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Geometric Study of 2D-Wave Equations in View of K-Symbol Airy Functions

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Abstract: The notion of k -symbol special functions has recently been introduced. This new concept offers many interesting geometric properties for these special functions including logarithmic convexity. The aim of the present paper is to exploit essentially two-dimensional wave propagation in the earth-ionosphere wave path using k -symbol Airy functions (KAFs) in the open unit disk. It is shown that the standard wave-mode working formula may be determined by orthogonality considerations without the use of intricate justifications of the complex plane. By taking into account the symmetry-convex depiction of the KAFs, the formula combination is derived.

Keywords: analytic function; inequalities; univalent function; open unit disk; symmetric differential operator; airy functions; normalization; complex wave equation; k -symbol calculus

MSC: 30C45; 30C15; 33C10



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1. Introduction

When Diaz and Pariguan [1] were assessing Feynman integrals, they introduced and researched k -gamma functions. Because they provide a generic integral representation of the relevant functions, these integrals are fundamentally important in high-energy physics [2]. K -gamma functions have since been developed which have a variety of consequences for mathematics and applications. In light of significant applications in quantum chemistry, Karwowski and Witek [3] employed k -special functions for determining the solution of the complex Schrodinger equation for the harmonium and similar designs. In their collected papers, there is a great deal of attention to the theory of measurement and combination versions for the k -maximizing factorial numbers that are used as examples as well as to the combinatorics of the Pochhammer k -symbol.

K -gamma functions were employed for combination analysis by Lackner and Lackner [4] in light of significant applications in statistics. Applications of various k -gamma function types have eliminated the major concerns, and, as a result, multiple publications analyzing k -gamma functions have been made available. Fractional calculus plays a vital role in simulating real-world issues [5]. It is perhaps surprising that k -gamma functions and associated k -Pochhammer symbols are also used in the field of fractional calculus functions. Fractional kinetic equations, including k -Mittag–Leffler functions, have been solved by Agarwal et al. [6]. In [7], Set et al. employed the k -calculus equivalent of the Riemann–Liouville singular kernel. More in-depth discussion can be found in [8,9]. Review of the literature on k -gamma functions has led us to conclude that, on the one hand, k -gamma functions have stimulated the study of mathematical ideas using novel methods, and on

the other hand, that the application of these functions in diverse situations is fundamental. The k -symbol calculus has recently been proposed as a tool for modifying, generalizing, and analyzing classes of analytic functions, such as differential, integral, and convolution operators in the open unit disk [10–13].

Airy functions (AFs), which are the solutions of $\aleph''(\xi) - \xi\aleph(\xi) = 0$, and Legendre functions, are frequently used in place of the propagating wave functions in the approximate solution due to their asymptotic expansions. In their investigation on the optics of a raindrop, Olivier and Soares provided a thorough justification for the Airy hypothesis [14]. The theory of electromagnetic diffraction, the propagation of radio waves, the propagation of light, and physical optics are all fields in which AFs play a vital role. Additionally, they are often employed in research, as described in [15]. Applications of AFs are discussed in relation to the two characteristics of symmetry and convexity. Studies using radiation exploit the symmetry characteristic (see [16–18]). The convexity feature is used in lens research (see [19–21]).

To solve a complex k -symbol wave equation on the open unit disk, we use the characteristics of k -symbol Airy functions. We first give the k -symbol Airy functions in the normalized form in order to describe how the solution of the wave equation behaves. Investigation of the geometric characteristics is made easier by this. We establish that the normalized formula has several interesting special functions. We then locate the symmetry-convex representation of the KAFs to investigate the propagation of two-dimensional waves in a complicated domain. To acquire the univalent solution, which is crucial for solving the complex wave equation, we seek to demonstrate a set of necessary conditions. It is demonstrated that the fundamental working formula for the wave theory may be derived from orthogonality considerations without the need for a thorough explanation in the complex plane. The formula is coupled with consideration of the symmetry-convex representation of the KAFs. The approach is presented in Section 2, the findings are detailed and discussed in Section 3, and conclusions are drawn in Section 4.

2. Approaches

Different ideas that are considered in the conclusion are covered below.

2.1. Normalized Airy Function

The Airy functions are formulated by the integral structure

$$\aleph(\xi) = \int_{-\infty}^{+\infty} \exp(i[\xi t + t^3/3]) dt$$

achieving the power series

$$\begin{aligned} \aleph_1(\xi) &= \left(\frac{1}{3^{2/3}\pi} \right) \sum_{n=0}^{\infty} \left(\frac{3^{n/3}\Gamma(\frac{n+1}{3}) \sin\left(\frac{2(n+1)\pi}{3}\right)}{\Gamma(n+1)} \right) \xi^n \\ &= \left(\frac{1}{3^{2/3}\pi} \right) \left(\Gamma\left(\frac{1}{3}\right) \sin\left(\frac{2\pi}{3}\right) \right) + \left(\frac{1}{3^{2/3}\pi} \right) \left(3^{1/3}\Gamma\left(\frac{2}{3}\right) \sin\left(\frac{4\pi}{3}\right) \right) \xi \\ &\quad + \left(\frac{1}{3^{2/3}\pi} \right) \sum_{n=2}^{\infty} \left(\frac{3^{n/3}\Gamma(\frac{n+1}{3}) \sin\left(\frac{2(n+1)\pi}{3}\right)}{\Gamma(n+1)} \right) \xi^n \\ &= \frac{1}{(3^{2/3}\Gamma(2/3))} - \frac{\xi}{(3^{1/3}\Gamma(1/3))} + \frac{\xi^3}{(6 \times 3^{2/3}\Gamma(2/3))} - \frac{\xi^4}{(12(3^{1/3}\Gamma(1/3)))} + O(\xi^5) \end{aligned}$$

and

$$\begin{aligned}\aleph_2(\xi) &= \left(\frac{1}{3^{1/6}\pi}\right) \sum_{n=0}^{\infty} \left(\frac{3^{n/3}\Gamma(\frac{n+1}{3}) \left| \sin\left(\frac{2(n+1)\pi}{3}\right) \right|}{\Gamma(n+1)} \right) \xi^n \\ &= \left(\frac{1}{3^{1/6}\pi}\right) \left(\Gamma\left(\frac{1}{3}\right) \left| \sin\left(\frac{2\pi}{3}\right) \right| \right) + \left(\frac{1}{3^{1/6}\pi}\right) \left(3^{1/3}\Gamma\left(\frac{2}{3}\right) \left| \sin\left(\frac{4\pi}{3}\right) \right| \right) \xi \\ &\quad + \left(\frac{1}{3^{1/6}\pi}\right) \sum_{n=2}^{\infty} \left(\frac{3^{n/3}\Gamma(\frac{n+1}{3}) \left| \sin\left(\frac{2(n+1)\pi}{3}\right) \right|}{\Gamma(n+1)} \right) \xi^n \\ &= \frac{1}{3^{1/6}\Gamma(2/3)} + \frac{3^{1/6}\xi}{\Gamma(1/3)} + \frac{\xi^3}{6 \times 3^{1/6}\Gamma(2/3)} + \frac{\xi^4}{4 \times 3^{5/6}\Gamma(1/3)} + O(\xi^5).\end{aligned}$$

By setting $g(0) = 0$ and $g'(0) = 1$, we aim to normalize Airy functions. We can examine the geometrical structure of these functions using this technique. The normalized power series are as follows:

$$\begin{aligned}\mathbb{Y}_1(\xi) &= \left(\frac{\aleph_1(\xi) - \left(\frac{1}{(3^{2/3}\Gamma(2/3))} \right)}{\left(-\frac{1}{(3^{1/3}\Gamma(1/3))} \right)} \right) \\ &= \xi - \frac{\xi^3\Gamma(1/3)}{(6(3^{1/3}\Gamma(2/3)))} + \dots \\ &:= \xi + \sum_{n=2}^{\infty} y_n \xi^n,\end{aligned}$$

where

$$\begin{aligned}y_n &:= \left(\frac{3^{(n-1)/3}\Gamma(\frac{n+1}{3}) \sin\left(\frac{2(n+1)\pi}{3}\right)}{\Gamma(\frac{2}{3}) \sin\left(\frac{4\pi}{3}\right) \Gamma(n+1)} \right) \\ &= -\frac{2 \times 3^{n-3/2} \sin(2/3\pi(n+1))\Gamma((n+1)/3)}{(\Gamma(2/3)\Gamma(n+1))},\end{aligned}$$

and

$$\begin{aligned}\mathbb{Y}_2(\xi) &= \left(\frac{\aleph_2(\xi) - \left(\frac{1}{3^{1/6}\pi} \right) \left(\Gamma\left(\frac{1}{3}\right) \left| \sin\left(\frac{2\pi}{3}\right) \right| \right)}{\left(\frac{1}{3^{1/6}\pi} \right) \left(3^{1/3}\Gamma\left(\frac{2}{3}\right) \left| \sin\left(\frac{4\pi}{3}\right) \right| \right)} \right) \\ &= \xi + \sum_{n=2}^{\infty} \left(\frac{3^{(n-1)/3}\Gamma(\frac{n+1}{3}) \left| \sin\left(\frac{2(n+1)\pi}{3}\right) \right|}{\Gamma(\frac{2}{3}) \left| \sin\left(\frac{4\pi}{3}\right) \right| \Gamma(n+1)} \right) \xi^n \\ &= \xi + \frac{\xi^3\Gamma(1/3)}{(6 \times 3^{1/3}\Gamma(2/3))} + \dots \\ &= \xi + \sum_{n=2}^{\infty} |y_n| \xi^n.\end{aligned}$$

2.2. K -Symbol Calculus

The k -symbol gamma function Γ_k , often known as the motivate gamma function, is formulated as follows [1]:

$$\Gamma_k(\xi) = \lim_{n \rightarrow \infty} \frac{n!k^n (nk)^{\frac{\xi}{k}-1}}{(\xi)_{n,k}},$$

where

$$(\xi)_{n,k} := \xi(\xi + k)(\xi + 2k) \dots (\xi + (n-1)k)$$

and

$$(\xi)_{n,k} = \frac{\Gamma_k(\xi + nk)}{\Gamma_k(\xi)}.$$

Based on the definition of Γ_k , we present the normalized k -symbol functions as follows:

$$\begin{aligned} [\mathbb{Y}_1]_k(\xi) &= \xi - \frac{\xi^3 \Gamma_k(1/3)}{(6(3^{1/3} \Gamma_k(2/3)))} + \dots \\ &:= \xi + \sum_{n=2}^{\infty} [y_n]_k \xi^n, \end{aligned}$$

where

$$[y_n]_k := -\frac{2 \times 3^{n-3/2} \sin(2/3\pi(n+1)) \Gamma_k((n+1)/3)}{(\Gamma_k(2/3) \Gamma_k(n+1))},$$

and

$$\begin{aligned} [\mathbb{Y}_2]_k(\xi) &= \xi + \frac{\xi^3 \Gamma_k(1/3)}{(6 \times 3^{1/3} \Gamma_k(2/3))} + \dots \\ &= \xi + \sum_{n=2}^{\infty} |[y_n]_k| \xi^n. \end{aligned}$$

The following outcomes demonstrate some characteristics of the k -symbol Airy functions (see Figure 1).

Proposition 1. The following outcomes are accurate for k -special functions

•

$$\begin{aligned} [\mathbb{Y}_1]_k(\xi) &= \frac{G_k(4/3)3^{1/3}/G_k(1/3)}{G_k(5/3)3^{2/3}/G_k(2/3)} \\ &\quad - \frac{G_k(4/3)3^{1/3}/G_k(1/3)}{3} \left([I_{-1/3}]_k \left(\frac{2\xi^{3/2}}{3} \right) (\xi^{3/2})^{1/3} - \frac{\xi [I_{1/3}]_k((2\xi^{3/2})/3)}{(\xi^{3/2})^{1/3}} \right), \end{aligned}$$

where G_k is the k -Barnes function satisfying $G_k(n) = \frac{(\Gamma_k(n))^{n-1}}{\kappa(n)}$ (κ is the κ function) and $[I_n]_k(\xi)$ is the k -modified Bessel function.

•

$$\begin{aligned} [\mathbb{Y}_1]_k(\xi) &= -\left(1/3 [J_{-1/3}]_k(2/3(-\xi)^{3/2}) ((-\xi)^{3/2})^{1/3} \right) (G_k(4/3)3^{1/3}/G_k(1/3)) \\ &\quad - \frac{G_k(4/3)3^{1/3}/G_k(1/3)}{G_k(5/3)3^{2/3}/G_k(2/3)} - \frac{\xi [J_{1/3}]_k(2/3(-\xi)^{3/2}) (G_k(4/3)3^{1/3})/G_k(1/3)}{3((-\xi)^{3/2})^{1/3}}, \end{aligned}$$

where $[J_n]_k(\xi)$ indicates the k -Bessel function.

•

$$[\mathbb{Y}_2]_k(\xi) = \frac{\xi [{}_0F_1]_k(4/3; \xi^3/9) 3^{1/6}}{\Gamma_k(1/3)} - \frac{1}{\Gamma_k(2/3) 3^{1/6}} + \frac{[{}_0F_1]_k(2/3; \xi^3/9)}{\Gamma_k(2/3) 3^{1/6}},$$

where $[{}_0F_1]_k$ represents the k -hypergeometric function.

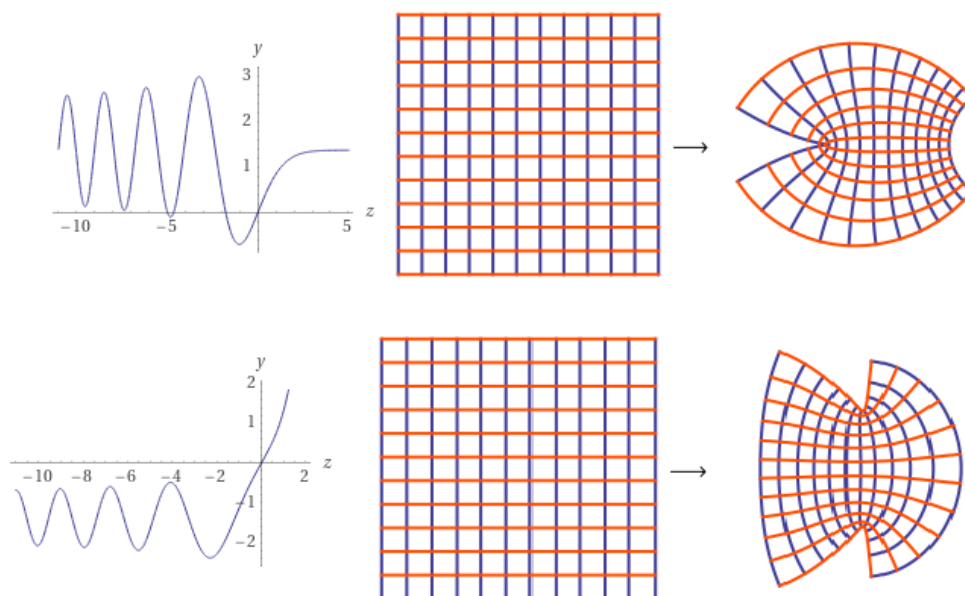


Figure 1. The graph of the normalized Airy functions $\mathbb{Y}_1, \mathbb{Y}_2$, respectively.

2.3. K -Airy Differential Operator

Using the normalized k -Airy functions, we then define the symmetric-convex differential operator. For an analytic function normalized in the open unit disk $\Lambda := \{\xi \in \mathbb{C} : |\xi| < 1\}$, we have the following structure:

$$v(\xi) = \xi + \sum_{n=2}^{\infty} a_n \xi^n,$$

The following power series is produced using the convoluted operator $(*)$ and the normalized Airy function $[\mathbb{Y}_1]_k(\xi)$

$$(v * [\mathbb{Y}_1]_k)(\xi) = ([\mathbb{Y}_1]_k * v)(\xi) = \xi + \sum_{n=2}^{\infty} a_n [y_n]_k \xi^n, \quad \xi \in \Lambda.$$

By considering the above convoluted product, we define the following normalized k -Airy symmetric-convex differential operator (KASCO):

$$\begin{aligned} [\Omega_\beta]_k(\xi) &= (1 - \beta) \xi (v * [\mathbb{Y}_1]_k)'(\xi) - \beta \xi (v * [\mathbb{Y}_1]_k)'(-\xi) \\ &= (1 - \beta) \left(\xi + \sum_{n=2}^{\infty} n a_n [y_n]_k \xi^n \right) - \beta \left(-\xi + \sum_{n=2}^{\infty} n a_n [y_n]_k (-1)^n \xi^n \right) \\ &= \xi + \sum_{n=2}^{\infty} n a_n [y_n]_k [(1 - \beta) + \beta (-1)^{n+1}] \xi^n \\ &:= \xi + \sum_{n=2}^{\infty} n a_n [y_n]_k \omega_n(\beta) \xi^n \quad \xi \in \Lambda, \end{aligned}$$

where

$$\omega_n(\beta) := [(1 - \beta) + \beta(-1)^{n+1}].$$

The m -dimensional KASCO is illustrated as follows:

$$\begin{aligned} [\Omega_\beta]_k^2(\zeta) &= [\Omega_\beta]_k([\Omega_\beta]_k)(\zeta) \\ &= (1 - \beta)\zeta([\Omega_\beta]_k)'(\zeta) - \beta\zeta([\Omega_\beta]_k)'(-\zeta) \\ &= (1 - \beta)\left(\zeta + \sum_{n=2}^{\infty} n^2 a_n [y_n]_k \omega_n(\beta) \zeta^n\right) - \beta\left(-\zeta + \sum_{n=2}^{\infty} n^2 a_n [y_n]_k \omega_n(\beta) (-1)^n \zeta^n\right) \\ &= \zeta + \sum_{n=2}^{\infty} n a_n [y_n]_k \omega_n(\beta) [(1 - \beta) + \beta(-1)^{n+1}] \zeta^n \\ &= \zeta + \sum_{n=2}^{\infty} n^2 a_n [y_n]_k \omega_n^2(\beta) \zeta^n \quad \zeta \in \Lambda. \end{aligned}$$

Generally, the m -formula is given by (see Figure 2)

$$[\Omega_\beta]_k^m(\zeta) = \zeta + \sum_{n=2}^{\infty} n^m a_n [y_n]_k \omega_n^m(\beta) \zeta^n \quad \zeta \in \Lambda. \quad (1)$$

Note that, under the consideration data $k = 1, \beta = 0$ and $[y_n]_k \approx 1$, this implies the Salagean differential operator [22].

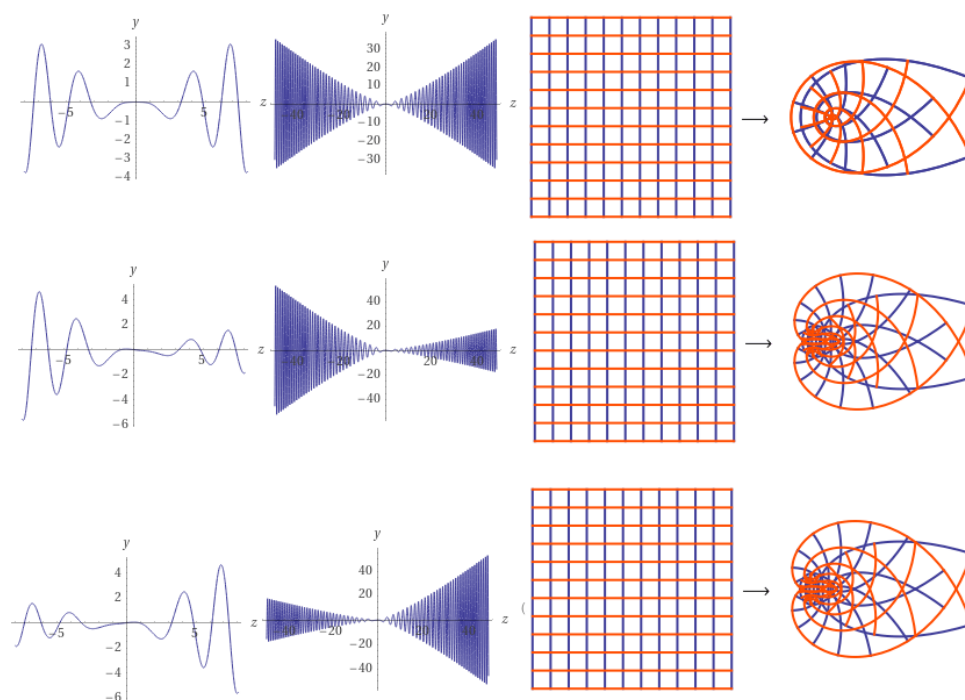


Figure 2. The graph of KASCO, when $m = k = 1, \beta = 1/2, 1/4, 3/4$ accordingly.

2.4. Univalent Solution of the k -Wave Equation

In an effort to develop the wave equation, we suggest utilizing the parametric Koebe function. The Koebe function is an extreme function that belongs to the family of convex univalent functions. The Koebe function $\sigma(\zeta) = \zeta / (1 - \zeta)^2$ maps the unit disk conformally onto the complex plane $\mathbb{C}$ with a slit along the disk $|\zeta| < 1/4$. We utilize the rotate Koebe function of the structure

$$\sigma_t(\zeta) = \frac{\zeta}{(1 - e^{it}\zeta)^2} = \zeta + \sum_{n=2}^{\infty} n e^{i(n-1)t} \zeta^n, \quad \zeta \in \Lambda.$$

The operator Ω_α^k acts on $\sigma(\xi)$, producing the following expansion

$$[\Omega_\beta]_k^m(\xi; t) = \xi + \sum_{n=2}^{\infty} n^{1+m} e^{i(n-1)t} [y_n]_k \omega_n^m(\beta) \xi^n \quad \xi \in \Lambda. \quad (2)$$

Using the operator (2), we proceed to formulate the complex wave equation. The complex wave equation is considered in the formula

$$\left(\frac{\partial^2}{\partial t^2} + \varepsilon^2 \frac{\partial^2}{\partial \xi^2} \right) [\Omega_\beta]_k^m(\xi; t) = \Sigma(\xi), \quad (3)$$

where $[\Omega_\beta]_k^m(\xi; t)$ indicates the m -iterative wave amplitude in Λ with the convex parameter $\beta \in [0, 1]$ and Σ is known as the non-linear functional of the wave under consideration owing $\Sigma(0) = 0$ and $\Sigma'(0) = 1$ (normalized function in Λ). A unique instance is examined in [23], when $\Sigma(\xi) = 0$ and $[\Omega_\beta]_k^m(\xi; t) = [\Omega_\beta]_k^m(\xi; t)$.

We provide a univalent outcome to the wave equation. The univalent result is significant in wave equations (see [24–27]). The wave peaks necessarily travel faster than the troughs and ultimately reach these levels since the solutions to the wave equations are known to be erroneous for infinite layers as they are not univalent functions. The primary requirement to achieve an analytic univalent solution fulfilling the inequality is covered in the next section $\Re([\Omega_\beta]_k^m(\xi; t)') > 0$ where $' = d/d\xi$. Alternatively, the answer is a complex domain Λ with a limited rotation function. In this instance, the gradients continue to increase, but eventually these effects start to occur, slowing this expansion. The precise behavior of the solution in Λ , which cannot be predicted from the wave equation, depends on the form of the dissipation components that are taken into account.

3. Results and Discussion

This section describes our findings for the univalent solution of Equation (3) for various hypotheses concerning Σ .

Proposition 2. Consider Equation (3). If the operator $[\Omega_\beta]_k^m(\xi; t)$ fulfils the symmetrical inequality

$$\Re \left(\frac{\xi [\Omega_\beta]_k^m(\xi; t)'}{[\Omega_\beta]_k^m(\xi; t) - [\Omega_\beta]_k^m(-\xi; t)} \right) > 0 \quad (4)$$

then $[\Omega_\beta]_k^m(\xi; t)$ is a univalent outcome for Equation (3).

Proof. The normalization formula of $[\Omega_\beta]_k^m(\xi; t)$ yields $[\Omega_\beta]_k^m(0; t) = 0$ and $[\Omega_\beta]_k^m(0; t)' = 1$. Replacing $-\xi$ by ξ in the inequality (4), we get

$$\Re \left(\frac{\xi [\Omega_\beta]_k^m(-\xi; t)'}{[\Omega_\beta]_k^m(\xi; t) - [\Omega_\beta]_k^m(-\xi; t)} \right) > 0. \quad (5)$$

Combining inequalities (4) and (5), we receive

$$\Re \left(\frac{\xi ([\Omega_\beta]_k^m(-\xi; t)' - [\Omega_\beta]_k^m(-\xi; t)')}{[\Omega_\beta]_k^m(\xi; t) - [\Omega_\beta]_k^m(-\xi; t)} \right) > 0. \quad (6)$$

This shows that $[\Omega_\beta]_k^m(\xi; t) - [\Omega_\beta]_k^m(-\xi; t)$ is univalent in Λ . In view of the Kaplan Theorem of uni-valency [28], we obtain $[\Omega_\beta]_k^m(\xi; t)$ is a univalent outcome of Equation (3). $\square$

Different conditions for $[\Omega_\beta]_k^m(\xi; t)$ to be univalently solvable are shown in the following outcomes.

Proposition 3. For Equation (3), assume that the operator $[\Omega_\beta]_k^m(\xi; t)$ violates the relation

$$\Re([\Omega_\beta]_k^m(\xi; t)' + \lambda(\xi)[\Omega_\beta]_k^m(\xi; t)'') > 0 \quad (7)$$

where $\lambda(\xi)$ is an analytic function in Λ with a non-negative real part. Then $[\Omega_\beta]_k^m(\xi; t)$ is a univalent outcome for Equation (3).

Proof. Assume that (7) is a true inequality. Formulate an admissible function $\Delta : \mathbb{C}^2 \rightarrow \mathbb{C}$, as follows:

$$\Delta(\rho, \varsigma) = \rho(\xi) + \lambda(\xi)\varsigma(\xi).$$

In view of the assumption (7), and by letting

$$\rho(\xi) := [\Omega_\beta]_k^m(\xi; t)', \quad \varsigma(\xi) := \xi[\Omega_\beta]_k^m(\xi; t)'',$$

we have that

$$\Re(\Delta([\Omega_\beta]_k^m(\xi; t)', \xi[\Omega_\beta]_k^m(\xi; t)'')) > 0.$$

According to Theorem 5 of [29], we conclude that

$$\Re([\Omega_\beta]_k^m(\xi; t)') > 0,$$

which leads to $[\Omega_\beta]_k^m(\xi; t)$ is a univalent solution of Equation (3). $\square$

Extra conditions on $[\Omega_\beta]_k^m(\xi; t)$ to be univalent. The following outcome is a relation between $[\Omega_\beta]_k^m(\xi; t)$ and $\Sigma(\xi)$ in Equation (3).

Proposition 4. Assume that Equation (3), where $\Sigma(\xi)$ is a bounded function in Λ , with

$$\inf\left(\frac{\Sigma(\xi_1) - \Sigma(\xi_2)}{\xi_1 - \xi_2}\right) > 0, \quad \xi_1, \xi_2 \in \Lambda.$$

If

$$\left| \frac{\xi}{[\Omega_\beta]_k^m(\xi; t)} - \frac{\xi}{\Sigma(\xi)} \right| \leq \frac{2 \inf\left(\frac{\Sigma(\xi_1) - \Sigma(\xi_2)}{\xi_1 - \xi_2}\right)}{[\sup_{\xi \in \Lambda}(\Sigma(\xi))]^2},$$

which leads to $[\Omega_\beta]_k^m(\xi; t)$ is a univalent solution for Equation (3).

Proof. Let $[\Omega_\beta]_k^m(\xi; t) = \xi + \sum_{n=2}^{\infty} \vartheta_n \xi^n$ and $\Sigma(\xi) = \xi + \sum_{n=2}^{\infty} \varphi_n \xi^n$. Formulate the function $F : \Lambda \rightarrow \Lambda$, as follows:

$$F(\xi) = \left[\frac{\xi}{[\Omega_\beta]_k^m(\xi; t)} - \frac{\xi}{\Sigma(\xi)} \right]''.$$

Clearly, $F(\xi)$ is analytic in Λ . Integrating both sides, we get

$$\left[\frac{\xi}{[\Omega_\beta]_k^m(\xi; t)} - \frac{\xi}{\Sigma(\xi)} \right]' = \varphi_2 - \vartheta_2 + \int_0^\xi F(\tau) d\tau.$$

Consequently, we have

$$\left[\frac{\xi}{[\Omega_\beta]_k^m(\xi; t)} - \frac{\xi}{\Sigma(\xi)} \right] = (\varphi_2 - \vartheta_2)\xi + \int_0^\xi ds \int_0^s F(\tau) d\tau.$$

Therefore, a calculation gives that

$$[\Omega_\beta]_k^m(\xi; t) = \frac{\Sigma(\xi)}{1 + (\varphi_2 - \vartheta_2)\Sigma(\xi) + \Sigma(\xi)(f(\xi)/\xi)'}$$

where

$$f(\xi) = \int_0^\xi ds \int_0^s F(\tau) d\tau.$$

A calculation yields that

$$\left(\frac{f(\xi)}{\xi} \right)' = \frac{1}{\xi^2} \int_0^\xi t f''(t) dt = \frac{1}{\xi^2} \int_0^\xi \tau F(\tau) d\tau.$$

By virtue of the assumption, we have

$$\begin{aligned} \left| \frac{f(\xi_2)}{\xi_2} - \frac{f(\xi_1)}{\xi_1} \right| &= \left| \int_{\xi_1}^{\xi_2} \left(\frac{f(\xi)}{\xi} \right)' d\xi \right| \\ &\leq \left(\frac{2 \inf \left(\frac{\Sigma(\xi_1) - \Sigma(\xi_2)}{\xi_1 - \xi_2} \right)}{[\sup_{\xi \in \Lambda} (\Sigma(\xi))]^2} \right) \left(\frac{|\xi_2 - \xi_1|}{2} \right), \end{aligned}$$

where $\xi_1 \neq \xi_2$. The next step is to prove that $[\Omega_\beta]_k^m(\xi_1; t) \neq [\Omega_\beta]_k^m(\xi_2; t)$ or

$$|[\Omega_\beta]_k^m(\xi_1; t) - [\Omega_\beta]_k^m(\xi_2; t)| > 0, \quad \xi_1 \neq \xi_2.$$

$$\begin{aligned} &|[\Omega_\beta]_k^m(\xi_1; t) - [\Omega_\beta]_k^m(\xi_2; t)| \\ &= \frac{\left| \Sigma(\xi_1) - \Sigma(\xi_2) + \Sigma(\xi_2) \Sigma(\xi_1) \left(\frac{f(\xi_2)}{\xi_2} - \frac{f(\xi_1)}{\xi_1} \right) \right|}{\left| 1 + (\varphi_2 - \vartheta_2) \Sigma(\xi_1) + \Sigma(\xi_1) \left(\frac{f(\xi_1)}{\xi_1} \right) \right| \left| 1 + (\varphi_2 - \vartheta_2) \Sigma(\xi_2) + \Sigma(\xi_2) \left(\frac{f(\xi_2)}{\xi_2} \right) \right|} \\ &> \frac{|\Sigma(\xi_1) - \Sigma(\xi_2)| - \inf \left(\frac{\Sigma(\xi_1) - \Sigma(\xi_2)}{\xi_1 - \xi_2} \right) (\xi_2 - \xi_1)}{\left| 1 + (\varphi_2 - \vartheta_2) \Sigma(\xi_1) + \Sigma(\xi_1) \left(\frac{f(\xi_1)}{\xi_1} \right) \right| \left| 1 + (\varphi_2 - \vartheta_2) \Sigma(\xi_2) + \Sigma(\xi_2) \left(\frac{f(\xi_2)}{\xi_2} \right) \right|} \\ &\geq 0. \end{aligned}$$

Consequently, we obtain that $[\Omega_\beta]_k^m(\xi; t)$ is a univalent solution of Equation (3) in Λ . $\square$

Some unique examples of Proposition 4 are as follows:

Corollary 1. *If*

$$\left| \left(\frac{\xi}{[\Omega_\beta]_k^m(\xi; t)} \right)'' \right| \leq 2,$$

then $[\Omega_\beta]_k^m(\xi; t)$ is a univalent solution.

Proof. By putting $\Sigma(\xi) = \xi$ in Proposition 4, we have the result. Note that this result is sharp when

$$[\Omega_\beta]_k^m(\xi; t) = \frac{\xi}{(1 + \xi)^{2+\ell}},$$

where

$$\left| \left(\frac{\xi}{[\Omega_\beta]_k^m(\xi; t)} \right)'' \right| = (2 + \ell)(1 + \ell)(1 + \xi)^\ell, \quad \ell > 0.$$

$\square$

By Corollary 1, we have

Corollary 2. *If*

$$[\Omega_\beta]_k^m(\xi; t) = \frac{\xi}{1 + \sum_{n=1}^{\infty} b_n \xi^n},$$

where

$$\sum_{n=2}^{\infty} n(n-1)|b_n| \leq 2,$$

then $[\Omega_\beta]_k^m(\xi; t)$ is a univalent solution.

The concluding remarks are presented below.

Remark 1.

- Solutions that are periodic exist because $n - 1$ is an integer with Keobe function. Since individual modes do not necessarily have to be periodic, this restriction is not required. Instead, the value of t will be determined by the boundary conditions. Furthermore, it is asserted that $\Re(t) > 0$ without sacrificing generality, and special emphasis is given to solutions that behave as $\exp(it)$. The waves in the direction of positive t are attenuated in this way. The form of the waves traveling in the direction of negative t is the same (symmetric sense).
- The way in which the concept is developed here readily lends itself to many generalizations. This represents an intriguing situation when the height of the top border varies along the direction of propagation. The normalized analytic function is seen as a function of $\xi \in \Lambda$ to obtain the normalized univalent solution in the complex model under study.
- It may be anticipated that a waveguide with slowly changing characteristics will not differ greatly from a waveguide with a constant cross-section based on fundamental principles. The structure of the modes may be used to identify a normalized waveguide with a univalent function. The ideal ground conductivity is now standardized to a value that is very near to unity.

4. Conclusions

A symmetric-convex differential formula of normalized Airy functions in the open unit disk was developed. This equation was taken into account as a differential operator working on a class of normalized analytic functions. The proposed operator (KASCO) was shown to be a solution to a wave equation in the following phase of this inquiry. We provided the necessary requirements for KASCO to be a univalent solution because we sought to analyze the geometric shape of the solution (symmetry and convexity). Based on the theory of the wave equation of a complex variable, the univalent solution is a particularly delicate property. Based on the theory of geometric functions, this characteristic leads to several geometric presentations for the solution.

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