axioms

MDPI

Article

New Fractional Integral Inequalities Pertaining to Caputo–Fabrizio and Generalized Riemann–Liouville Fractional Integral Operators

Muhammad Tariq [1], Omar Mutab Alsalami [2], Asif Ali Shaikh [1], Kamsing Nonlaopon [3,*] and Sotiris K. Ntouyas [4,5]

1 Department of Basic Sciences and Related Studies, Mehran University of Engineering and Technology, Jamshoro 76062, Pakistan
2 Department of Electrical Engineering, College of Engineering, Taif University, Taif 21944, Saudi Arabia
3 Department of Mathematics, Faculty of Science, Khon Kaen University, Khon Kaen 40002, Thailand
4 Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece
5 Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Department of Mathematics, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
* Correspondence: nkamsi@kku.ac.th; Tel.: +66-8-6642-1582

Abstract: Integral inequalities have accumulated a comprehensive and prolific field of research within mathematical interpretations. In recent times, strategies of fractional calculus have become the subject of intensive research in historical and contemporary generations because of their applications in various branches of science. In this paper, we concentrate on establishing Hermite–Hadamard and Pachpatte-type integral inequalities with the aid of two different fractional operators. In particular, we acknowledge the critical Hermite–Hadamard and related inequalities for n-polynomial s-type convex functions and n-polynomial s-type harmonically convex functions. We practice these inequalities to consider the Caputo–Fabrizio and the k-Riemann–Liouville fractional integrals. Several special cases of our main results are also presented in the form of corollaries and remarks. Our study offers a better perception of integral inequalities involving fractional operators.

Keywords: Hermite–Hadamard inequality; convex function; harmonically convex function; Caputo–Fabrizio fractional operator; fractional integral inequality

Citation: Tariq, M.; Alsalami, O.M.; Shaikh, A.A.; Nonlaopon, K.; Ntouyas, S.K. New Fractional Integral Inequalities Pertaining to Caputo–Fabrizio and Generalized Riemann–Liouville Fractional Integral Operators. *Axioms* **2022**, *11*, 618. https://doi.org/10.3390/axioms11110618

Academic Editor: Simeon Reich

Received: 28 September 2022
Accepted: 31 October 2022
Published: 7 November 2022

1. Introduction

The convex function is a class of significant functions popularly accepted in mathematical analysis. This class represents prominent parts of the theory of inequality. Moreover, convex functions have been widely used in many research fields such as optimization, engineering, physics, financial activities, etc. In optimization, the concept of generalized convexity along with inequality theory is often used. Hermite–Hadamard integral inequalities containing convex functions are an intense research topic for many mathematicians because of their relevance and efficiency in use.

Convex functions have a very strong association with integral inequalities. Recently, several mathematicians have explored the close relationship and correlated work on symmetry and convexity. It is also explained that while working on any one of the concepts, work tends to be applied to the other one too. Many familiar and relevant inequalities are modifications of convex functions. In the literature, there are some well-known inequalities such as the Hermite–Hadamard inequality and the Jensen inequality that interpret the geometrical meaning of convex functions. In this paper, we concentrate on presenting new versions of fractional integral inequalities through n-polynomial s-type convex functions and n-polynomial s-type harmonically convex functions. To begin the discussion, let us recall the definition of a convex function.

In 1905, Jensen presented the meaning of convex function as follows:

Definition 1 ([1,2])**.** *A function* $\Phi\colon [\mathfrak{a}_1, \mathfrak{a}_2] \to \mathbb{R}$ *is called convex if*

$$\Phi(\ell x + (1-\ell)y) \leq \ell\Phi(x) + (1-\ell)\Phi(y),$$

holds for every $x, y \in [\mathfrak{a}_1, \mathfrak{a}_2]$ *and* $\ell \in [0,1]$.

The well-known Hermite–Hadamard inequality is given as follows:

Theorem 1 (see [3])**.** *Consider* $\Phi : \mathbb{T} \subseteq \mathbb{R} \to \mathbb{R}$ *to be a convex function with* $\mathfrak{a}_1 < \mathfrak{a}_2$ *and* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$. *Then, the following inequality holds:*

$$\Phi\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) \leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1} \int_{\mathfrak{a}_1}^{\mathfrak{a}_2} \Phi(x)\mathrm{d}x \leq \frac{\Phi(\mathfrak{a}_1) + \Phi(\mathfrak{a}_2)}{2}. \tag{1}$$

Definition 2 (see [4])**.** *A function* $\Phi\colon \mathbb{T} \to \mathbb{R}$ *is said to be a harmonically convex function if*

$$\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1 + (1-\ell)\mathfrak{a}_2}\right) \leq \ell\Phi(\mathfrak{a}_2) + (1-\ell)\Phi(\mathfrak{a}_2), \tag{2}$$

holds for all $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$ *and* $\ell \in [0,1]$.

2. Preliminaries

The set $\mathbb{T} \subseteq \mathbb{R} \setminus \{0\}$ is called convex if $\ell x + (1-\ell)y \in \mathbb{T}$ for $x, y \in \mathbb{T}$ and $\ell \in [0,1]$ and the set $\mathbb{S} \subseteq \mathbb{R} \setminus \{0\}$ as harmonically convex if $\frac{xy}{\ell x + (1-\ell)y} \in \mathbb{S}$ for all $x, y \in \mathbb{S}$ and $\ell \in [o, 1]$. From now on, we always assume $\mathbb{T}$ to be a convex set and $\mathbb{S}$ as a harmonically convex set.

Many researchers have generalized and extended the Hermite–Hadamard inequality using different convexities. For example, Dragomir et al. [5], Qi et al. [6] and Kirmaci et al. [7] proved some refinements of Hermite–Hadamard inequality for differentiable functions and presented some applications of the main results for special means and trapezoidal rules. Furthermore, the related inequalities for s-convex functions were investigated in articles [8,9]. Özcan et al. [10] improved the refinements of Hermite–Hadamard type inequalities using improved Holder's inequality. Moreover, this inequality was also improved for interval-valued preinvex functions in [11]. Recently, a group of mathematicians, namely Toplu, Kadakal and İşcan [12], presented a very important class of convex function, i.e., the n-polynomial convex function, which is given as:

Let $n \in \mathbb{N}$. A function $\Phi : \mathbb{T} \to \mathbb{R}$ is said to be an n-polynomial convex function on $\mathbb{T}$, if

$$\Phi(\ell x + (1-\ell)y) \leq \frac{1}{n}\sum_{\wp=1}^{n}\left[1-(1-\ell)^{\wp}\right]\Phi(x) + \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\ell^{\wp}\right]\Phi(y),$$

for all $x, y \in \mathbb{T}$ and $\ell \in [0,1]$.

In the same paper, they also proved the following Hermite–Hadamard inequality employing this new generalized notion of convexity.

Theorem 2 (see [12])**.** *Suppose* $\Phi\colon \mathbb{T} \to \mathbb{R}$ *is an* n*-polynomial convex function,* $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$ *with* $\mathfrak{a}_1 < \mathfrak{a}_2$ *and* Φ *is a Lebesgue integrable function on* $[\mathfrak{a}_1, \mathfrak{a}_2]$. *Then the following integral inequality holds:*

$$\frac{2^{-1}n}{n + 2^{-n} - 1}\Phi\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) \leq \frac{1}{\mathfrak{a}_2 - \mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\Phi(x)\mathrm{d}x \leq \frac{\Phi(\mathfrak{a}_1) + \Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\frac{\wp}{\wp+1}. \tag{3}$$

If we set $n = 1$ *in the inequality* (3)*, then the classical Hermite–Hadamard inequality* (1) *for a convex function is recovered.*

Inspired by the above-mentioned article, Awan et al. [13], extended the concept of n-polynomial convexity and presented a generalized version of a harmonically convex function, i.e., an n-polynomial harmonically convex function, given as:

A function $\Phi : \mathbb{S} \to \mathbb{R}^+$ is said to be an n-polynomial harmonically convex if for all $x, y \in \mathbb{S}$, $n \in N$ and $\ell \in [0,1]$, the following inequality holds.

$$\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1 + (1-\ell)\mathfrak{a}_2}\right) \leq \frac{1}{n}\sum_{\wp=1}^{n}(1-(1-\ell)^{\wp})\Phi(\mathfrak{a}_2) + \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\ell^{\wp}\right]\Phi(\mathfrak{a}_1).$$

In the same paper, the following new version of Hermite–Hadamard inequality was established.

Theorem 3 (see [13]). *Suppose $\Phi\colon \mathbb{S} \to \mathbb{R}^+$ is an n-polynomial harmonically convex function. If $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{S}$ with $0 < \mathfrak{a}_1 < \mathfrak{a}_2$ and $\Phi \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_2]$, then the following integral inequality holds.*

$$\frac{2^{-1}n}{n+2^{-n}-1}\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right) \leq \frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\frac{\Phi(x)}{x^2}dx \leq \frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\frac{\wp}{\wp+1}.$$

Definition 3 ([14]). *A function $\Phi\colon \mathbb{T} \to \mathbb{R}$ is said to be an n-polynomial s-type convex function for $n \in \mathbb{N}$. If for $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$ with $\ell, s \in [0,1]$, the following inequality satisfies.*

$$\Phi(\ell x + (1-\ell)y) \leq \frac{1}{n}\sum_{\wp=1}^{n}\left[1-(s(1-\ell))^{\wp}\right]\Phi(x) + \frac{1}{n}\sum_{\wp=1}^{n}\left[1-(s\ell)^{\wp}\right]\Phi(y). \tag{4}$$

Theorem 4 (see [14]). *Let $\Phi\colon \mathbb{S} \to \mathbb{R}$ be an n-polynomial s-type convex function. If $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$ with $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$ with $\mathfrak{a}_1 < \mathfrak{a}_2$. If $\Phi \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_2]$, then the following integral inequality holds.*

$$\frac{2^{-1}}{\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right) \leq \frac{1}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\Phi(x)dx \leq \frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\left[\frac{\wp+1-s^{\wp}}{\wp+1}\right]. \tag{5}$$

Integral inequalities have been indispensable in establishing the uniqueness of solutions for certain fractional differential equations. Sarikaya et al. [15] introduced the fractional version of Hermite–Hadamard inequality employing a Riemann–Liouville fractional operator. Motivated by this article many mathematicians used different notions of fractional operators to generalize inequalities such as Hermite–Hadamard, Ostrowski, Simpson, Opial, Jensen-Mercer, etc. To carry forward our investigation about fractional calculus, we start with the notion of the Caputo–Fabrizio fractional operator.

Note: From now on, we will use $\mathcal{M}(\lambda) > 0$ as a normalization function satisfying $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

Let $\mathcal{L}^2(\mathfrak{a}_1, \mathfrak{a}_2)$ be the space of square integrable function on the interval $(\mathfrak{a}_1, \mathfrak{a}_2)$ and

$$H'(\mathfrak{a}_1, \mathfrak{a}_2) = \left\{g/g \in \mathcal{L}^2(\mathfrak{a}_1, \mathfrak{a}_2) \quad and \quad g' \in \mathcal{L}^2(\mathfrak{a}_1, \mathfrak{a}_2)\right\}.$$

If $\Phi \in H'(\mathfrak{a}_1, \mathfrak{a}_2)$, $\mathfrak{a}_1 < \mathfrak{a}_2$ and $\lambda) \in [0,1]$, then the left- and right-sided Caputo–Fabrizio fractional integral operator ${}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_1}$ and ${}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}$ are defined as:

Definition 4 (see [16,17]). *Let $\Phi \in H'(\mathfrak{a}_1, \mathfrak{a}_2)$, $\mathfrak{a}_1 < \mathfrak{a}_2$, $\lambda \in [0,1]$, then the definition of the left fractional integral in the sense of Caputo and Fabrizio becomes*

$$\left({}^{C\mathcal{F}}_{\mathfrak{a}_1}\mathrm{I}^{\lambda}\Phi\right)(\varphi) = \frac{(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(\varphi) + \frac{\lambda}{\mathcal{M}(\lambda)}\int_{\mathfrak{a}_1}^{\varphi}\Phi(x)\mathrm{d}x, \tag{6}$$

$$\left({}^{C\mathcal{F}}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi\right)(\varphi) = \frac{(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(\varphi) + \frac{\lambda}{\mathcal{M}(\lambda)}\int_{\varphi}^{\mathfrak{a}_2}\Phi(x)\mathrm{d}x, \tag{7}$$

where $\mathcal{M} : [0,1] \to (0,\infty)$ is a normalization function satisfying $\mathcal{M}(0) = \mathcal{M}(1) = 1$.

Gürbüz et al. [16] used Caputo–Fabrizio fractional integrals to establish the following Hermite–Hadamard inequality.

Theorem 5 (see [16])**.** *Let $\Phi : \mathbb{T} \to \mathbb{R}$ be a convex function on $\mathbb{T}$. If $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$ with $\mathfrak{a}_1 < \mathfrak{a}_2$ and Φ is a Lebesgue integral function on $[\mathfrak{a}_1, \mathfrak{a}_2]$, then the following double inequality holds:*

$$\Phi\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) \leq \frac{\mathcal{M}(\lambda)}{\lambda(\mathfrak{a}_2 - \mathfrak{a}_1)}\left[\left({}^{C\mathcal{F}}\mathrm{I}^{\lambda}_{\mathfrak{a}_1}\Phi\right)(k) + \left({}^{C\mathcal{F}}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi\right)(k) - \frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(k)\right] \leq \frac{\Phi(\mathfrak{a}_1) + \Phi(\mathfrak{a}_2)}{2},$$

where $\lambda \in [0,1], k \in [\mathfrak{a}_1, \mathfrak{a}_2]$.

Theorem 6. *Let $\Phi : \mathbb{T} \to \mathbb{R}$ be a Lebesgue integrable function on $[\mathfrak{a}_1, \mathfrak{a}_2]$ with $\mathfrak{a}_1 < \mathfrak{a}_2$ and $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$. If Φ is an n-polynomial convex function then,*

$$\frac{2^{-1}n}{n + 2^{-n} - 1}\Phi\left(\frac{\mathfrak{a}_1 + \mathfrak{a}_2}{2}\right) \leq \frac{\mathcal{M}(\lambda)}{\lambda(\mathfrak{a}_2 - \mathfrak{a}_1)}\left[{}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_1}\Phi(r) + {}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi(r) - \frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\right] \leq \frac{\Phi(\mathfrak{a}_1) + \Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\frac{\wp}{\wp + 1},$$

where $\lambda \in [0,1], r \in [\mathfrak{a}_1, \mathfrak{a}_2]$ and $\mathcal{M}(\lambda) > 0$, is a normalization function.

Fractional derivatives and integral operators have recently been used to generalize existing kernels. Nwaeze et al. [18] proved fractional versions of Hermite–Hadamard inequality for n-polynomial convex and n-polynomial harmonically convex functions. Sahoo et al. [19] established some new Hermite–Hadamard type fractional inequalities for (h-m) convex functions. Abdeljawad et al. [20] used local fractional integrals to present inequalities for generalized (s,m)-convex functions. Ostrowski-type inequalities are also investigated using fractional operators in [21,22]. Further refinements of Hermite–Hadamard inequalities are done for Wright-generalized Bessel functions [23], polynomial convex functions [24] and for strongly convexity via Atangana–Baleanu operators [25].

The Caputo–Fabrizio fractional derivative was introduced by Caputo and Fabrizio [26] in 2015. The advantage of this proposition was due to the necessity of accepting a model that describes structures with various scales. Recently, it has been seen that many mathematicians are showing their interest in using the Caputo fractional derivative and Caputo–Fabrizio fractional integral to establish fractional integral inequalities such as Hermite–Hadamard, Ostrowski, etc. The persistence of this article is to employ the Caputo–Fabrizio fractional integral operator and k-Riemann–Liouville fractional operator to investigate some new types of integral inequalities involving n-polynomial convex and n-polynomial harmonically convex functions, which have been presented earlier using various fractional operators such as Riemann–Liouville, Atangana–Baleanu, Katugampola, generalized fractional operators, etc. The results presented could be remedial to prove the existence and uniqueness of some fractional differential equations.

Now we recall that the left- and right-side k-Riemann–Liouville fractional operator ${}_k\mathrm{I}^{\lambda}_{\mathfrak{a}_1+}$ and ${}_k\mathrm{I}^{\lambda}_{\mathfrak{a}_2-}$ of order $\lambda > 0$ for a real valued continuous function $\Phi(x)$ are defined by (see [27,28]).

$${}_k\mathrm{I}^{\lambda}_{\mathfrak{a}_1+}\Phi(x) = \frac{1}{k\Gamma(\lambda)}\int_{\mathfrak{a}_1}^{x}(x-t)^{\frac{\lambda}{k}-1}\Phi(\ell)dt \quad x > \mathfrak{a}_1,$$

and

$${}_k\mathrm{I}^{\lambda}_{\mathfrak{a}_2-}\Phi(x) = \frac{1}{k\Gamma(\lambda)}\int_{x}^{\mathfrak{a}_2}(t-x)^{\frac{\lambda}{k}-1}\Phi(\ell)dt \quad x < \mathfrak{a}_2.$$

When $k > 0$ and Γ_k is the k-gamma function given by

$$\Gamma_k(x) = \int_0^x \ell^{x-1}\exp^{\frac{-\ell k}{k}}d\ell \quad Re(x) > 0,$$

with the properties $\Gamma_k(x+k) = x\Gamma_k(x)$ and $\Gamma_k(k) = 1$ if $k = 1$ we simply write ${}_1\mathrm{I}^{\lambda}_{\mathfrak{a}_1+}\Phi = \mathrm{I}^{\lambda}_{\mathfrak{a}_1+}\Phi$ and ${}_1\mathrm{I}^{\lambda}_{\mathfrak{a}_1+}\Phi = \mathrm{I}^{\lambda}_{\mathfrak{a}_1+}\Phi$. The beta function is defined by

$$\beta(u,v) = \int_0^1 \ell^{u-1}(1-\ell)^{v-1}d\ell \quad for \quad R_e(u) > 0, R_e(v) > 0. \tag{8}$$

The novelty of this article is that it deals with inequalities of Hermite–Hadamard and Pachpatte type for higher-order convexity, i.e., n-polynomial s-type convex and n-polynomial s-type harmonically convex functions employing two different types of fractional integral operators. The rest of the article has the following structure: after studying some necessary concepts about fractional calculus and Hermite–Hadamard type inequalities, in Section 3, we present new variants of Hermite–Hadamard-type inequality via Caputo–Fabrizio fractional operators for n-polynomial s-type convex functions. Next, Section 4 is dedicated to establishing Hermite–Hadamard inequalities for n-polynomial s-type harmonically convex functions via k-Riemann–Liouville fractional operators. A brief conclusion and future scopes of the present work is given in the last Section 5.

3. Main Results

Theorem 7. *Let $\Phi : \mathbb{T} \to \mathbb{R}$ be an n-polynomial s-type convex function on $\mathbb{T}$ with $\mathfrak{a}_1 < \mathfrak{a}_2$ and $\mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{T}$. If Φ is a Lebesgue integrable function on $[\mathfrak{a}_1, \mathfrak{a}_2]$, then*

$$\begin{aligned}\frac{2^{-1}n}{\sum\limits_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right) &\leq \frac{\mathcal{M}(\lambda)}{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}\left[{}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_1}\Phi(r) + {}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi(r) - \frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\right] \\ &\leq \frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\frac{\wp+1-s^{\wp}}{\wp+1},\end{aligned}$$

where $\lambda \in [0,1], s \in [0,1], r \in [0,1]$ and $\mathcal{M}(\lambda) > 0$ is a normalization function.

Proof. Given that Φ is n-polynomial s-type convex function. It follows from Equation (5) that

$$\begin{aligned}\frac{n}{\sum\limits_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right) &\leq \frac{2}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\Phi(x)dx \\ &= \frac{2}{\mathfrak{a}_2-\mathfrak{a}_1}\left[\int_{\mathfrak{a}_1}^{r}\Phi(x)dx + \int_{r}^{\mathfrak{a}_2}\Phi(x)dx\right].\end{aligned} \tag{9}$$

Multiplying both sides of Equation (9) by $\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}$ gives

$$\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}\frac{n}{\sum\limits_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right) \leq \frac{\lambda}{\mathcal{M}(\lambda)}\left[\int_{\mathfrak{a}_1}^{r}\Phi(x)dx + \int_{r}^{\mathfrak{a}_2}\Phi(x)dx\right]. \tag{10}$$

By adding $\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)$ to both sides of Equation (10), we obtain

$$\begin{aligned}&\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r) + \frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}\frac{n}{\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right) \leq \frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r) \\ &+ \frac{\lambda}{\mathcal{M}(\lambda)}\left[\int_{\mathfrak{a}_1}^{r}\Phi(x)dx + \int_{r}^{\mathfrak{a}_2}\Phi(x)dx\right] \\ &= \left[\frac{(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r) + \frac{\lambda}{\mathcal{M}(\lambda)}\int_{\mathfrak{a}_1}^{r}\Phi(x)dx\right] + \left[\frac{(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r) + \frac{\lambda}{\mathcal{M}(\lambda)}\int_{r}^{\mathfrak{a}_2}\Phi(x)dx\right] \\ &= {}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_1}\Phi(r) + {}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi(r).\end{aligned}$$

This implies that

$$\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)+\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}\frac{n}{\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right)\leq^{CF}\mathrm{I}_{\mathfrak{a}_1}^{\lambda}\Phi(r)+^{CF}\mathrm{I}_{\mathfrak{a}_2}^{\lambda}\Phi(r). \tag{11}$$

On the other hand from Equation (5), we also obtain

$$\frac{2}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\Phi(x)dx\leq\frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\left[\frac{\wp+1-s^{\wp}}{\wp+1}\right]. \tag{12}$$

If we multiply Equation (12) by $\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}$ and then add $\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)$ to the resulting inequality, we obtain

$$^{CF}\mathrm{I}_{\mathfrak{a}_1}^{\lambda}\Phi(r)+^{CF}\mathrm{I}_{\mathfrak{a}_2}^{\lambda}\Phi(r)\leq\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{\mathcal{M}(\lambda)}\frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\left[\frac{\wp+1-s^{\wp}}{\wp+1}\right]+\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r). \tag{13}$$

Hence, the desired result is obtained by combining Equations (11) and (13). □

Remark 1. *By taking $s=1$, Theorem 7 becomes Theorem 6.*

Corollary 1. *By taking $n=1$, Theorem 7 becomes the following inequality,*

$$\begin{aligned}&\Phi\left(\frac{\mathfrak{a}_1+\mathfrak{a}_2}{2}\right)\\&\leq\frac{2-s}{\lambda}\frac{\mathcal{M}(\lambda)}{\mathfrak{a}_2-\mathfrak{a}_1}\left[{}_{\mathfrak{a}_1}^{CF}\mathrm{I}^{\lambda}\Phi(r)+^{CF}\mathrm{I}_{\mathfrak{a}_2}^{\lambda}\Phi(r)-\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\right]\leq\frac{(2-s)^2}{2}[\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)].\end{aligned}$$

Remark 2. *By taking $n=s=1$, then Theorem 7 becomes Theorem 5.*

Theorem 8. *Suppose $\Phi,\Upsilon:\mathbb{T}\to\mathbb{R}$ is functions such that $\Phi\,\Upsilon$ is integrable on $[\mathfrak{a}_1,\mathfrak{a}_2]$ with $\mathfrak{a}_1<\mathfrak{a}_2$ and $\mathfrak{a}_1,\mathfrak{a}_2\in\mathbb{T}$. If Φ is n_1-polynomial s-type convex function and Φ is an n_2-polynomial s-type convex function, then the following inequality holds:*

$$\begin{aligned}&\frac{\mathcal{M}(\lambda)}{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}\left[{}_{\mathfrak{a}_1}^{CF}I^{\lambda}\Phi(r)\Upsilon(r)+^{CF}\mathrm{I}_{\mathfrak{a}_2}^{\lambda}\Phi(r)\Upsilon(r)-\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\Upsilon(r)\right]\\&\leq\int_0^1[\Delta_1(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_1)+\Delta_2(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_2)+\Delta_3(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1)+\Delta_4(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_2)]d\ell,\end{aligned}$$

where $\lambda\in[0,1]$ and $r\in[\mathfrak{a}_1,\mathfrak{a}_2]$ and $\mathcal{M}(\lambda)>0$ is a normalization function and

$$\begin{aligned}\Delta_1(\ell)&=\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}[1-(s(1-\ell))^{\wp}]\sum_{\wp=1}^{n_2}[1-(s(1-\ell))^{\wp}],\\\Delta_2(\ell)&=\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}[1-(s\ell)^{\wp}]\sum_{\wp=1}^{n_2}[1-(s\ell)^{\wp}],\\\Delta_3(\ell)&=\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}[1-(s\ell)^{\wp}]\sum_{\wp=1}^{n_2}[1-(s(1-\ell))^{\wp}],\\\Delta_4(\ell)&=\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}[1-(s(1-\ell))^{\wp}]\sum_{\wp=1}^{n_2}[1-(s\ell)^{\wp}].\end{aligned}$$

Proof. Let Φ be n_1-polynomial s-type convex function and Υ is n_2-polynomial s-type convex function

$$\Phi\Big(\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2\Big)\leq\frac{1}{n_1}\sum_{\wp=1}^{n_1}\left[1-\big(s(1-\ell)\big)^{\wp}\right]\Phi(\mathfrak{a}_1)+\frac{1}{n_1}\sum_{\wp=1}^{n_1}\left[1-\big(s\ell\big)^{\wp}\right]\Phi(\mathfrak{a}_2). \tag{14}$$

$$\Upsilon\Big(\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2\Big)\leq\frac{1}{n_2}\sum_{\wp=1}^{n_2}\Big[1-\big(s(1-\ell)\big)^{\wp}\Big]\Upsilon(\mathfrak{a}_1)+\frac{1}{n_2}\sum_{\wp=1}^{n_2}\Big[1-(s\ell)^{\wp}\Big]\Upsilon(\mathfrak{a}_2). \quad (15)$$

Multiplying (14) and (15).

$$\begin{aligned}
&\Phi\Big(\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2\Big)\Upsilon\Big(\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2\Big)\\
&\leq\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_2}\Big[1-\big(s(1-\ell)\big)^{\wp}\Big]\sum_{\wp=1}^{n_1}\Big[1-\big(s(1-\ell)\big)^{\wp}\Big]\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_1)\\
&+\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_2}\Big[1-(s\ell)^{\wp}\Big]\sum_{\wp=1}^{n_1}\Big[1-(s\ell)^{\wp}\Big]\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_2)\\
&+\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}\Big[1-(s\ell)^{\wp}\Big]\sum_{\wp=1}^{n_2}\Big[1-\big(s(1-\ell)\big)^{\wp}\Big]\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1)\\
&+\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}\Big[1-(s\ell)^{\wp}\Big]\sum_{\wp=1}^{n_2}\Big[1-\big(s(1-\ell)\big)^{\wp}\Big]\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1). \quad (16)\\
&=\Delta_1(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_1)+\Delta_2(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_2)+\Delta_3(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1)+\Delta_4(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_2).
\end{aligned}$$

This implies that

$$\begin{aligned}
&\Phi\Big(\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2\Big)\Upsilon\Big(\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2\Big)\\
&\quad\leq\Delta_1(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_1)+\Delta_2(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_2)+\Delta_3(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1)+\Delta_4(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_2).
\end{aligned}$$

Integrating both sides of (16) with respect to over $[0,1]$ results to

$$\begin{aligned}
\frac{2}{\mathfrak{a}_2-\mathfrak{a}_1}\int_{\mathfrak{a}_1}^{\mathfrak{a}_2}\Phi(x)\Upsilon(x)dx&\leq 2\int_0^1\Big[\Delta_1(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_1)+\Delta_2(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_2)\\
&+\Delta_3(\ell)\Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1)+\Delta_4(\ell)\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_2)\Big]d\ell\\
&=N(\mathfrak{a}_1,\mathfrak{a}_2).
\end{aligned}$$

Consequently,

$$\frac{2}{\mathfrak{a}_2-\mathfrak{a}_1}\left[\int_{\mathfrak{a}_1}^{r}\Phi(x)\Upsilon(x)dx+\int_{r}^{\mathfrak{a}_2}\Phi(x)\Upsilon(x)dx\right]\leq N(\mathfrak{a}_1,\mathfrak{a}_2). \quad (17)$$

Now, multiplying (17) by $\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}$ and then adding $\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)$ to the result, we obtain

$$\begin{aligned}
&\frac{\lambda}{\mathcal{M}(\lambda)}\left[\int_{\mathfrak{a}_1}^{r}\Phi(x)\Upsilon(x)dx+\int_{r}^{\mathfrak{a}_2}\Phi(x)\Upsilon(x)dx\right]+\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\Upsilon(r)\\
&\leq\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}N(\mathfrak{a}_1,\mathfrak{a}_2)+\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\Upsilon(r).
\end{aligned}$$

Hence,

$${}^{CF}_{\mathfrak{a}_1}\mathrm{I}^{\lambda}\Phi(r)\Upsilon(r)+{}^{CF}\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi(r)\Upsilon(r)\leq\frac{\lambda(\mathfrak{a}_2-\mathfrak{a}_1)}{2\mathcal{M}(\lambda)}N(\mathfrak{a}_1,\mathfrak{a}_2)+\frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\Upsilon(r).$$

From which we obtain the intended inequality. □

Remark 3. *If we put $s=1$ in Theorem 8, we get Theorem 6.*

Remark 4. *If we put $n_1=n_2=1, s=1$ in Theorem 8, we obtain Theorem 4.*

Corollary 2. *If we put $n_1 = n_2 = 1$, in Theorem 8, then*

$$\frac{2\mathcal{M}(\lambda)}{\lambda(\mathfrak{a}_2 - \mathfrak{a}_1)}\left[{}^{CF}_{\mathfrak{a}_1}\mathrm{I}^{\lambda}\Phi(r)\Upsilon(r) + {}^{CF}\,\mathrm{I}^{\lambda}_{\mathfrak{a}_2}\Phi(r)\Upsilon(r) - \frac{2(1-\lambda)}{\mathcal{M}(\lambda)}\Phi(r)\Upsilon(r)\right]$$
$$\leq \frac{2}{3}(3(1-s)+s^3)[\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_1) + \Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_2)]$$
$$+ \frac{1}{3}(6(1-s)+s^2)[\Phi(\mathfrak{a}_1)\Upsilon(\mathfrak{a}_2) + \Phi(\mathfrak{a}_2)\Upsilon(\mathfrak{a}_1)].$$

4. Further Estimations via *n*-Polynomial Harmonically *s*-Type Convex Function

Theorem 9. *Suppose $\Phi : \mathbb{S} \to \mathbb{R}^+$ be an n-polynomial harmonically s-type convex function on $\mathbb{S}$ with $\mathfrak{a}_1 < \mathfrak{a}_2$ and $\Phi \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_2]$ and $\mathfrak{a}_1, \mathfrak{a}_2 > 0$, $s \in [0,1]$. Then, the following fractional inequality holds:*

$$\frac{1}{\sum\limits_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}$$
$$\leq \Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\frac{\Gamma_k(\lambda)+k}{n}\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_k\mathrm{I}^{\lambda}_{\frac{1}{\mathfrak{a}_2}+}\Phi\circ\Psi\left(\frac{1}{\mathfrak{a}_1}\right) + {}_k\mathrm{I}^{\lambda}_{\frac{1}{\mathfrak{a}_1}-}\Phi\circ\Psi\left(\frac{1}{\mathfrak{a}_2}\right)\right]$$
$$\leq \frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n^2}\sum_{\wp=1}^{n}\left[2 - \frac{s^{\wp}\lambda}{\lambda+ik} - \frac{s^{\wp}\lambda}{k}\beta\left(\frac{\lambda}{k},\wp+1\right)\right],$$

where $\Psi(r) = \frac{1}{r}$ and β is the beta function.

Proof. Given that Φ is n-polynomial s-type convex function,

$$\Phi\left(\frac{2xy}{x+y}\right) \leq \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right][\Phi(x)+\Phi(y)]. \tag{18}$$

Now, let $x = \frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}$ and $y = \frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}$ then (18) becomes,

$$\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right) \leq \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}. \tag{19}$$

Multiplying both sides of Equation (19) by $\ell^{\frac{\lambda}{k}-1}$ and integrating with respect to ℓ over $[0,1]$, we obtain

$$\int_0^1 \ell^{\frac{\lambda}{k}-1}\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)d\ell$$
$$\leq \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]\int_0^1 \ell^{\frac{\lambda}{k}-1}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}d\ell$$
$$= \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]\left\{\int_0^1 \ell^{\frac{\lambda}{k}-1}\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\int_0^1 \ell^{\frac{\lambda}{k}-1}\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)d\ell\right\}$$
$$= \frac{1}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]$$
$$\times\left[\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\int_{\frac{1}{\mathfrak{a}_2}}^{\frac{1}{\mathfrak{a}_1}}\left(\frac{1}{\mathfrak{a}_1}-r\right)^{\frac{\lambda}{k}-1}\Phi\left(\frac{1}{r}\right)dr + \left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\int_{\frac{1}{\mathfrak{a}_2}}^{\frac{1}{\mathfrak{a}_1}}\left(r-\frac{1}{\mathfrak{a}_2}\right)^{\frac{\lambda}{k}-1}\Phi\left(\frac{1}{r}\right)dr\right]$$

$$= \frac{k\Gamma_k(\lambda)}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}$$
$$\times\left[\frac{1}{k\Gamma_k(\lambda)}\int_{\frac{1}{\mathfrak{a}_2}}^{\frac{1}{\mathfrak{a}_1}}\left(\frac{1}{\mathfrak{a}_1}-r\right)^{\frac{\lambda}{k}-1}\Phi\left(\frac{1}{r}\right)dr+\frac{1}{k\Gamma_k}(\lambda)\int_{\frac{1}{\mathfrak{a}_2}}^{\frac{1}{\mathfrak{a}_1}}\left(r-\frac{1}{\mathfrak{a}_2}\right)^{\frac{\lambda}{k}-1}\Phi\left(\frac{1}{r}\right)dr\right]$$

$$= \frac{k\Gamma_k(\lambda)}{n}\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\circ\Psi)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\circ\Psi)(\frac{1}{\mathfrak{a}_2})\right],$$

where $\Psi(r)=\frac{1}{r}$, this implies that

$$\frac{1}{\sum_{\wp=1}^{n}[1-(\frac{s}{2})^{\wp}]}\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)$$
$$\leq\frac{\Gamma_k(\lambda+k)}{n}\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\circ\Psi)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\circ\Psi)(\frac{1}{\mathfrak{a}_2})\right]. \tag{20}$$

Next, substituting $x=\mathfrak{a}_1, y=\mathfrak{a}_2$ in (4) gives

$$\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\leq\frac{1}{n}\sum_{\wp=1}^{n}[1-s(1-\ell)^{\wp}]\Phi(\mathfrak{a}_2)+\frac{1}{n}\sum_{\wp=1}^{n}[1-(s\ell)^{\wp}]\Phi(\mathfrak{a}_1). \tag{21}$$

Reversing the role of $\mathfrak{a}_1$ and $\mathfrak{a}_2$ in (21)

$$\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\leq\frac{1}{n}\sum_{\wp=1}^{n}[1-(s(1-\ell)^{\wp})]\Phi(\mathfrak{a}_1)+\frac{1}{n}\sum_{\wp=1}^{n}[1-(s\ell)^{\wp}]\Phi(\mathfrak{a}_2). \tag{22}$$

Adding (20) and (21) and multiplying the resulting inequality by $\ell^{\frac{\lambda}{k}-1}$, then integrating with respect to $\ell\in[0,1]$, we obtain

$$\int_0^1\ell^{\frac{\lambda}{k}-1}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}d\ell$$
$$\leq\frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\int_0^1\left[2\ell^{\frac{\lambda}{k}-1}-\ell^{\frac{\lambda}{k}-1}(s(1-\ell))^{\wp}-(s\ell)^{\wp}\ell^{\frac{\lambda}{k}-1}\right]d\ell$$
$$\leq\frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n}\sum_{\wp=1}^{n}\left[2\frac{k}{\lambda}-\frac{s^{\wp}k}{\lambda+ik}-s^{\wp}\beta\left(\frac{\lambda}{k},\wp+1\right)\right]. \tag{23}$$

Again from (23), one has

$$\frac{\Gamma_k(\lambda+k)}{n}\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\circ\Psi)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\circ\Psi)(\frac{1}{\mathfrak{a}_2})\right]$$
$$\leq\frac{\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)}{n^2}\sum_{\wp=1}^{n}\left[2-\frac{s^{\wp}\lambda}{\lambda+ik}-\frac{s^{\wp}\lambda}{k}\beta\left(\frac{\lambda}{k},\wp+1\right)\right].$$

Combining (20) and (22) leads us to the desired result. □

Remark 5. *If we take $s=1$ and $\lambda=k=1$, then Theorem 9 reduces to Theorem 3.*

Remark 6. *If we take $\lambda=k=1$, then Theorem 9 reduces to Theorem 4.*

Remark 7. *If we take $n=1$, $s=1$ $\lambda=k=1$ in Theorem 9, then the classical Hermite–Hadamard type inequality for harmonic convex function is recovered.*

Remark 8. *If we take $n = \lambda = k = 1$ in Theorem 9, then the classical Hermite–Hadamard inequality for harmonic s-type convex function is recovered.*

Corollary 3. *If we set $n = 1$ in Theorem 9, then we have the following inequality.*

$$
\begin{aligned}
&\frac{1}{\left[1-\left(\frac{s}{2}\right)\right]}\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\\
&\leq \frac{\Gamma_k(\lambda+k)}{n}\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_k\mathrm{I}^{\lambda}_{\left(\frac{1}{\mathfrak{a}_2}\right)^+}(\Phi\circ\Psi)\left(\frac{1}{\mathfrak{a}_1}\right)+{}_k\mathrm{I}^{\lambda}_{\left(\frac{1}{\mathfrak{a}_1}\right)^-}(\Phi\circ\Psi)\left(\frac{1}{\mathfrak{a}_2}\right)\right]\\
&\leq [\Phi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)]\left[2-\frac{s\lambda}{\lambda+k}-\frac{s\lambda}{k}\beta\left(\frac{\lambda}{k},2\right)\right].
\end{aligned}
$$

Theorem 10. *Suppose $\Phi, \Psi : S \to \mathbb{R}^+$ be two functions such that $\Phi\Psi \in \mathcal{L}[\mathfrak{a}_1, \mathfrak{a}_2]$ and $\mathfrak{a}_1, \mathfrak{a}_2 > 0$, $\mathfrak{a}_1, \mathfrak{a}_2 \in S$. If Φ is an n_1-polynomial harmonically s-type convex function and Ψ is an n_2-polynomial harmonically s-type convex function with $\lambda, k > 0$, then the following inequality holds:*

$$
\begin{aligned}
&\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_k\mathrm{I}^{\lambda}_{\left(\frac{1}{\mathfrak{a}_2}\right)^+}(\Phi\Psi\circ h)\left(\frac{1}{\mathfrak{a}_1}\right)+{}_k\mathrm{I}^{\lambda}_{\left(\frac{1}{\mathfrak{a}_1}\right)^-}(\Phi\Psi\circ h)\left(\frac{1}{\mathfrak{a}_2}\right)\right]\\
&\leq \frac{D(\mathfrak{a}_1,\mathfrak{a}_2)}{k\Gamma_k(\lambda)}\int_0^1 \ell^{\frac{\lambda}{k}-1}[\Delta_1(\ell)+\Delta_4(\ell)]d\ell+\frac{F(\mathfrak{a}_1,\mathfrak{a}_2)}{k\Gamma_k(\lambda)}\int_0^1 \ell^{\frac{\lambda}{k}-1}[\Delta_2(\ell)+\Delta_4(\ell)]d\ell,
\end{aligned}
$$

where $D(\mathfrak{a}_1,\mathfrak{a}_2) = \Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2)$, $F(\mathfrak{a}_1,\mathfrak{a}_2) = \Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)+\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1)$, $h(r) = \frac{1}{r}$ and $\Delta_1(\ell), \Delta_2(\ell), \Delta_3(\ell)$ and $\Delta_4(\ell)$ are defined in Theorem 8.

Proof. Since Φ is an n_1-polynomial harmonically s-type convex function and Ψ is an n_2-polynomial harmonically s-type convex function, we have

$$
\begin{aligned}
&\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\\
&\leq \frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}\left[1-(s(1-\ell))^{\wp}\right]\sum_{\wp=1}^{n_2}\left[1-\left(s(1-\ell)\right)^{\wp}\right]\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2)\\
&+\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}\left[1-(s(1-\ell))^{\wp}\right]\sum_{\wp=1}^{n_2}\left[1-(s\ell)^{\wp}\right]\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1)\\
&+\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_2}\left[1-(s\ell)^{\wp}\right]\sum_{\wp=1}^{n_1}\left[1-\left(s(1-\ell)\right)^{\wp}\right]\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)\\
&+\frac{1}{n_1}\frac{1}{n_2}\sum_{\wp=1}^{n_1}\left[1-(s\ell)^{\wp}\right]\sum_{\wp=1}^{n_2}\left[1-(s\ell)^{\wp}\right]\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1)\\
&=\Delta_1(\ell)\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2)+\Delta_2(\ell)\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1)+\Delta_3(\ell)\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)+\Delta_4(\ell)\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1).
\end{aligned}
$$

This gives

$$
\begin{aligned}
&\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\\
&\leq \Delta_1(\ell)\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2)+\Delta_2(\ell)\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1)+\Delta_3(\ell)\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)+\Delta_4(\ell)\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1). \quad (24)
\end{aligned}
$$

Similarly, we also have

$$
\begin{aligned}
&\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\\
&\leq \Delta_1(\ell)\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1)+\Delta_2(\ell)\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)+\Delta_3(\ell)\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1)+\Delta_4(\ell)\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2). \quad (25)
\end{aligned}
$$

Adding (24) and (25)

$$\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)$$
$$\leq (\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2))[\Delta_1(\ell)+\Delta_4(\ell)]$$
$$+(\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)+\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1))[\Delta_2(\ell)+\Delta_3(\ell)].$$

Multiplying both sides of (17) by $\ell^{\frac{\lambda}{k}-1}$ and then integrating with respect to ℓ over [0,1], one obtains

$$k\Gamma_k(\lambda)\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_2})\right]$$
$$\int_0^1\ell^{\frac{\lambda}{k}-1}\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)d\ell$$
$$+\int_0^1\ell^{\frac{\lambda}{k}-1}\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)d\ell$$
$$\leq(\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_1)+\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_2))\int_0^1\ell^{\frac{\lambda}{k}-1}[\Delta_1(\ell)+\Delta_4(\ell)]d\ell$$
$$+(\Phi(\mathfrak{a}_1)\Psi(\mathfrak{a}_2)+\Phi(\mathfrak{a}_2)\Psi(\mathfrak{a}_1))\int_0^1\ell^{\frac{\lambda}{k}-1}[\Delta_2(\ell)+\Delta_3(\ell)]d\ell$$
$$=D(\mathfrak{a}_1,\mathfrak{a}_2)\int_0^1\ell^{\frac{\lambda}{k}-1}[\Delta_1(\ell)+\Delta_4(\ell)]d\ell+F(\mathfrak{a}_1,\mathfrak{a}_2)\int_0^1\ell^{\frac{\lambda}{k}-1}[\Delta_2(\ell)+\Delta_3(\ell)]d\ell.$$

Hence, the proof is completed. □

Corollary 4. *Suppose $\Phi,\Psi\colon S\to\mathbb{R}^+$ are functions such that $\Phi\Psi\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_2]$ and $\mathfrak{a}_1,\mathfrak{a}_2>0$, $\mathfrak{a}_1,\mathfrak{a}_2\in S$. If Φ and Ψ are n_1-polynomial harmonically s-type convex functions, then the following fractional inequality holds:*

$$\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_2})\right]$$
$$\leq\frac{D(\mathfrak{a}_1,\mathfrak{a}_2)}{\Gamma_k(\lambda)}\left[\frac{1+(1-s)^2}{\lambda}+\frac{2s^2}{\lambda+2k}-\frac{2s^2}{\lambda+k}\right]+\frac{F(\mathfrak{a}_1,\mathfrak{a}_2)}{\Gamma_k(\lambda)}\left[\frac{2(1-s)}{\lambda}+\frac{2s^2}{\lambda+k}-\frac{2s^2}{\lambda+2k}\right].$$

Proof. Let $n_1=n_2=1$ $\quad\Delta_1(\ell)=[1-s(1-\ell)]^2$, $\Delta_4(\ell)=[1-s\ell]^2$ and $\Delta_3(\ell)=\Delta_4(\ell)=[(1-s)+s^2(\ell-\ell^2)]$.

The result follows using Theorem 10. □

Remark 9. *If we put $s=1$ in Corollary 4, then Corollary 2 [18] is recovered.*

Theorem 11. *Suppose $\Phi,\Psi\colon \mathrm{S}\to\mathbb{R}^+$ be functions such that $\Phi\Psi\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_2]$ with $\mathfrak{a}_1,\mathfrak{a}_2>0$ and $\mathfrak{a}_1,\mathfrak{a}_2\in\mathrm{S}$. If Φ is n_1-polynomial harmonically s-type convex function, Ψ is n_2-polynomial harmonically s-type convex function and $\lambda,k>0$. Then the following fractional inequality holds:*

$$\frac{n_1n_2}{\sum_{\wp=1}^{n_1}[1-(\frac{2}{s})]\sum_{\wp=1}^{n_2}[1-(\frac{2}{s})]}\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)$$
$$\leq\Gamma_k(\lambda+k)\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_2})\right]$$
$$+\frac{\lambda}{k}\int_{0^1}\ell^{\frac{\lambda}{k}-1}\Big\{[\Lambda_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\Lambda_{n_2}(\ell)]D(\mathfrak{a}_1,\mathfrak{a}_2)$$
$$+[\Lambda_{n_1}(\ell)\Lambda_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)]F(\mathfrak{a}_1,\mathfrak{a}_2)\Big\}d\ell,$$

where h is defined as in Theorem 9, $\Lambda_n = \frac{1}{n}\sum_{\wp=1}^{n}[1-(s(1-\ell))^{\wp}]$ *and* $\bar{\Lambda}_n = \frac{1}{n}\sum_{\wp=1}^{n}[1-(s\ell)^{\wp}]$.

Proof. Please note that $\bar{\Lambda}_n\left(\frac{1}{2}\right) = \Lambda_n\left(\frac{1}{2}\right) = E_n = \frac{\sum_{\wp=1}^{n}\left[1-\left(\frac{s}{2}\right)^{\wp}\right]}{n}$.

Now, let $\ell \in [0,1]$, hence from (10), one obtains

$$\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right) \le E_{n_1}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\},$$

and

$$\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right) \le E_{n_2}\left\{\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)+\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}.$$

Now,

$$\begin{aligned}
&\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\\
&\le E_{n_1}E_{n_2}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\right.\\
&\left.+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}\\
&+E_{n_1}E_{n_2}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right.\\
&\left.+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\right\}\\
&\le E_{n_1}E_{n_2}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\right.\\
&\left.+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}\\
&+E_{n_1}E_{n_2}\left\{[\Lambda_{n_1}(\ell)\Phi(\mathfrak{a}_2)+\bar{\Lambda}_{n_1}(\ell)\Phi(\mathfrak{a}_2)][\Lambda_{n_2}(\ell)\Psi(\mathfrak{a}_1)+\bar{\Lambda}_{n_2}(\ell)\Psi(\mathfrak{a}_2)]\right.\\
&\left.+[\Lambda_{n_1}(\ell)\Phi(\mathfrak{a}_1)+\bar{\Lambda}_{n_1}(\ell)\Phi(\mathfrak{a}_2)][\Lambda_{n_2}(\ell)\Psi(\mathfrak{a}_2)+\bar{\Lambda}_{n_2}(\ell)\Psi(\mathfrak{a}_1)]\right\}\\
&=E_{n_1}E_{n_2}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\right.\\
&\left.+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}}\right)\right\}\\
&+E_{n_1}E_{n_2}\left\{[\Lambda_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\Lambda_{n_2}(\ell)]D(\mathfrak{a}_1,\mathfrak{a}_2)\right.\\
&\left.+[\Lambda_{n_1}(\ell)\Lambda_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)]F(\mathfrak{a}_1,\mathfrak{a}_2)\right\}.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right) \le\, & E_{n_1}E_{n_2}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\right.\\
&\left.+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}\\
&+E_{n_1}E_{n_2}\left\{[\Lambda_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\Lambda_{n_2}(\ell)]D(\mathfrak{a}_1,\mathfrak{a}_2)\right.\\
&\left.+[\Lambda_{n_1}(\ell)\Lambda_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)]F(\mathfrak{a}_1,\mathfrak{a}_2)\right\}. \qquad (26)
\end{aligned}$$

Multiplying both sides of (26) by $\ell^{\frac{\lambda}{k}-1}$ and integrating the resulting inequality with respect to ℓ over $[0,1]$ one has

$$
\begin{aligned}
&\frac{k}{\lambda}\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\\
&=\int_0^1 \ell^{\frac{\lambda}{k}-1}\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\\
&\leq E_{n_1}E_{n_2}\int_0^1 \ell^{\frac{\lambda}{k}-1}\left\{\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_1+(1-\ell)\mathfrak{a}_2}\right)\right.\\
&\left.+\Phi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\Psi\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\ell\mathfrak{a}_2+(1-\ell)\mathfrak{a}_1}\right)\right\}\\
&+E_{n_1}E_{n_2}\int_0^1 \ell^{\frac{\lambda}{k}-1}\Big\{[\Lambda_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\Lambda_{n_2}(\ell)]D(\mathfrak{a}_1,\mathfrak{a}_2)\\
&+[\Lambda_{n_1}(\ell)\Lambda_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)]F(\mathfrak{a}_1,\mathfrak{a}_2)\Big\}.\\
&=E_{n_1}E_{n_2}\left\{k\Gamma_k(\lambda)\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_2})\right]\right\}\\
&+E_{n_1}E_{n_2}\int_{0^1}\ell^{\frac{\lambda}{k}-1}\Big\{[\Lambda_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\Lambda_{n_2}(\ell)]D(\mathfrak{a}_1,\mathfrak{a}_2)\\
&+[\Lambda_{n_1}(\ell)\Lambda_{n_2}(\ell)+\bar{\Lambda}_{n_1}(\ell)\bar{\Lambda}_{n_2}(\ell)]F(\mathfrak{a}_1,\mathfrak{a}_2)\Big\}d\ell.
\end{aligned}
$$

The required result follows. □

Corollary 5. *Let $\Phi,\Psi : S\to\mathbb{R}^+$ be two functions such that $\Phi\Psi\in\mathcal{L}[\mathfrak{a}_1,\mathfrak{a}_2]$ and $\mathfrak{a}_1,\mathfrak{a}_2>0$, $\mathfrak{a}_1,\mathfrak{a}_2\in S$. If Φ and Ψ are n_1-polynomial harmonically s-type convex functions with $\lambda,k>0$, then*

$$
\begin{aligned}
&\Phi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\Psi\left(\frac{2\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_1+\mathfrak{a}_2}\right)\\
&\leq\left(1-\frac{s}{2}\right)\Gamma_k(\lambda+k)\left(\frac{\mathfrak{a}_1\mathfrak{a}_2}{\mathfrak{a}_2-\mathfrak{a}_1}\right)^{\frac{\lambda}{k}}\left[{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_2})^+}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_1})+{}_kI^{\lambda}_{(\frac{1}{\mathfrak{a}_1})^-}(\Phi\Psi\circ h)(\frac{1}{\mathfrak{a}_2})\right]\\
&+\left(1-\frac{s}{2}\right)^2\left\{\left[2(1-s)+\frac{2s^2\lambda}{\lambda+k}-\frac{2s^2\lambda}{\lambda+2k}\right]D(\mathfrak{a}_1,\mathfrak{a}_2)\right.\\
&\left.+\left[(1+(1-s))^2-\frac{2s^2\lambda}{\lambda+k}+\frac{2s^2\lambda}{\lambda+2k}\right]F(\mathfrak{a}_1,\mathfrak{a}_2)\right\}.
\end{aligned}
$$

Proof. Let $n_1=n_2=1$, then $\Lambda_{n_1}(\ell)=\Lambda_{n_2}(\ell)=1-s(1-\ell)$ and $\bar{\Lambda}_{n_1}(\ell)=\bar{\Lambda}_{n_2}(\ell)=1-s\ell$. The intended result follows using Theorem 11. □

Remark 10. *If we put $s=1$ in Corollary 5, then we obtain Corollary 3 [18].*

5. Conclusions and Future Scope

As per recent trends, incorporating different fractional operators into the theory of inequalities is a new area of interest among several researchers. Several mathematicians have worked on the generalizations of some well-known inequalities to offer new bounds and new applications using new methods. In this manuscript:

(1) We presented and concentrated several fractional inequalities of the Caputo–Fabrizio operator for an n-polynomial s-type convex function and k-Riemann–Liouville fractional integral operator for an n-polynomial harmonically s-type convex function.
(2) New version of Hermite–Hadamard inequality and Pachpatte-type inequality are obtained via Caputo–Fabrizio fractional integral operators.

(3) Some special cases of the presented results have been in the form of corollaries and remarks.

In the future, we intend to generalize the theory of inequality for concepts such as interval-valued analysis, quantum calculus, fuzzy interval-valued calculus and time-scale calculus.

Author Contributions: Conceptualization, M.T. and A.A.S.; Data curation, O.M.A. and S.K.N.; Formal analysis, M.T.; Funding acquisition, K.N.; Investigation, S.K.N.; Methodology, K.N. and S.K.N.; Resources, M.T. and O.M.A.; Software, M.T. and K.N.; Supervision, K.N.; Validation, A.A.S. and S.K.N.; Visualization, M.T.; Writing—original draft, M.T.; Writing—review and editing, M.T. and O.M.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was supported by the Fundamental Fund of Khon Kaen University, Thailand.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the Khon Kaen University, Thailand.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Jensen, J.L.W.V. Sur les fonctions convexes et les inegalites entre les valeurs moyennes. *Acta Math.* **1905**, *30*, 175–193. [CrossRef]
2. Niculescu, C.P.; Persson, L.E. *Convex Functions and Their Applications*; Springer: New York, NY, USA, 2006.
3. Hadamard, J. Étude sur les propriétés des fonctions entiéres en particulier d'une fonction considérée par Riemann. *J. Math. Pures Appl.* **1893**, *58*, 171–215.
4. İşcan, İ. Hermite-Hadamard type inequalities for harmonically convex functions. *Hacet. J. Math. Stat.* **2014**, *43*, 935–942. [CrossRef]
5. Dragomir, S.S.; Agarwal, R.P. Two inequalities for differentiable mappings and applications to special means fo real numbers and to trapezoidal formula. *Appl. Math. Lett.* **1998**, *11*, 91–95 [CrossRef]
6. Xi, B.Y.; Qi, F. Some integral inequalities of Hermite–Hadamard type for convex functions with applications to means. *J. Funct. Spaces. Appl.* **2012**, *2012*, 980438. [CrossRef]
7. Kirmaci, U.S.; Özdemir, M.E. On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula. *Appl. Math. Comput.* **2004**, *153*, 361–368. [CrossRef]
8. Hudzik, H.; Maligranda, L. Some remarks on s–convex functions. *Aequationes Math.* **1994**, *1994*, 48100–48111.
9. Dragomir, S.S.; Fitzpatrik, S. The Hadamard inequality for s–convex functions in the second sense. *Demonstr. Math.* **1999**, *32*, 687–696. [CrossRef]
10. Özcan, S.; İşcan, İ. Some new Hermite-Hadamard type integral inequalities for the s–convex functions and theirs applications. *J. Inequal. Appl.* **2019**, *201*, 201. [CrossRef]
11. Srivastava, H.M.; Sahoo, S.K.; Mohammed, P.O.; Baleanu, D.; Kodamasingh, B. Hermite–Hadamard Type Inequalities for Interval-Valued Preinvex Functions via Fractional Integral Operators. *Int. J. Comput. Intell. Syst*. **2022**, *15*, 8. [CrossRef]
12. Toplu, T.; Kadakal, M.; İşcan, İ. On n-polynomial convexity and some relatd inequalities. *AIMS Math.* **2020**, *5*, 1304–1318. [CrossRef]
13. Awan, M.U.; Akhtar, N.; Iftikhar, S.; Noor, M.A.; Chu, Y.M. New Hermite?Hadamard type inequalities for n-polynomial harmonically convex functions. *J. Inequl. Appl.* **2020**, *2020*, 125. [CrossRef]
14. Rashid, S.; İşcan, İ.; Baleanu, D.; Chu, Y.M. Generation of new fractional inequalities via n–polynomials s–type convexity with applications. *Adv. Differ. Equ.* **2020**, *2020*, 264. [CrossRef]
15. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
16. Gürbüz, M.; Akdemir, A.O.; Rashid, S.; Set, E. Hermite-Hadamard inequality for fractional integrals of Caputo-Fabrizio type and related inequalities. *J. Inequl. Appl*. **2020**, *2020*, 172. [CrossRef]
17. Abdeljawad, T.; Baleanu, D. On fractional derivatives with exponential kernel and their discrete versions. *Rep. Math. Phys.* **2017**, *80*, 11–27. [CrossRef]
18. Nwaeze, E.R.; Khan, M.A.; Ahmadian, A.; Ahmad, M.N.; Mahmood, A.K. Fractional inequalities of the Hermite-Hadamard type for m-polynomial convex and harmonically convex functions. *AIMS Math.* **2021**, *6*, 1889–1904. [CrossRef]
19. Sahoo, S.K.; Ahmad, H.; Tariq, M.; Kodamasingh, B.; Aydi, H.; De la Sen, M. Hermite-Hadamard type inequalities involving k-fractional operator for $(\bar{h}, m)$-convex functions. *Symmetry* **2021**, *13*, 1686. [CrossRef]
20. Abdeljawad, T.; Rashid, S.; Hammouch, Z.; Chu, Y.M. Some new local fractional inequalities associated with generalized (s, m)-convex functions and applications. *Adv. Differ. Equ.* **2020**, *2020*, 406. [CrossRef]
21. Sahoo, S.K.; Tariq, M.; Ahmad, H.; Nasir, J.; Aydi, H.; Mukheimer, A. New Ostrowski-type fractional integral inequalities via generalized exponential-type convex functions and applications. *Symmetry* **2021**, *13*, 1429. [CrossRef]
22. Tariq, M.; Sahoo, S.K.; Nasir, J.; Aydi, H.; Alsamir, H. Some Ostrowski type inequalities via n-polynomial exponentially s-convex functions and their applications. *AIMS Math.* **2021**, *6*, 13272–13290. [CrossRef]

23. Ali, S.; Mubeen, S.; Ali, R.S.; Rahman, G.; Morsy, A.; Nisar, K.S.; Purohit, S.D.; Zakarya, M. Dynamical significance of generalized fractional integral inequalities via convexity. *AIMS Math.* **2021**, *6*, 9705–9730. [CrossRef]
24. Akdemir, A.O.; Butt, S.I.; Nadeem, M.; Alessandra, M. Some new integral inequalities for a general variant of polynomial convex functions. *AIMS Math.* **2022**, *7*, 20461–20489. [CrossRef]
25. Kizil, S.; Ardic, M.A. Inequalities for strongly convex functions via Atangana-Baleanu Integral Operators. *Turk. J. Sci.* **2021**, *6*, 96–109.
26. Caputo, M.; Fabrizio, M. A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **2015**, *1*, 73–85.
27. Wu, S.; Iqbal, S.; Aamir, M.; Samraiz, M.; Younus, A. On some Hermite-Hadamard inequalities involving k-fractional operators. *J. Ineq. Appl.* **2021**, *2021*, 32. [CrossRef]
28. Mubeen, S.; Habibullah, G.M. k-Fractional integrals and application. *Int. J. Contemp. Math. Sci.* **2012**, *7*, 89–94.