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Article

Analytic Approximate Solution in the Neighborhood of a Moving Singular Point of a Class of Nonlinear Equations

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Abstract: The paper considers a class of nonlinear differential equations which are not solvable in quadratures in general case. The author's technology for solving such equations contains six problems. In this article, the solution to one of these problems is given, a real area in which it is possible to calculate an analytical approximate solution in the case of an approximate value of a moving singular point is obtained. Obtained results are based on the application of elements of differential calculus in finding estimates for the approximate solution error. Theoretical provisions are confirmed by numerical calculations, which characterize their reliability.

Keywords: aspect of nonlinearity; perturbation of a moving singular point; real domain; a-priori estimate

MSC: 34G20; 34A05

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1. Introduction

In solving many technical problems, researchers come up with various differential equations, both linear and non-linear. In the case of linear equations, there are no problems, but in the case of nonlinear differential equations, some aspects of this class of equations have to be taken into account, for example, the presence of moving singular points, which is a sufficient condition for the impossibility of solving these equations in quadratures. Note that all currently known classical, analytical and numerical methods are not fit to solve this type of equations. This situation pushes the development of research towards methods able to solve similar classes of equations.

Such equations emerge, for example, when studying wave processes in elastic beams or rods [1,2], or when observing breaking forces in building structures [3]. There are publications on the solvability of such equations in particular cases [4–11], the development of the author's method in [12–14] and the asymptotic approach [15,16]. It should be noted that the asymptotic approach does not allow obtaining the results presented in this study. In this paper, we continue the study of the considered class of equations presented in [17–19], where at the first stage the problem of the uniqueness of the solution of the considered equation for a neighborhood of a moving singular point is solved. Then, an analytical approximate solution is obtained, and a-priori error estimates are proven. Further, the results of the influence of the perturbation of a moving singular point on the analytical approximate solution are obtained. In the course of these studies of the above stages, we observe a significant decrease in the area where it is possible to carry out calculations for an analytical approximate solution. In the present work, by applying the elements of differential calculus in estimating the error, it is possible to significantly expand this area. This fact is the novelty of this study. Obtained theoretical positions are confirmed by numerical experiments.

2. Research Methods

For the classical Cauchy problem

$$y'''(x) = y^7(x) + r(x), \tag{1}$$

$$\begin{cases} y(x_0) = y_0, \\ y'(x_0) = y_1, \\ y''(x_0) = y_2, \end{cases} \tag{2}$$

in [17], an analytical approximate solution was constructed for a neighborhood of a moving singular point in the way as follows

$$y(x) = (x^* - x)^{-\frac{1}{2}} \cdot \sum_{n=0}^{\infty} C_n (x^* - x)^{\frac{n}{2}}. \tag{3}$$

Since the currently existing methods allow to calculate the value of the moving singular point approximately, instead of (3) we obtain a new structure of the solution:

$$\tilde{y}_N(x) = (\tilde{x}^* - x)^{-\frac{1}{2}} \cdot \sum_{n=0}^{N} \tilde{C}_n (\tilde{x}^* - x)^{\frac{n}{2}}, \tag{4}$$

where $\tilde{C}_n$ are the perturbed values of the coefficients.

When estimating the error of solution (4) in [19], we note the narrowing of the domain for solution (4) in comparison with the existence theorem in [17]. Applying an element of differential calculus in estimating the error of solution (4), it is possible to significantly approach the area in [17], but for an approximate solution (4).

Theorem 1. *We require the following conditions to be met:*

1. $r(x) \in C^\infty$*(the function is infinitely differentiable) in the domain* $|\tilde{x}^* - x| < \rho_1$, *where* $0 < \rho_1 = const$;
2. $\exists M_n : \frac{|r^{(n)}(\tilde{x}^*)|}{n!} \le M_n, M_n = const$;
3. $\tilde{x}^* \le x^*$;
4. *The following estimate is made for* $\tilde{x}^* : |\tilde{x}^* - x^*| \le \Delta\tilde{x}^*$;
5. $\Delta\tilde{x}^* < \frac{1}{4(M+1)^4}$.

In this case, the approximate solution (4) of problem (1)–(2) has the following error estimate

$$\Delta\tilde{y}_N \le \Delta_0 + \Delta_1 + \Delta_2 + \Delta_3,$$

in the domain

$$F = F_1 \cap F_2 \cap F_3,$$

where

$$\Delta_0 \le \frac{\sqrt[6]{15}}{2\sqrt[6]{8}} \frac{\Delta\tilde{x}^*}{|\tilde{x}_1^* - x|^{3/2}},$$

$$\Delta_1 \le \frac{8M(M+1)^{2(N+1)}}{1 - (M+1)^6|\tilde{x}^* - x|^{\frac{3}{2}}} |\tilde{x}^* - x|^{\frac{N}{2}}$$

$$\times \left(\frac{1}{N(N-2)(N-4)} + \frac{|\tilde{x}^* - x|^{\frac{1}{2}}}{(N+1)(N-1)(N-3)} + \frac{|\tilde{x}^* - x|}{(N+2)N(N-2)} \right)$$

in case of $N + 1 = 3k$,

$$\Delta_1 \le \frac{8M(M+1)^{2N}}{1 - (M+1)^6|\tilde{x}^* - x|^{\frac{3}{2}}} |\tilde{x}^* - x|^{\frac{N}{2}}$$

$$\times\left(\frac{1}{N(N-2)(N-4)}+\frac{|\tilde{x}^*-x|^{\frac{1}{2}}}{(N+1)(N-1)(N-3)}+\frac{|\tilde{x}^*-x|}{(N+2)N(N-2)}\right)$$

for the variant $N+1=3k+1$, *and*

$$\Delta_1\leq\frac{8M(M+1)^{2(N-1)}}{1-(M+1)^6|\tilde{x}^*-x|^{\frac{3}{2}}}|\tilde{x}^*-x|^{\frac{N}{2}}$$

$$\times\left(\frac{1}{N(N-2)(N-4)}+\frac{|\tilde{x}^*-x|^{\frac{1}{2}}}{(N+1)(N-1)(N-3)}+\frac{|\tilde{x}^*-x|}{(N+2)N(N-2)}\right)$$

where $N+1=3k+2$.

$$\Delta_2\leq\frac{8\Delta M(M+\Delta M+1)^6|\tilde{x}_2^*-x|^{3/2}}{1-2^3(M+\Delta M+1)^{12}\left|\tilde{x}_2^*-x\right|^3}$$

$$\times\left(1+(M+\Delta M+1)^6|\tilde{x}_2^*-x|+(M+\Delta M+1)^6|\tilde{x}_2^*-x|^2\right)$$

$$\times\left(1+(M+\Delta M+1)^6|\tilde{x}_2^*-x|^{3/2}\right),$$

$$\Delta_3\leq\frac{4\Delta\tilde{x}^*M(M+\Delta M+1)^6|\tilde{x}_2^*-x|^{3/2}}{\left(1-(M+\Delta M+1)^6\left|\tilde{x}_2^*-x\right|^{3/2}\right)^2}$$

$$\times\left(3+(M+\Delta M+1)^6|\tilde{x}_2^*-x|(7|\tilde{x}_2^*-x|+5)-2(M+\Delta M+1)^{12}|\tilde{x}_2^*-x|^{5/2}\right.$$

$$\left.\times\ (2|\tilde{x}_2^*-x|+1)\right),$$

where $\rho_2=\min\left\{\rho_1,\frac{1}{4(M+1)^4},\frac{1}{4(M+\Delta M+1)^4}\right\}$, $M=\max\left\{\sup\limits_n\left\{\frac{|r^{(n)}(\tilde{x}^*)|}{n!}\right\},|y_0|,|y_1|,|y_2|\right\}$,

$$\Delta M=\ \max\left\{\sup_n\left\{\frac{\left|r^{(n)}(\tilde{x}^*)\right|}{n!}\right\},|y_0|,|y_1|,|y_2|\right\}\cdot\Delta\tilde{x}^*,\ n=0,\ 1,\ 2,\ ...,$$

$$F_1=\{x:x<\tilde{x}_1^*\},F_2=\left\{x:|\tilde{x}^*-x|<\frac{1}{4(M+1)^4}\right\},$$

$$F_3=\left\{x:|\tilde{x}_2^*-x|<\frac{1}{4(M+\Delta M+1)^4}\right\},\tilde{x}_1^*=\tilde{x}^*-\Delta\tilde{x}^*,\tilde{x}_2^*=\tilde{x}^*+\Delta\tilde{x}^*.$$

Proof. Based on the classical approach, we have:

$$\Delta\tilde{y}_N(x)=|y(x)-\tilde{y}_N(x)|\leq|y(x)-\tilde{y}(x)|+|\tilde{y}(x)-\tilde{y}_N(x)|.$$

Let us estimate the expression $|y(x)-\tilde{y}(x)|$ using differential calculus methods [20]:

$$|y(x)-\tilde{y}(x)|\leq\sup_U\left|\frac{\partial\tilde{y}(x)}{\partial\tilde{x}^*}\right|\Delta\tilde{x}^*+\sum_0^\infty\sup_U\left|\frac{\partial\tilde{y}(x)}{\partial\tilde{C}_n}\right|\Delta\tilde{C}_n,$$

where $U=\{x:|\tilde{x}^*-x|<\Delta\tilde{x}^*\}$.

We go ahead to analyze the expression:

$$\sup_U\left|\frac{\partial\tilde{y}(x)}{\partial\tilde{x}^*}\right|=\sup_U\left|\sum_0^\infty\tilde{C}_n\frac{n-1}{2}(\tilde{x}^*-x)^{\frac{n-3}{2}}\right|\leq\sum_0^\infty\sup_U|\tilde{C}_n|\frac{n-1}{2}\sup_U|\tilde{x}^*-x|^{\frac{n-3}{2}}.$$

Taking into account the expansion of the function $r(x)$ into a regular series, according to the condition of the theorem, $r(x) = \sum\limits_{n=0}^{\infty} A_n(x-\tilde{x}^*)^n$, we can note down in the general case as $\tilde{C}_n$ the way $\tilde{C}_n = \tilde{C}_n(A_0, A_1, \ldots, A_m)$. We remind you of the estimates for $\tilde{C}_n$ [17]:

$$\left|\tilde{C}_{3n}\right| \leq \frac{8M(M+\Delta M+1)^{6n}}{(3n-1)(3n-3)(3n-5)},$$

$$\left|\tilde{C}_{3n+1}\right| \leq \frac{8M(M+\Delta M+1)^{6n}}{3n(3n-2)(3n-4)},$$

$$\left|\tilde{C}_{3n+2}\right| \leq \frac{8M(M+\Delta M+1)^{6n}}{(3n+1)(3n-1)(3n-3)}.$$

As well as estimates for $\Delta\tilde{C}_n$ [19]:

$$\left|\Delta\tilde{C}_{3n}\right| \leq \frac{8\Delta M(M+\Delta M+1)^{6n}}{(3n-1)(3n-3)(3n-5)},$$

$$\left|\Delta\tilde{C}_{3n+1}\right| \leq \frac{8\Delta M(M+\Delta M+1)^{6n}}{3n(3n-2)(3n-4)},$$

$$\left|\Delta\tilde{C}_{3n+2}\right| \leq \frac{8\Delta M(M+\Delta M+1)^{6n}}{(3n+1)(3n-1)(3n-3)}.$$

Now let us estimate the expression $\sup_U\left|\tilde{C}_n\right|$ by taking into account the recurrence relations given in [17], and performing a series of transformations; we will obtain:

$$\sup_U\left|\tilde{C}_n\right| \leq \tilde{C}_n(|A_0+\Delta A_0|, |A_1+\Delta A_1|, \ldots, |A_m+\Delta A_m|) \leq \tfrac{8M(M+\Delta M+1)^{6\left[\frac{n}{3}\right]}}{(n-1)(n-3)(n-5)} = \mathrm{T}_n,$$

where

$$\Delta M = \max\left\{\sup_n\left\{\frac{\left|r^{(n)}(\tilde{x}^*)\right|}{n!}\right\}, |y_0|, |y_1|, |y_2|\right\}\cdot\Delta\tilde{x}^*,\ n = 0,\ 1,\ 2,\ \ldots,$$

$$M = \max\left\{\sup_n\left\{\frac{\left|r^{(n)}(\tilde{x}^*)\right|}{n!}\right\}, |y_0|, |y_1|, |y_2|\right\}.$$

Taking into account the fact that

$$\sup_U|\tilde{x}^*-x|^{\frac{n-3}{2}} \leq \begin{cases} |\tilde{x}_1^*-x|^{\frac{n-3}{2}}, n = 0,1,2 \\ |\tilde{x}_2^*-x|^{\frac{n-3}{2}}, n > 2 \end{cases} \quad (\textit{where } \tilde{x}_1^* = \tilde{x}^*-\Delta\tilde{x}^*, \tilde{x}_2^* = \tilde{x}^*+\Delta\tilde{x}^*),$$

and

$$\sup_U\left|\frac{\partial\tilde{y}(x)}{\partial\tilde{C}_n}\right| = \sup_U|\tilde{x}^*-x|^{\frac{n-1}{2}} \leq \begin{cases} |\tilde{x}_1^*-x|^{\frac{n-1}{2}}, n = 0 \\ |\tilde{x}_2^*-x|^{\frac{n-1}{2}}, n > 0 \end{cases},$$

as well as

$$C_0 = \tilde{C}_0 = \pm\sqrt[6]{\frac{15}{8}}, C_1 = C_2 = \ldots = C_6 = \tilde{C}_1 = \ldots = \tilde{C}_6 = 0$$

we obtain an estimate as follows for

$$|y(x)-\tilde{y}(x)|$$

$$|y(x)-\tilde{y}(x)| \le \frac{\sqrt[6]{15}}{2\sqrt[6]{8}}\frac{\Delta\tilde{x}^*}{|\tilde{x}_1^*-x|^{3/2}} + \Delta\tilde{x}^*\sum_7^\infty \frac{n-1}{2}\mathrm{T}_n|\tilde{x}_2^*-x|^{\frac{n-3}{2}} + \sum_7^\infty \Delta\tilde{C}_n|\tilde{x}_2^*-x|^{\frac{n-1}{2}}.$$

Then

$$|y(x)-\tilde{y}_N(x)| \le \frac{\sqrt[6]{15}}{2\sqrt[6]{8}}\frac{\Delta\tilde{x}^*}{|\tilde{x}_1^*-x|^{3/2}} + \sum_{n=N+1}^\infty |\tilde{C}_n||\tilde{x}^*-x|^{\frac{n-1}{2}} + \sum_7^\infty \Delta\tilde{C}_n|\tilde{x}_2^*-x|^{\frac{n-1}{2}}$$

$$+\Delta\tilde{x}^*\sum_7^\infty \frac{n-1}{2}\mathrm{T}_n|\tilde{x}_2^*-x|^{\frac{n-3}{2}} = \Delta_0+\Delta_1+\Delta_2+\Delta_3.$$

Thus, we obtain the result $\Delta_0 = \frac{\sqrt[6]{15}}{2\sqrt[6]{8}}\frac{\Delta\tilde{x}^*}{|\tilde{x}_1^*-x|^{3/2}}$.

For the estimate Δ_1, let us use the Theorem 2 in [17]:

$$\Delta_1 \le \frac{8M(M+1)^{2(N+1)}}{1-(M+1)^6|\tilde{x}^*-x|^{\frac{3}{2}}}|\tilde{x}^*-x|^{\frac{N}{2}}$$

$$\times\left(\frac{1}{N(N-2)(N-4)} + \frac{|\tilde{x}^*-x|^{\frac{1}{2}}}{(N+1)(N-1)(N-3)} + \frac{|\tilde{x}^*-x|}{(N+2)N(N-2)}\right)$$

in the case of $N+1=3k$,

$$\Delta_1 \le \frac{8M(M+1)^{2N}}{1-(M+1)^6|\tilde{x}^*-x|^{\frac{3}{2}}}|\tilde{x}^*-x|^{\frac{N}{2}}$$

$$\times\left(\frac{1}{N(N-2)(N-4)} + \frac{|\tilde{x}^*-x|^{\frac{1}{2}}}{(N+1)(N-1)(N-3)} + \frac{|\tilde{x}^*-x|}{(N+2)N(N-2)}\right)$$

For the variant $N+1=3k+1$, and

$$\Delta_1 \le \frac{8M(M+1)^{2(N-1)}}{1-(M+1)^6|\tilde{x}^*-x|^{\frac{3}{2}}}|\tilde{x}^*-x|^{\frac{N}{2}}$$

$$\times\left(\frac{1}{N(N-2)(N-4)} + \frac{|\tilde{x}^*-x|^{\frac{1}{2}}}{(N+1)(N-1)(N-3)} + \frac{|\tilde{x}^*-x|}{(N+2)N(N-2)}\right)$$

where $N+1=3k+2$.

Data are fair in the domain $F_2 = \left\{x : |\tilde{x}^*-x| < \frac{1}{4(M+1)^4}\right\}$.

Then, we proceed to the estimates for Δ_2:

$$\sum_7^\infty |\Delta\tilde{C}_n||\tilde{x}_2^*-x|^{\frac{n-1}{2}} = \sum_4^\infty |\Delta\tilde{C}_{2n}||\tilde{x}_2^*-x|^{\frac{2n-1}{2}} + \sum_4^\infty |\Delta\tilde{C}_{2n-1}||\tilde{x}_2^*-x|^{n-1} = \Delta_{21}+\Delta_{22}.$$

Let us consider the case of fractional powers, taking into account the regularity of estimates for $\Delta\tilde{C}_n$ and obtained estimates $\sup_U|\tilde{x}^*-x|^{\frac{n-1}{2}}$:

$$
\begin{aligned}
\Delta_{21} = \sum_4^\infty \left|\Delta\tilde{C}_{2n}\right| |\tilde{x}_2^* - x|^{\frac{2n-1}{2}} &\leq \sum_1^\infty \left|\Delta\tilde{C}_{6n-2}\right| |\tilde{x}_2^* - x|^{\frac{6n-3}{2}} \\
&+ \sum_1^\infty \left|\Delta\tilde{C}_{6n}\right| |\tilde{x}_2^* - x|^{\frac{6n-1}{2}} + \sum_1^\infty \left|\Delta\tilde{C}_{6n+2}\right| |\tilde{x}_2^* - x|^{\frac{6n+1}{2}} \\
&\leq \sum_1^\infty \frac{8\Delta M(M+\Delta M+1)^{12n-6}}{(6n-3)(6n-5)(6n-7)} |\tilde{x}_2^* - x|^{3n-3/2} + \sum_1^\infty \frac{8\Delta M(M+\Delta M+1)^{12n}}{(6n-1)(6n-3)(6n-5)} |\tilde{x}_2^* - x|^{3n-1/2} \\
&+ \sum_1^\infty \frac{8\Delta M(M+\Delta M+1)^{12n-6}}{(6n+1)(6n-1)(6n-3)} |\tilde{x}_2^* - x|^{3n+1/2} \leq \frac{8\Delta M(M+\Delta M+1)^6 \left|\tilde{x}_2^*-x\right|^{3/2}}{1-(M+\Delta M+1)^{12}\left|\tilde{x}_2^*-x\right|^3} \\
&+ \frac{8\Delta M(M+\Delta M+1)^{12} \left|\tilde{x}_2^*-x\right|^{5/2}}{1-(M+\Delta M+1)^{12}\left|\tilde{x}_2^*-x\right|^3} + \frac{8\Delta M(M+\Delta M+1)^6 \left|\tilde{x}_2^*-x\right|^{7/2}}{1-(M+1)^{12}\left|\tilde{x}_2^*-x\right|^3} \\
&= \frac{8\Delta M(M+\Delta M+1)^6 \left|\tilde{x}_2^*-x\right|^{3/2}}{1-(M+\Delta M+1)^{12}\left|\tilde{x}_2^*-x\right|^3} \\
&\times \left(1 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x| + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|^2\right).
\end{aligned}
$$

The situation is similar for the case of integer degrees Δ_{22}:

$$
\Delta_{22} \leq \frac{8\Delta M(M+\Delta M+1)^{12} |\tilde{x}_2^* - x|^3}{1-(M+\Delta M+1)^{12}\left|\tilde{x}_2^* - x\right|^3}
$$

$$
\times \left(1 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x| + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|^2\right).
$$

Thus, we obtain the final expression for the estimate Δ_2:

$$
\Delta_2 \leq \frac{8\Delta M(M+\Delta M+1)^6 |\tilde{x}_2^* - x|^{3/2}}{1-2^3(M+\Delta M+1)^{12}\left|\tilde{x}_2^* - x\right|^3}
$$

$$
\times \left(1 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x| + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|^2\right)
$$

$$
\times \left(1 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|^{3/2}\right).
$$

It remains to show the validity of the estimate Δ_3:

$$
\begin{aligned}
\Delta_3 \leq \Delta\tilde{x}^* \left(\sum_0^\infty \left|\tilde{C}_n\right| |\tilde{x}_2^* - x|^{\frac{n-1}{2}}\right)' &= \Delta\tilde{x}^* \left(\sum_0^\infty \frac{8M(M+\Delta M+1)^{6\left[\frac{n}{3}\right]}}{(n-1)(n-3)(n-5)} |\tilde{x}_2^* - x|^{\frac{n-1}{2}}\right)' \\
&\leq \Delta\tilde{x}^* \left(\frac{8M(M+\Delta M+1)^6 \left|\tilde{x}_2^*-x\right|^{3/2}}{1-(M+\Delta M+1)^{12}\left|\tilde{x}_2^*-x\right|^3}\right. \\
&\times \left(1 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x| + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|^2\right) \\
&\left.\times \left(1 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|^{3/2}\right)\right)' = \frac{4\Delta\tilde{x}^*\Delta M(M+\Delta M+1)^6 \left|\tilde{x}_2^*-x\right|^{3/2}}{\left(1-(M+\Delta M+1)^6\left|\tilde{x}_2^*-x\right|^{3/2}\right)^2} \\
\times \Big(3 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|(7|\tilde{x}_2^* - x| + 5) &- 2(M+\Delta M+1)^{12} |\tilde{x}_2^* - x|^{5/2}(2|\tilde{x}_2^* - x| + 1)\Big).
\end{aligned}
$$

After transformations, we obtain an estimate for Δ_3:

$$
\Delta_3 \leq \frac{4\Delta\tilde{x}^* M(M+\Delta M+1)^6 |\tilde{x}_2^* - x|^{3/2}}{\left(1-(M+\Delta M+1)^6\left|\tilde{x}_2^* - x\right|^{3/2}\right)^2}
$$

$$
\times \left(3 + (M+\Delta M+1)^6 |\tilde{x}_2^* - x|(7|\tilde{x}_2^* - x| + 5) - 2(M+\Delta M+1)^{12} |\tilde{x}_2^* - x|^{5/2}\right.
$$

$$
\times \ (2|\tilde{x}_2^* - x| + 1)).
$$

The estimates Δ_2 and Δ_3 work in the domain $F_3 = \left\{x : |\tilde{x}_2^* - x| < \frac{1}{4(M+\Delta M+1)^4}\right\}$.

Finally, we obtain the domain $F = F_1 \cap F_2 \cap F_3$. $\square$

3. Discussion of Results

Example 1. *Let us calculate the Cauchy problem (1)–(2), with the conditions* $r(x) = 0$, $y(0) = 0.998$, $y'(0) = 0.998$, $y''(0) = 0.998$, $\tilde{x}^* = 0.93157$, *and* $\Delta\tilde{x}^* = 0.0002$. *We consider the problem (1)–(2) which, under these conditions, is not solvable in quadratures. Applying the obtained results, we calculate the approximate solution (4) for* $N = 9$ *to notify the argument* x_1 *by taking into account the domain* $F = F_1 \cap F_2 \cap F_3$, *to which the theorem [19] may be applied* $\rho = 0.015625$ *(a designation accepted in [19]). The calculation results are presented in Table 1.*

Table 1. Characteristics of the approximate solution.

x_1	$\tilde{y}_9(x_1)$	Δ_1	Δ_2	Δ_3
0.92	10.3204	0.04	0.07	0.0005

Where $\tilde{y}_9(x_1)$ *is an analytical approximate solution (4),* Δ_1 *is an error estimate obtained according to the work [19],* Δ_2 *is an error estimate obtained in this paper;* Δ_3 *is an a-posteriori estimate for the solution (4). In the case of* $\Delta_3 = 0.0005$, *the proven theorem requires* $N = 14$. *The terms from 10 to 14 in the total do not affect the accuracy of the result—*$\varepsilon = 0.0005$; *this means that, when* $N = 9$ *we obtain* $\tilde{y}_9(x_1)$ *with accuracy* $\varepsilon = 0.0005$. *It should be noted that the a-priori estimates in this paper and in [19] are of the same order.*

Example 2. *Consider the Cauchy problem (1)–(2), with the conditions such as* $r(x) = 0$, $y(0) = 0.998$, $y'(0) = 0.998$, $y''(0) = 0.998$, $\tilde{x}^* = 0.93157$, *and* $\Delta\tilde{x}^* = 0.0002$. *We consider the value that satisfies the conditions of this theorem, located in the region* $F = F_1 \cap F_2 \cap F_3$*—*$\rho_2 = 0.015355$, $x_2 = 0.9163$, *but not falling under the results of the theorem in [19]. The calculation results are presented in Table 2.*

Table 2. Characteristics of the approximate solution according to the theory of this article.

x_2	$\tilde{y}_9(x_2)$	Δ_1'	Δ_2'
0.9163	8.9841	0.02	0.0009

Where $\tilde{y}_9(x_2)$ *is an approximate solution (4);* Δ_1' *is an error estimate, obtained in this paper; and* Δ_2' *is an a-posteriori solution error estimate (4). For the variant* $\Delta_2' = 0.0009$, *according to the theorem proven in the paper, it turns out that* $N = 13$. *Terms from 10 to 13 do not affect the accuracy of calculations completed* $\varepsilon = 0.0009$. *This means that when* $N = 9$, *the obtained result with the value* $\tilde{y}_9(x_2)$ *has an accuracy of* $\varepsilon = 0.0009$.

4. Conclusions

The author's approach to the method of analytical approximate solution finds further development of the example of the considered class of nonlinear equations. The article proves the error estimate for the analytical approximate solution of the considered equation with moving singular points. Conducted studies allow to substantially approach the region obtained in [17] in proving the uniqueness of the solution of the nonlinear equation under consideration. The exact boundaries of the area of application of the approximate solution in the neighborhood of the approximate value of the moving singular point were obtained. The presented studies were confirmed by numerical experiments.

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