanation of the violation of some laws in weak interactions (weak decay time is above 10^{-13} s [15]; typical weak interaction time is 10^{-10} s [16]), while these laws hold in strong interactions (strong decay time is around 10^{-23} s [15]).

2. Pseudo-Euclidean Space $\mathbb{E}_{3,3}$ as an Image of the Spinor Space

Consider the pseudo-Euclidean space $\mathbb{E}_{3,3}$, and let η_{ij} designate the metric there, i.e.,

$$\eta_{ij} = \begin{cases} 0, & i \neq j, \\ +1, & i = j = 1, 2, 3, \\ -1, & i = j = 4, 5, 6. \end{cases} \tag{4}$$

The squared interval in $\mathbb{E}_{3,3}$ is given by

$$s^2 = \eta_{kl} x^k x^l, \quad k, l = 1, \dots, 6, \tag{5}$$

where $\vec{x} = (x_1, \dots, x_6)$ is a vector in $\mathbb{E}_{3,3}$, and summation is meant by the same upper and lower indices. The group of proper motions of metric (5) in $\mathbb{E}_{3,3}$ is given by a group of proper pseudo-Euclidean rotations $SO(3,3)$.

Let us now introduce the four-dimensional complex space \mathbb{C}^4 , whose elements are the four-component complex vectors, called spinors $\vec{\zeta} = (\zeta_1, \dots, \zeta_4)$, and the space \mathbb{C}^4 itself is referred to as spinor space [17]. Now, denote $t^1 = x^4, t^2 = x^5, t^3 = x^6$.

For any vector $\vec{x} \in \mathbb{E}_{3,3}$, one can find such a spinor $\vec{\zeta} \in \mathbb{C}^4$ such that the following will hold:

$$\begin{aligned} t^1 &= \zeta^1 \bar{\zeta}^2 + \bar{\zeta}^1 \zeta^2, \quad t^2 = \frac{\zeta^1 \bar{\zeta}^2 - \bar{\zeta}^1 \zeta^2}{i}, \quad t^3 = \zeta^1 \bar{\zeta}^1 - \zeta^2 \bar{\zeta}^2, \\ x^1 &= \zeta^3 \bar{\zeta}^4 + \bar{\zeta}^3 \zeta^4, \quad x^2 = \frac{\zeta^3 \bar{\zeta}^4 - \bar{\zeta}^3 \zeta^4}{i}, \quad x^3 = \zeta^3 \bar{\zeta}^3 - \zeta^4 \bar{\zeta}^4, \end{aligned} \tag{6}$$

where $\bar{\zeta}^\mu$ is the complex conjugate to the ζ^μ spinor component and i is the imaginary unit. Formula (6) may be rewritten in a more elegant form. To do so, consider the complex matrix algebra $M(4, \mathbb{C})$ in spinor space \mathbb{C}^4 . In this algebra, let us choose the matrices

$$\hat{\sigma}^p = \begin{cases} \begin{pmatrix} \sigma^p & 0 \\ 0 & 0 \end{pmatrix}, & p = 1, 2, 3, \\ \begin{pmatrix} 0 & 0 \\ 0 & \sigma^{p-3} \end{pmatrix}, & p = 4, 5, 6, \end{cases} \tag{7}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{8}$$

are Pauli matrices. The matrices (7) form the six-dimensional basis in the subalgebra L of the algebra $M(4, \mathbb{C})$. The following commutation relationships take place:

$$\hat{\sigma}^k \hat{\sigma}^l - \hat{\sigma}^l \hat{\sigma}^k = 2i \hat{\sigma}^m \varepsilon_{klm}, \tag{9}$$

where $k, l, m = 1, \dots, 6$, ε_{klm} is the completely antisymmetric tensor, product $\hat{\sigma}^k \hat{\sigma}^l$ is zero if indices of the pair belong to different triplets ($k = 1, 2, 3; l = 4, 5, 6$), and

$$\hat{\sigma}^k \hat{\sigma}^l + \hat{\sigma}^l \hat{\sigma}^k = 2\delta^{kl} p^m, \tag{10}$$

where p^m is the two-dimensional orthogonal projector in the spinor space \mathbb{C}^4 . At that, if $k, l = 1, 2, 3$, then $m = 2$; if $k, l = 4, 5, 6$ then $m = 1$, $p^1(\mathbb{C}^4) = \{\vec{\zeta} \in \mathbb{C}^4; \vec{\zeta} = (\zeta^1, \zeta^2, 0, 0)\}$, $p^2(\mathbb{C}^4) = \{\vec{\zeta} \in \mathbb{C}^4; \vec{\zeta} = (0, 0, \zeta^3, \zeta^4)\}$.

According to the relationships (8), (9), and (10), the Lie algebra (7) is reducible. To each pair $(\hat{\sigma}^m, \vec{\zeta}) \in M(4, \mathbb{C}) \times \mathbb{C}^4$ there is a corresponding m -th coordinate of the vector $\vec{x} \in \mathbb{E}_{3,3}$ according to the formula:

$$x^m = \langle \vec{\zeta}, \hat{\sigma}^m \vec{\zeta} \rangle, \quad m = 1, \dots, 6, \tag{11}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{C}^4 , given by $\langle \vec{\zeta}, \vec{\eta} \rangle = \delta_{\nu\mu} \zeta^\nu \eta^\mu$.

One can readily notice that the representations (6) and (11) are equivalent. Thus, the material coordinates of the vectors of pseudo-Euclidean space $\mathbb{E}_{3,3}$ can be represented as the average values of the Hermitian operators of the form (7) upon spinors in \mathbb{C}^4 space.

3. Hidden Groups of Proper Motions of the Metric

From Formula (11), it follows that, if an arbitrary vector $\vec{x} \in \mathbb{E}_{3,3}$ with coordinates (x^1, \dots, x^6) is set, then in the general case, there exists a pair $(\hat{\sigma}_m, \vec{\zeta}) \in M(4, \mathbb{C}) \times \mathbb{C}^4$, determinable to within the unitary equivalence with respect to the group $SU(4)$, and the relation (11) holds. This means that the pseudo-Euclidean metric (5) in spinor space \mathbb{C}^4 is invariant relative to the action of the group $SU(4)$, which may be regarded as the hidden proper motions of this metric. Before studying the relation of the group $SU(4)$ with the conserved quantum characteristics of elementary particles, we consider simpler groups, leading us to conservation laws of such characteristics as spin, “weak” isospin, electric charge, and “weak” hypercharge.

Consider the two-parameter group of unitary transformations $U(1) \times U(1)$, which is represented in $M(4, \mathbb{C})$ as unitary matrices of the following kind:

$$\begin{pmatrix} e^{i\varphi} & 0 & 0 & 0 \\ 0 & e^{i\varphi} & 0 & 0 \\ 0 & 0 & e^{i\psi} & 0 \\ 0 & 0 & 0 & e^{i\psi} \end{pmatrix}, \tag{12}$$

and which operates in \mathbb{C}^4 . The transformations of the group $U(1) \times U(1)$ leave the right-hand sides of the relations (6) invariant, i.e., the coordinates of the vectors in $\mathbb{E}_{3,3}$ remain unchanged under such transformations. Therefore, the metric (5) itself of the space $\mathbb{E}_{3,3}$ remains invariant. We will say that the transformations of the kind (12) of the group $U(1) \times U(1)$ represent hidden motions of the metric (5). Group $U(1) \times U(1)$ generates two conservation laws. The first law generated by the operator $\frac{1}{i} \frac{\partial}{\partial \varphi}$ will be interpreted as the law of conservation of “weak” hypercharge. The second law, induced by the generator of the group $\frac{1}{i} \frac{\partial}{\partial \psi}$ will be interpreted as the law of conservation of the electric charge. The appearance of the “weak” hypercharge conservation law is related to the hidden symmetries in the three-dimensional temporal subspace, which for the sake of brevity will be referred to as isospace.

Let us now proceed to the consideration of the more complicated unitary group. The representation of the unitary group with the Lie algebra, which is determined by the generators (7) and which operates in the space \mathbb{C}^4 , is quite reducible and may be expressed in the form of the direct sum of irreducible representations $SU(2) \times SU(2)$. Each of the irreducible representations corresponds to the group $SU(2)$ of unitary unimodular matrices U of dimension two, i.e., $U^\dagger U = 1, \det U = 1$. In the case of the first irreducible representation, such matrices may be represented in the form $U = e^{i\sigma_k a_k}, k = 1, 2, 3$, where σ_k are the Hermitian Pauli matrices and a_k are arbitrary real numbers. These matrices implement the identical representation of dimension two in the two-dimensional isospin space $p^1(\mathbb{C}^4)$ with elements $\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ put over the two basis spinors $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. In the case of the second irreducible representation, we will obtain the similar group of unitary unimodular matrices $U = e^{i\sigma_l a_l}, l = 4, 5, 6$, where the same Pauli matrices σ_l appear as generators according to the relation (7). These matrices realize a group of dimension two in the two-dimensional spinor space $p^2(\mathbb{C}^4)$ with elements $\begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix}$ put over two basis spinors. Thus, the isospace is the spin space related to the three-dimensional temporal subspace. Therefore, such an important characteristic of elementary particles as weak isospin has a pure geometric nature, and its conservation law is related to the invariance of the metric of the six-dimensional space $\mathbb{E}_{3,3}$ with respect to the group of rotations in the three-dimensional temporal subspace.

4. Group $SU(4)$ and the Conserved Quantum Characteristics It Generates

It has been shown above that the group $SU(4)$ leaves the metric of space $\mathbb{E}_{3,3}$ invariant. Let us now proceed to study in more detail the properties of the group $SU(4)$ and its Lie algebra for the necessity of giving their physical interpretations. The gen-

erators $\lambda_i, i = 1, \dots, 15$ of the Lie algebra may be represented as 15 traceless Hermitian four-dimensional matrices:

$$\begin{aligned}
 \lambda_1 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 \lambda_4 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_5 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\
 \lambda_7 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \lambda_9 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\
 \lambda_{10} &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \lambda_{11} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \lambda_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \\
 \lambda_{13} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \lambda_{14} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \lambda_{15} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{pmatrix}.
 \end{aligned} \tag{13}$$

We note that the Lie algebra for the group $SU(4)$ contains the Gell-Mann subalgebra for the group $SU(3)$, assigned by the generators $\lambda_1, \dots, \lambda_8$, as well as the Pauli subalgebra for the group $SU(2)$, assigned by generators $\lambda_1, \dots, \lambda_3$. Let us introduce the following designations:

$$\begin{aligned}
 F_i &= \frac{1}{2} \lambda_i, \quad i = 1, \dots, 15, \\
 I_{\pm} &= F_1 \pm iF_2, \quad I_3 = F_3, \\
 V_{\pm} &= F_4 \pm iF_5, \quad U_{\pm} = F_6 \pm iF_7, \quad Y = \frac{2}{\sqrt{3}} F_8 \\
 N_{\pm} &= F_9 \pm iF_{10}, \quad M_{\pm} = F_{11} \pm iF_{12}, \quad W_{\pm} = F_{13} \pm iF_{14}, \quad B = \frac{4}{\sqrt{6}} F_{15},
 \end{aligned} \tag{14}$$

where I_{\pm} are the raising and lowering operators for isospin projection, Y is the Hermitian hypercharge operator, and B is the Hermitian baryon charge operator. Among the operators given by (13), there is not one for the electrical charge Q . It can be assigned by the following Hermitian traceless matrix:

$$Q = \frac{1}{3} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

Then, the relation:

$$Q = I_3 + \frac{Y}{2}, \tag{16}$$

takes place, which was discovered phenomenologically by Gell-Mann [18] and Nishijima [19].

Hermitian operators I_3, Y, Q , and B satisfy the following commutation relations:

$$[I_3, Y] = [I_3, Q] = [I_3, B] = [Y, Q] = [Y, B] = [Q, B] = 0, \tag{17}$$

i.e., all four operators commute. This means that the physical characteristics of elementary particles given by the eigenvalues of these operators are simultaneously observable. Let us now find the expressions to describe the relationship of the commuting Hermitian

operators. We write out the commutators for the operators of creation and annihilation given by (14):

$$\begin{aligned}
 [I_+, I_-] &= 2I_3, \quad [V_+, V_-] = I_3 + \frac{3}{2}Y, \quad [U_+, U_-] = -I_3 + \frac{3}{2}Y \\
 [N_+, N_-] &= I_3 + \frac{1}{2}Y + B, \quad [M_+, M_-] = -I_3 + \frac{1}{2}Y + B, \\
 [W_+, W_-] &= -Y + B.
 \end{aligned}
 \tag{18}$$

Operator Q may be used also, for instance $[V_+, V_-] = 3Q - 2I_3$.

Let us tabulate the commutators, which are useful for constructing finite-dimensional representations of $SU(4)$ group. The first component is given by the row, and the second is assigned as the column. The commutators of I_3, Y, Q , and B with the rest are presented in the Table 1.

Table 1. Commutators of operators (14).

$\langle \cdot, b \rangle$ $\langle a, \cdot \rangle$	I_{\pm}	V_{\pm}	U_{\pm}	N_{\pm}	M_{\pm}	W_{\pm}
I_3	$\pm I_{\pm}$	$\pm \frac{1}{2}V_{\pm}$	$\mp \frac{1}{2}U_{\pm}$	$\pm \frac{1}{2}N_{\pm}$	$\mp \frac{1}{2}M_{\pm}$	0
Y	0	$\pm V_{\pm}$	$\pm U_{\pm}$	$\pm \frac{1}{3}N_{\pm}$	$\pm \frac{1}{3}M_{\pm}$	$\mp \frac{2}{3}W_{\pm}$
Q	$\pm I_{\pm}$	$\pm V_{\pm}$	0	$\pm \frac{2}{3}N_{\pm}$	$\mp \frac{1}{3}M_{\pm}$	$\mp \frac{1}{3}W_{\pm}$
B	0	0	0	$\pm \frac{4}{3}N_{\pm}$	$\pm \frac{4}{3}M_{\pm}$	$\pm \frac{4}{3}W_{\pm}$

From the commutation relations given above, it follows that operators I_+, V_+, U_-, N_+ , and M_- are raising and $I_-, V_-, U_+, N_-,$ and M_+ lowering the eigenvalues of I_3 . Operators $U_+, V_+, N_+, M_+,$ and W_- are raising and $U_-, V_-, N_-, M_-,$ and W_+ are lowering the eigenvalues of Y . Operators $I_+, V_+, N_+, M_-,$ and W_- are raising and $I_-, V_-, N_-, M_+,$ and W_+ are lowering the eigenvalues of Q . Operators $N_+, M_+,$ and W_+ are raising, $N_-, M_-,$ and W_- are lowering, and $I_{\pm}, V_{\pm},$ and U_{\pm} keep unchanged the eigenvalues of B .

Commutators of the rest of operators are given in Table 2.

Table 2. Commutators of operators (14), continued.

$\langle \cdot, b \rangle$ $\langle a, \cdot \rangle$	V_+	V_-	U_+	U_-	N_+	N_-	M_+	M_-	W_+	W_-
I_+	0	$-U_-$	V_+	0	0	$-M_-$	N_+	0	0	0
I_-	U_+	0	0	$-V_-$	M_+	0	0	$-N_-$	0	0
V_+	0	(18)	0	I_+	0	$-W_-$	0	0	N_+	0
V_-	(18)	0	$-I_-$	0	W_+	0	0	0	0	$-N_-$
U_+	0	I_-	0	(18)	0	0	0	$-W_-$	M_+	0
U_-	$-I_+$	0	(18)	0	0	0	W_+	0	0	$-M_-$
N_+	0	$-W_+$	0	0	0	(18)	0	I_+	0	V_+
N_-	W_-	0	0	0	(18)	0	$-I_-$	0	$-V_-$	0
M_+	0	0	0	$-W_+$	0	I_-	0	(18)	0	U_+
M_-	0	0	W_-	0	$-I_+$	0	(18)	0	$-U_-$	0
W_+	$-N_+$	0	$-M_+$	0	0	V_-	0	U_-	0	(18)
W_-	0	N_-	0	M_-	$-V_+$	0	$-U_+$	0	(18)	0

It is worth noting here that the studied quantum characteristics $I_3, Q, Y,$ and B or the notion of quarks are not derived or dependent on the standard theory. For instance, isospin was introduced in 1932 by Heisenberg to explain symmetries of the then newly discovered neutron [20]. They were introduced before, and the standard theory presents some scheme of them. Here, we propose an alternative description based on the six-dimensional $\mathbb{E}_{3,3}$ space.

5. Simplest Irreducible Representation of $SU(4)$

Using the commutation relations from Tables 1 and 2, we will construct, as an example, the simplest irreducible finite-dimensional representation for the $SU(4)$ group. The states that are contained in the representations are characterized by a set of values of the quantum characteristics (I_3, Y, Q, B) . Transitions between various states are introduced with the aid of the raising and lowering operators given above.

Let us introduce four orthogonal vectors:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \tag{19}$$

in the four-dimensional space induced by the set of quantum characteristics (I_3, Y, Q, B) . These vectors are the eigenvectors of the Hermitian operators $I_3, Y, Q,$ and $B,$ given by the matrices of the type of (14) and with the following eigenvalues: $I_3: (\frac{1}{2}, -\frac{1}{2}, 0, 0), Y: (\frac{1}{3}, \frac{1}{3}, -\frac{2}{3}, 0), Q: (\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}, 0), B: (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$. Let the state $\Psi_u = \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}}$ be set. It is determined as a representation vector, which satisfies the following relations:

$$\begin{aligned} I_3 \Psi_u &= \frac{1}{2} \Psi_u, \\ Y \Psi_u &= \frac{1}{3} \Psi_u, \\ Q \Psi_u &= \frac{2}{3} \Psi_u, \\ B \Psi_u &= \frac{1}{3} \Psi_u. \end{aligned} \tag{20}$$

This state is an eigenvector of operators $I_3, Y, Q,$ and B and corresponds to the u quark. Subjecting this state to the operator V_- , we obtain new state $\Psi_s = \Psi_{0, -\frac{2}{3}, -\frac{1}{3}, \frac{1}{3}}$, which corresponds to the s quark:

$$\begin{aligned} I_3 V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} &= (V_- I_3 - \frac{1}{2} V_-) \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} = 0 \cdot V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}}, \\ Y V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} &= (V_- Y - V_-) \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} = -\frac{2}{3} V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}}, \\ Q V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} &= (V_- Q - V_-) \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} = -\frac{1}{3} V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}}, \\ B V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} &= V_- B \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}} = \frac{1}{3} V_- \Psi_{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{3}}. \end{aligned} \tag{21}$$

This can be written briefly as

$$V_- \Psi_u = \Psi_s. \tag{22}$$

The state with the same set of characteristics corresponds to the b quark. It may be obtained by applying $N_- W_+$ operators subsequently to Ψ_u :

$$N_- W_+ \Psi_u = \Psi_b. \tag{23}$$

Next, applying the U_+ operator to the Ψ_s state and using the commutators from Table 2, one can obtain a state $\Psi_d = \Psi_{-\frac{1}{2}, \frac{1}{3}, -\frac{1}{3}, \frac{1}{3}}$, corresponding to the d quark:

$$U_+ \Psi_s = \Psi_d . \tag{24}$$

From the Ψ_d state using the I_+ operator, one may transfer to the Ψ_u state, or with the help of the M_- operator, a state $\Psi_a = \Psi_{0,0,0,-1}$ may be obtained, which corresponds to the anti-baryon with the characteristics of $I_3 = 0, Y = 0, Q = 0,$ and $B = -1$:

$$I_+ \Psi_d = \Psi_u , \quad M_- \Psi_d = \Psi_a . \tag{25}$$

Four states $\Psi_u, \Psi_s, \Psi_d,$ and Ψ_a establish a space of the simplest irreducible finite-dimensional representation of the $SU(4)$ group. In this space, a state $\Psi_c = \Psi_{0, \frac{4}{3}, \frac{2}{3}, \frac{1}{3}}$ also exists. It corresponds to the c quark and can be reached from Ψ_d by the V_+ operator or from Ψ_u with the help of the U_+ operator:

$$V_+ \Psi_d = \Psi_c , \quad U_+ \Psi_u = \Psi_c . \tag{26}$$

The state with the same set of characteristics $(I_3, Y, Q, B),$ which is related to the t quark, can be obtained from Ψ_d and Ψ_u as

$$N_+ W_- \Psi_d = \Psi_t , \quad M_+ W_- \Psi_u = \Psi_t . \tag{27}$$

These results can be illustrated in the form of the transition diagram (Figure 1).

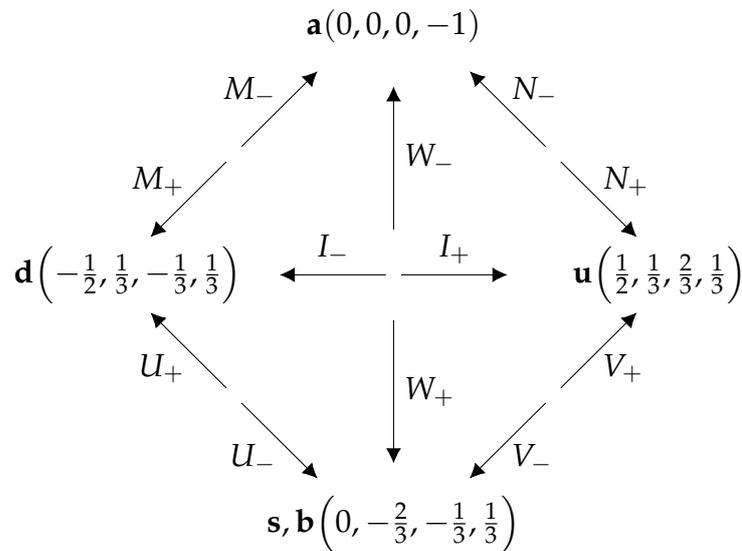


Figure 1. Simplest irreducible non-trivial representation of the $SU(4)$ group.

Figure 1 shows the simplest irreducible non-trivial representation of the $SU(4)$ group, containing four quarks in three states and one anti-baryon with the proper state $\Psi_{0,0,0,-1}$. This figure can be thought of as a tetrahedron in some abstract 3D space. The triangle containing the quarks forms a plane with baryon charge $B = 1/3$. The operators $U_{\pm}, V_{\pm},$ and I_{\pm} are moving between states in this plane. The other three operator pairs $(M_{\pm}, N_{\pm}, W_{\pm})$ are changing the baryon charge.

6. Representation of the Metric of the $\mathbb{E}_{3,3}$ Space Using Hyperbolic Complex Numbers

Let us define the algebra \mathbb{H} of hyperbolic complex numbers as a two-dimensional R -module with a pair of generatrices $\{1, j\}$ and the following multiplication table:

$$\begin{array}{c|cc} & 1 & j \\ \hline 1 & 1 & j \\ \hline j & j & 1 \end{array} \tag{28}$$

The elements $h \in \mathbb{H}$ are written in the form $h = 1x + jt$, where $x, t \in \mathbb{R}$, and j is the imaginary unity in \mathbb{H} . Real numbers $\Re h = x$ and $\Im h = t$ are called the real and imaginary parts of the hyperbolic complex number h , respectively. The involutive operation of complex conjugation is defined as $h = x + jt \rightarrow \bar{h} = x - jt$. In a plane, the algebra of hyperbolic complex numbers induces a two-dimensional pseudo-Euclidean geometry with metric $\eta_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Consider now the n -dimensional space of hyperbolic complex numbers \mathbb{H}^n . Here, we can introduce the scalar product of vectors $\langle \cdot, \cdot \rangle$. If $\vec{h} = (h^1, \dots, h^n)$, $\vec{g} = (g^1, \dots, g^n) \in \mathbb{H}^n$, then the scalar product is given by a bilinear form as:

$$\langle \vec{h}, \vec{g} \rangle = h^1 \bar{g}^1 + \dots + h^n \bar{g}^n \tag{29}$$

The form (29) is not positively defined. In the case of \mathbb{H}^3 , we have

$$\langle \vec{h}, \vec{h} \rangle = h^1 \bar{h}^1 + h^2 \bar{h}^2 + h^3 \bar{h}^3 \tag{30}$$

Taking into account that $h^k = x^k + jt^k, k = 1, 2, 3$, we obtain

$$\langle \vec{h}, \vec{h} \rangle = (x^1)^2 + (x^2)^2 + (x^3)^2 - (t^1)^2 - (t^2)^2 - (t^3)^2, \tag{31}$$

i.e., the scalar product of the hyperbolic complex vectors from \mathbb{H}^3 assigns the square bilinear form (pseudo-Euclidean metric) in the $\mathbb{E}_{3,3}$ space. Now, we consider some symmetry groups of the form (30).

7. Hyperbolic Groups of Unitary Symmetry and Their Representations

The metric of the six-dimensional pseudo-Euclidean space $\mathbb{E}_{3,3}$ is invariant relative to a number of hidden symmetry groups, which emerge as a result of representing the pseudo-Euclidean metric with the aid of hyperbolic complex numbers in the \mathbb{H}^3 space according to (30). Consider the unitary hyperbolic group $U(1, \mathbb{H}^3)$, which operates in the \mathbb{H}^3 space. It is a three-parameter group of H -unitary matrices:

$$U = \begin{pmatrix} e^{j\varphi_1} & 0 & 0 \\ 0 & e^{j\varphi_2} & 0 \\ 0 & 0 & e^{j\varphi_3} \end{pmatrix}, \tag{32}$$

which leave the bilinear form (30) invariant. The Hermitian-conjugated matrix

$$U^\dagger = \begin{pmatrix} e^{-j\varphi_1} & 0 & 0 \\ 0 & e^{-j\varphi_2} & 0 \\ 0 & 0 & e^{-j\varphi_3} \end{pmatrix} \tag{33}$$

is inverse of U , and $UU^\dagger = 1$. The Lie algebra of this group is commutative, and its basis is formed as

$$e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{34}$$

The identical representation (32) of the group $U(1, \mathbb{H}^3)$ is reducible as a direct sum of irreducible representations, which operate in invariant single-dimension subspaces \mathbb{H} :

$$U(1, \mathbb{H}^3) = U(1, \mathbb{H}) \times U(1, \mathbb{H}) \times U(1, \mathbb{H}) . \tag{35}$$

The generators of this group induce three conservation laws. Running a little bit ahead, we note that these conservation laws are associated with the three colour quantum characteristics of quarks. A more detailed discussion can be found in [21]. The unitary transformations of the group $U(1, \mathbb{H})$ in the \mathbb{H} space correspond to the Lorentz transformations in the pseudo-Euclidean space $\mathbb{E}_{1,1}$.

Consider now the hyperbolic group of unitary matrices $SU(2, \mathbb{H})$, which operates in the three-dimensional hyperbolic space \mathbb{H}^3 . This group $SU(2, \mathbb{H})$ consists of hyperbolic matrices U with dimension four, which are unitary unimodular, i.e., they satisfy the conditions $U^\dagger U = 1, |\det U| = 1$. Such a matrix may be represented as $U = e^{i\sigma_k a_k}, U^\dagger = e^{-j\sigma_k a_k}$, where σ_k are Hermitian traceless matrices, having the form of

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{36}$$

where a_k are arbitrary real numbers.

The matrices (36) form a three-dimensional basis in the Lie algebra of the $SU(2, \mathbb{H})$ group and differ from the Pauli matrices only by replacing the imaginary unit i with the hyperbolic imaginary unit j . The basis elements (36) of the Lie algebra satisfy the following commutation relations:

$$[\sigma_k, \sigma_l] = 2j\kappa_{klm}\sigma_m , \tag{37}$$

where κ_{klm} is a third-rank tensor with values $\kappa_{123} = 1, \kappa_{132} = 1, \kappa_{231} = 1, \kappa_{312} = -1, \kappa_{213} = -1, \kappa_{321} = -1$.

The structural constants of the Lie algebra of the group $SU(2, \mathbb{H})$ coincide with that of the group $SU(2)$ to within a sign.

From the components of the six-dimensional vector $\vec{x} = (x^1, x^2, x^3, t^1, t^2, t^3) \in \mathbb{E}_{3,3}$, one can choose three so as to avoid all three being of the same type, i.e., triplets (x^1, x^2, x^3) and (t^1, t^2, t^3) are excluded. There are eighteen such triplets. They can be joined in pairs so as to have all six components in a pair. For instance, for the triplet (x^1, x^3, t^2) , its pair will be (x^2, t^1, t^3) . Thus, there are nine pairs.

For each triplet of type (x^k, x^l, t^m) , which contains two spatial coordinates, we assign a matrix:

$$Y = \begin{pmatrix} x^k & x^l - jt^m \\ x^l + jt^m & -x^k \end{pmatrix}, \tag{38}$$

and to its pair $(x^n, t^p, t^q), n, p, q = 1, 2, 3, n \neq k, l, m \neq p, q$, we assign a matrix:

$$Y^C = \begin{pmatrix} t^p & t^q - jx^n \\ t^q + jx^n & -t^p \end{pmatrix}. \tag{39}$$

The following relation takes place:

$$\det Y^C - \det Y = (x^1)^2 + (x^2)^2 + (x^3)^2 - (t^1)^2 - (t^2)^2 - (t^3)^2 . \tag{40}$$

For any hyperbolic unitary matrices $U_1, U_2 \in SU(2, \mathbb{H})$, the independent unitary transformations:

$$Y' = U_1^\dagger Y U_1, Y^{C'} = U_2^\dagger Y^C U_2 \tag{41}$$

leave the bilinear square form in the right side of (40) invariant, by the equality:

$$\det Y^{C'} - \det Y' = \det Y^C - \det Y . \tag{42}$$

Thus, the unitary transformations over the pairs of matrices Y and Y^C from the $SU(2, \mathbb{H})$ group correspond to the pseudo-orthogonal transformations in the space $\mathbb{E}_{3,3}$, leaving the pseudo-Euclidean metric invariant.

There are altogether nine groups of this kind, where each representation of such groups expands into the direct sum of two irreducible conjugated representations, which operate in the three-dimensional subspaces of the six-dimensional space–time. To these eighteen representations of the hyperbolic unitary symmetry groups there should correspond eighteen conservation laws. Running ahead, we note that the conserved quantum characteristics may be interpreted as quark flavours. The fact that eighteen representations are paired means that quark flavours appear in pairs (u, d) , (s, c) , (b, t) , etc.

Now, consider the plane $B = 1/3$ in Figure 1 and the operators U_{\pm} , V_{\pm} , and I_{\pm} . It is easy to see that applying them as in (24), (22), and (25) repeatedly and in various sequences, one can construct an infinite hexagonal grid in the plane $B = 1/3$. Let us select four grid nodes where six known quarks $(u, d, s, b, c, \text{ and } t)$ reside and the grid nodes immediately adjacent to those; see Figure 2.

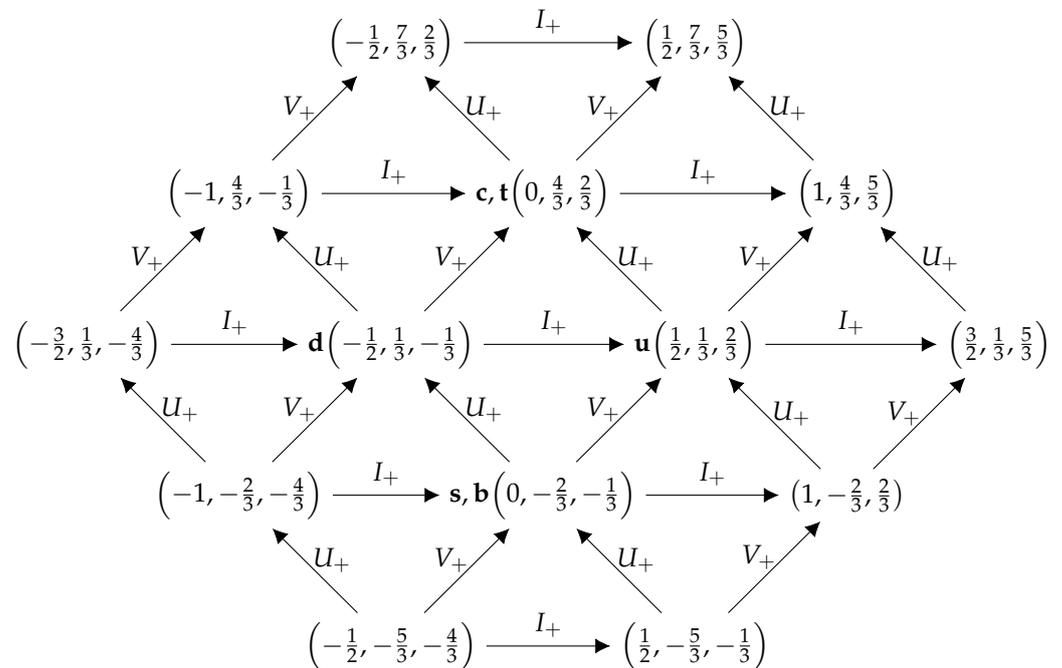


Figure 2. Diagram of 18 quarks.

There are 14 nodes involved. Taking into account that the states in the internal nodes are multiplied [22], there appear to be in total 18 states described by this grid. The internal nodes contain known quarks. Node $(0, \frac{4}{3}, \frac{2}{3})$ corresponds to quarks c and t , and node $(0, -\frac{2}{3}, -\frac{1}{3})$ represents quarks s and b . However, internal nodes $(-\frac{1}{2}, \frac{1}{3}, -\frac{1}{3})$ obtain only one d quark, and the node $(\frac{1}{2}, \frac{1}{3}, \frac{2}{3})$ obtains a single u . One can suppose that other quarks compose particles with very short decay periods and very big masses. Thus, they have not been discovered yet. We may call the proposed quark, which shares a node with u as h (hazy) and the one paired with d as v (vague). Presumably, h and v quarks form the most long-living particles among others currently undiscovered. Each of the outer nodes of the diagram contain a single quark.

The Klein–Fock–Gordon equation:

$$(\square + \mu^2)\Psi(x) = 0, \quad \mu = mc/\hbar \tag{43}$$

in the space $\mathbb{E}_{3,3}$ can be written as

$$\left(\frac{\partial^2}{\partial t_1^2} + \frac{\partial^2}{\partial t_2^2} + \frac{\partial^2}{\partial t_3^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} \right) \Psi = \mu^2 \Psi \tag{44}$$

and treated as an eigenvalue problem. The spectrum of the operator contains particle masses. The masses of the $u - d - s$ quark triplet, forming long-living particles, were estimated in [23]. Other components of spectrum might be found further.

8. Reduction to Unisotropic Space

So far, in this paper, we have discussed the fully symmetric \mathbb{H}^3 space or the isomorphic $\mathbb{E}_{3,3}$ one, i.e., the space with signature (3, 3). However, the commonly known space where we live has a signature of (1, 3). The question arises about how we come from six to four dimensions and what are the related effects. Here, we present three of them: weak interaction asymmetry, treatment of electric charge, and decreasing the number of quarks.

Within the common formalism, there is no explanation of the causes of the violation of the conservation laws for hypercharge and isospin in weak interactions. In the proposed approach, this phenomenon has a rather simple explanation. According to the above, the laws of conservation of isospin and hypercharge appear due to the presence of spherical symmetry in the three-dimensional temporal subspace. If that space would always remain isotropic, these laws would be rigorous. However, if a preferable direction appears in the three-dimensional temporal subspace, then the spherical symmetry is broken and the laws are violated. Strong interactions run over a time of order 10^{-24} s. One can propose that, as this scale, the temporal subspace remains spherically symmetric, and hence, the conservation laws of hypercharge and isospin stay valid. Weak interactions run much slower with typical intervals of 10^{-9} s, and the existence of the preferable time axis cannot be neglected. This breaks the spherical symmetry, and the conservation laws associated with it are violated. Only laws based on axial symmetry remain.

Two temporal dimensions may be compactified according to the following model introduced by the authors in an application to electromagnetism [14]. At a small time scale, all directions in the temporal subspace (t_1, t_2, t_3) are equal, and the subspace has spherical symmetry. For a bigger scale, one direction is selected, and let it be the axis t_1 . The plane (t_2, t_3) then has a one-parametric group of rotations, where the parameter can be an angle of rotation around the origin. Selecting the parameter as $\frac{\omega}{c} t^1$, one can write the following compactification rule:

$$\begin{aligned} t^2(t^1) &= t^2(0) \cos\left(\frac{\omega}{c} t^1\right) - t^3(0) \sin\left(\frac{\omega}{c} t^1\right), \\ t^3(t^1) &= t^2(0) \sin\left(\frac{\omega}{c} t^1\right) + t^3(0) \cos\left(\frac{\omega}{c} t^1\right). \end{aligned} \tag{45}$$

Integral curves $t^2(t^1), t^3(t^1)$ compose a vector field, which in the plane (t^2, t^3) is $(-\frac{\omega}{c} t^3, \frac{\omega}{c} t^2)$. In the plane of the tangent splitting, there is a circulation of this field along the circle of constant radius r_0 : $\omega^2 r_0^2 = c^2$.

Traditionally, Maxwell’s equations of electromagnetism include electric charge and currents as ontology phenomena without a clear mathematical definition. The proposed approach gives a more rigorous and consistent understanding of electromagnetism equations linking the different components of the electromagnetic tensor in six-dimensional space. Within the framework of this model, a purely geometrical interpretation of the concept of the electromagnetic field and point electric charge can be derived. The appearance of the point electric charge is associated with the circulation of the vector potential around a dedicated time axis in the three-dimensional temporal subspace. This is given in more detail in [14].

The number of quarks was derived as the number of combinations of coordinate axes, which satisfy the condition (42). One can note that the obtained result rests on the

assumption that the temporal subspace is isotropic, i.e., there is no preferable time axis. However, if this is not so and such a preferable time direction exists, let it be t_1 , then the spherical symmetry in the temporal subspace is violated and only axial symmetry remains. In this case, only three pairs are left:

$$\begin{aligned}(t_1, x_1, x_2) &\leftrightarrow (t_2, t_3, x_3); \\(t_1, x_1, x_3) &\leftrightarrow (t_2, t_3, x_2); \\(t_1, x_2, x_3) &\leftrightarrow (t_2, t_3, x_1).\end{aligned}\tag{46}$$

To each pair there corresponds a pair of conjugated matrices (38) and (39), inducing six conservation laws for quark flavours. Therefore, only six quarks are possible in a space with signature (1, 3).

9. Conclusions

In the present paper, we established the relationship between the conserved quantum characteristics of elementary particles and the internal (hidden) symmetries of the six-dimensional pseudo-Euclidean space $\mathbb{E}_{3,3}$, which, as has been supposed, may be a real physical space of the microcosm, limited by time intervals of 10^{-24} s. Proceeding from this concept, it is possible to explain the phenomenon of the violation of the conservation laws for hypercharge and isospin in weak interactions and keeping these laws in strong interactions, as well as to predict the number of possible quarks. It is shown that the conservation laws of the electric and baryon charges, hypercharge, spin, and isospin are induced by the unitary symmetry group $SU(4)$, which represents the most general group of hidden symmetries of the $\mathbb{E}_{3,3}$ space. The properties of the metric in this space allow predicting the number of quarks to be equal to 18.

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