axioms

Article

On Entire Function Solutions to Fermat Delay-Differential Equations

Xue-Ying Zhang, Ze-Kun Xu and Wei-Ran Lü *

College of Science, China University of Petroleum, Qingdao 266580, China
* Correspondence: luwr@upc.edu.cn

Abstract: This paper concerns the existence and precise expression form of entire solutions to a certain type of delay-differential equation. The significance of our results lie in that we generalize and supplement the related results obtained recently.

Keywords: entire function; meromorphic function; nonlinear delay-differential equation; growth

MSC: 34M05; 34M10; 30D35

Citation: Zhang, X.-Y.; Xu, Z.-K.; Lü, W.-R. On Entire Function Solutions to Fermat Delay-Differential Equations. *Axioms* **2022**, *11*, 554. https://doi.org/10.3390/axioms11100554

Academic Editor: Ioannis G. Tsoulos

Received: 31 August 2022
Accepted: 8 October 2022
Published: 14 October 2022

1. Introduction and Main Results

We assume that the reader is familiar with Nevanlinna theory of meromorphic functions f in $\mathbb{C}$. In this paper, $S(r, f)$, as usual, denotes any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \to \infty$ outside a possible exceptional set of finite logarithmic measures (see, for example, [1–3]). The order and the hyper-order of f are defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} \quad \text{and} \quad \rho_2(f) = \limsup_{r\to\infty} \frac{\log\log T(r, f)}{\log r}.$$

The motivation of this paper arose from the study of the following equation:

$$f^n + g^n = 1 \tag{1}$$

over $\mathbb{C}$, where $n \geq 1$ is an integer. For $n \geq 2$, the entire or meromorphic solutions of Equation (1) were completely analyzed by Baker [4], Gross [5–7] and Montel [8]. For the convenience of the reader, we summarize the related results as follows:

Theorem 1. *The solutions f and g for Equation (1) are characterized as follows:*

(1) If $n = 2$, then the entire solutions are $f = \cos(h)$ and $g = \sin(h)$, where h is an entire function, and the meromorphic solutions are $f = \frac{1-\beta^2}{1+\beta^2}$ and $g = \frac{2\beta}{1+\beta^2}$, where β is a nonconstant meromorphic function.

(2) If $n > 2$, then there are no nonconstant entire solutions.

(3) If $n = 3$, then the meromorphic solutions are $f = \frac{1+\frac{\wp'(h(z))}{\sqrt{3}}}{2\wp(h(z))}$ and $g = \frac{1-\frac{\wp'(h(z))}{\sqrt{3}}}{2\wp(h(z))}\eta$, where h is a nonconstant entire function, $\eta^3 = 1$ and $\wp$ is denoted as the Weierstrass $\wp - function$ that satisfies $(\wp')^2 = 4\wp^3 - 1$ under appropriate periods.

(4) If $n > 3$, then there are no nonconstant meromorphic solutions.

In 2004, Yang and Li [9] considered the existence of the entire solutions to Equation (1) with $g := f'$ and showed that the differential equation $f^2 + (f')^2 = 1$ has transcendental entire solutions only in the form $f(z) = \frac{1}{2}(Pe^{\alpha z} + \frac{1}{P}e^{-\alpha z})$, where P and α are nonzero constants.

Later, Liu and Yang [10] treated the mixture of f, f' and the shift of f and obtained two results:

Theorem 2. *If $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$, then the equation*

$$f^2 + 2\omega f f' + (f')^2 = 1 \tag{2}$$

has no transcendental meromorphic solutions.

Theorem 3. *If $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$, then the finite-order transcendental entire solutions of the equation*

$$f^2(z) + 2\omega f(z) f(z+c) + f^2(z+c) = 1 \tag{3}$$

for $\omega \in \mathbb{C}$ must be of the first order.

Recently, Xu et al. [11] considered the related questions in $\mathbb{C}^2$. In 2019, Han and Lü [12] gave the description of meromorhic solutions to the functional Equation (1) when $g = f'$ and 1 is replaced by $e^{\alpha z+\beta}$, where $\alpha, \beta \in \mathbb{C}$. Now, for $f^2 + (f')^2 = e^{\alpha z+\beta}$, the function f must be entire, where $f(z) = e^{\frac{\alpha z+\beta}{2}} \sin(h(z))$ and $f'(z) = e^{\frac{\alpha z+\beta}{2}} \cos(h(z))$. Thereby, either $\alpha = 0$ and $h' = 1$ or h is a constant.

In 2021, Luo, Xu and Hu [13] proved the following two results:

Theorem 4. *Let $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$, where g is a nonconstant polynomial. If the differential equation*

$$f^2 + 2\omega f f' + (f')^2 = e^g \tag{4}$$

admits a transcendental entire solution $f(z)$ of a finite order, then $g(z)$ must be of the form $g(z) = az + b$, where a and b are constants.

Theorem 5. *Let $\omega \in \mathbb{C}$ and $\omega^2 \neq 1, 0$, where g is a nonconstant polynomial. If the difference equation*

$$[f(z+c)]^2 + 2\omega f(z) f(z+c) + [f(z)]^2 = e^{g(z)} \tag{5}$$

admits a transcendental entire solution $f(z)$ of a finite order, then $g(z)$ must be of the form $g(z) = az + b$, where a, b and $c\ (\neq 0)$ are constants.

When viewing the above Theorems, we find that the authors required the restrictive condition that "$\rho(f) < \infty$".

Now, a natural and very interesting question will be posed:

Problem 1. *Can we characterize the entire solutions of Equations (2) and (3) with $\rho_2(f) < 1$? Moreover, what can be said if the polynomial "$g(z)$" is replaced by "any entire function, for the sake, say, $2g(z)$?"*

In the following paragraphs, we will consider the above questions and obtain the following results, which improve and complement some related results (see, for example, Refs. [14–21] and the references therein).

Theorem 6. *Let a, b, $\omega\ (\neq 0, \pm 1)$ and $c (\neq 0)$ be constants, $k \geq 1$ be an integer, $L(f) = af(z+c) + bf(z)$ with $(a, b) \neq (0, 0)$ and g be a nonconstant entire function with $\rho(g) < 1$. If the equation*

$$[L(f)]^2 + 2\omega Ł(f) f^{(k)}(z) + [f^{(k)}(z)]^2 = e^{2g(z)} \tag{6}$$

has an entire solution f with $\rho_2(f) < 1$, then $g = \alpha z + \beta$, where $\alpha\ (\neq 0)$ and β are constants:

(1) If $a = 0$, then f is of the form

$$f(z) = c_0 e^{\alpha z} \quad or \quad f(z) = c_0 e^{\mu z} + d_0 e^{\gamma z} + c_k, \tag{7}$$

in which $\omega_2 \mu^k = \omega_1 \gamma^k$.

(2) If $a \neq 0$, then

$$f(z) = c_0 e^{\alpha z} + c_k \quad or \quad f(z) = c_0 e^{\mu z} + d_0 e^{\gamma z} + c_k, \tag{8}$$

where c_0, d_0, c_k, μ and gamma are constants, $2\alpha = \mu + \gamma$, $\omega_1 = -\omega + \sqrt{\omega^2 - 1}$ and $\omega_2 = -\omega - \sqrt{\omega^2 - 1}$. Moreover, $ae^{\mu} + b = \omega_2 \mu^k$, $ae^{\gamma} + b = \omega_1 \gamma^k$ and $a + b = 0$.

Theorem 7. *Assume that g is a constant and the assumptions in Theorem 6 remain the same. If f is a transcendental entire solution to Equation (6) with $\rho_2(f) < 1$, then*

$$f(z) = \frac{e^g}{\omega_2 - \omega_1} \left[\frac{1}{u^k} e^{uz+v} - \frac{1}{(-u)^k} e^{-uz-v} \right] + c_k,$$

Here, c_k is an arbitrary constant. Moreover, $ae^{-uc} + b = \omega_1 (-u)^k$, $e^{uc} + b = \omega_2 u^k$ and $a + b = 0$.

2. Preliminaries

To prove our results, the following lemmas are needed:

Lemma 1 (see, for example, [22])**.** *Let f be a non-constant meromorphic function with $\rho_2(f) < 1$ and c_1, $c_2 \in \mathbb{C}$. Then, we have*

$$m\left(r, \frac{f(z + c_1)}{f(z + c_2)}\right) = S(r, f),$$

outside of a possible exceptional set with a finite logarithmic measure.

Lemma 2 ([2], Lemma 3.3)**.** *Suppose that f is meromorphic and transcendental in the plane and that*

$$[f(z)]^n P(z) = Q(z),$$

where P and Q are differential polynomials in f and the degree of Q is at most n. Then, we have

$$m(r, P) = S(r, f) \text{ as } r \to \infty.$$

Lemma 3 ([3], Theorem 1.52)**.** *Let us say that f_j $(1 \leq j \leq n)$ and $g_j (1 \leq j \leq n)(n \geq 2)$ are entire functions satisfying the following conditions:*

(i) $\sum_{j=1}^{n} f_j(z) e^{g_j}(z) \equiv 0$;

(ii) The orders of f_j are less than that of $e^{g_h(z) - g_k(z)}$ for $1 \leq j \leq n$ and $1 \leq h < k \leq n$. Then, $f_j(z) \equiv 0$ for $1 \leq j \leq n$.

Lemma 4 ([3], Lemma 5.1)**.** *Let f be a non-constant periodic meromorphic function. Then, $\rho(f) \geq 1$.*

3. Proof of Theorem 6

Proof. Assume that in Equation (6) exists an entire solution f with $\rho_2(f) < 1$. Since g is a nonconstant entire function, f must be transcendental, and Equation (6) can be transformed to be

$$[L(f) - \omega_1 f^{(k)}][L(f) - \omega_2 f^{(k)}] = e^{2g}, \tag{9}$$

in which $\omega_1 = -\omega + \sqrt{\omega^2 - 1}$ and $\omega_2 = -\omega - \sqrt{\omega^2 - 1}$. Therefore, it follows by Equation (9) that

$$L(f) - \omega_1 f^{(k)} = e^{g+p}, \; L(f) - \omega_2 f^{(k)} = e^{g-p}, \tag{10}$$

where p is an entire function. Moreover, Lemma 1 and the logarithmic derivative lemma tell us that

$$\begin{aligned} T(r, e^p) = m(r, e^p) \leq & m(r, e^g) + m(r, L(f) - \omega_1 f^{(k)}) + S(r, f) \\ \leq & m(r, e^g) + m(r, f) + m\Big(r, \frac{L(f) - \omega_1 f^{(k)}}{f}\Big) + S(r, f) \\ \leq & T(r, e^g) + T(r, f) + S(r, f), \end{aligned}$$

which through [3] (Theorem 1.14 and the corollary of Theorem 1.19) will give $\rho(p) < 1$ because $\rho_2(f) < 1$ and $\rho(g) < 1$.

Now, set $g + p = r$ and $g - p = s$. Then, using Equation (10), we must have

$$L(f) = \frac{\omega_2 e^r - \omega_1 e^s}{\omega_2 - \omega_1}, \quad f^{(k)}(z) = \frac{e^r - e^s}{\omega_2 - \omega_1}. \tag{11}$$

Thus, by the definition of $L(f)$ and Equation (11), we obtain

$$ae^{r(z+c)} - ae^{s(z+c)} + (b - p_2\omega_2)e^{r(z)} + (p_1\omega_1 - b)e^{s(z)} \equiv 0, \tag{12}$$

in which $p_2 = (r')^k + p_{k-1}(r')$, $p_{k-1}(r')$ denotes a differential polynomial of r' with $\deg p_{k-1}(r') \leq k - 1$ and $p_1 = (s')^k + p_{k-1}(s')$, where $p_{k-1}(s')$ denotes a differential polynomial of s' with $\deg p_{k-1}(s') \leq k - 1$. Obviously, $\rho(p_1) < 1$ and $rho(p_2) < 1$.

Case 1. Suppose that $a = 0$. Then, Equation (12) becomes

$$(b - p_2\omega_2)e^{r(z)-s(z)} + p_1\omega_1 - b \equiv 0, \tag{13}$$

which yields $r(z) - s(z) = 2p(z)$ as a constant when $b - p_1\omega_1 \not\equiv 0$.

Now, we deal with the case where $b - p_1\omega_1 \not\equiv 0$, and thus $r(z) - s(z) = 2p(z)$ is a constant. Thereby, we have $r' = s' = g'$. If we let $r(z) - s(z) = 2p(z) = \tau$, where τ is a constant, then by Equation (13), we deduce that

$$(\omega_1 - \omega_2 e^\tau)(g')^k + (\omega_1 - \omega_2 e^\tau)p_{k-1}(g') + b(e^\tau - 1) \equiv 0. \tag{14}$$

Since $b \neq 0$ and $\omega_1 \neq \omega_2$, it is easy to see that $\omega_1 - \omega_2 e^\tau \neq 0$. Thus, if g' is a transcendental entire function, it follows by Equation (14) and Lemma 2 that $m(r, g') = S(r, g')$, which is impossible. Consequently, g' is a polynomial. Moreover, Equation (14) tells us that g' is a constant, and we set $g(z) = \alpha z + \beta$. By Equation (11), we obtain

$$L(f) = bf(z) = \frac{\omega_2 e^{\alpha z + \beta + \tau/2} - \omega_1 e^{\alpha z + \beta - \tau/2}}{\omega_2 - \omega_1}$$

and

$$f(z) = c_0 e^{\alpha z},$$

where c_0 is a nonzero constant.

Next, we assume that $b - p_2\omega_2 \equiv 0$. Then, $b - p_1\omega_1 \equiv 0$, and

$$(s')^k + p_{k-1}(s') - \frac{b}{\omega_1} \equiv 0,$$

which, combined with Lemma 2, gives $m(r,s') = S(r,s')$ if s' is a transcendental entire function. This, of course, is impossible. Thus, s' must be a polynomial. In the same arguments, we obtain that r' is also a polynomial. Thereby, using $b - p_2\omega_2 \equiv 0$ and $b - p_1\omega_1 \equiv 0$, and again, a simple computation shows that r' and s' are constants. Thus, we can set $s(z) = \gamma z + \delta$, $r(z) = \mu z + \nu$ and $g(z) = \frac{(\gamma+\mu)z+\delta+\nu}{2} =: \alpha z + \beta$, where γ, δ, μ and *nu* are constants. Moreover, it follows from $b - p_2\omega_2 \equiv 0$ and $b - p_1\omega_1 \equiv 0$ that $\omega_2\mu^k = \omega_1\gamma^k$. This shows that $\gamma \neq \mu$, and we can deduce that $p(z) = \frac{(\mu-\gamma)z+\nu-\delta}{2}$. In this case, by Equation (11), we have

$$f(z) = \frac{\omega_2 e^{\mu z+\nu} - \omega_1 e^{\gamma z+\delta}}{b(\omega_2 - \omega_1)} := c_0 e^{\mu z} + d_0 e^{\gamma z},$$

where c_0 and d_0 are nonzero constants and $\omega_2\mu^k = \omega_1\gamma^k$.

Case 2. Assume that $a \neq 0$. When applying Lemma 3 to Equation (12), we find at least one of $r(z+c) - s(z)$, $r(z+c) - r(z)$, $r(z+c) - s(z+c)$, $s(z+c) - s(z)$ or $s(z+c) - r(z)$ is a constant.

To prove our result, the following cases will be considered:

Subcase 2.1. Suppose that $r(z+c) - s(z)$ is a constant, such as A. Thus, Equation (12) gives

$$ae^A - ae^{s(z+c)-s(z)} + (b - p_2\omega_2)e^{r(z)-s(z)} + p_1\omega_1 - b \equiv 0. \tag{15}$$

Since $a \neq 0$, then by applying Lemma 3 to Equation (15) again, we then see that at least one of $s(z+c) - s(z)$, $r(z) - s(z)$ or $s(z+c) - r(z)$ must be a constant.

Let us first consider the case where $s(z+c) - s(z)$ is a constant, which immediately follows that $s'(z)$ is a periodic function. Note that we have proven $\rho(s) < 1$, and consequently, Lemma 4 yields that $s'(z)$ is a constant, such as γ. Thus, $s(z) = \gamma z + \delta$. By $r(z+c) - s(z) = A$, we can obtain $r(z) = \gamma z + \nu$, and $g(z) = \frac{(\gamma+\gamma)z+\delta+\nu}{2} = \gamma z + \frac{\delta+\nu}{2} := \alpha z + \beta$, where δ and *nu* are constants. Moreover, it follows by $2p = r - s$ that p is also a constant. Again, with the help of Equation (11), it is easy to see that

$$f^{(k)}(z) = \frac{e^{g+p} - e^{g-p}}{\omega_2 - \omega_1} = \frac{e^p - e^{-p}}{\omega_2 - \omega_1} e^{\alpha z+\beta},$$

which gives

$$f(z) = c_0 e^{\alpha z} + c_1 z^{k-1} + \cdots + c_{k-1}z + c_k, \tag{16}$$

where $c_0\ (\neq 0), c_1, \cdots, c_k$ are constants.

By substituting Equation (16) into the first expression of Equation (11), we have

$$\begin{aligned} &ac_0 e^{\alpha(z+c)} + ac_1(z+c)^{k-1} + ac_2(z+c)^{k-2} + \cdots + ac_{k-1}(z+c) + ac_k \\ &+bc_0 e^{\alpha z} + bc_1 z^{k-1} + bc_2 z^{k-2} + \cdots + bc_{k-1}z + bc_k \equiv \frac{\omega_2 e^{\alpha z+\nu} - \omega_1 e^{\alpha z+\delta}}{\omega_2 - \omega_1}. \end{aligned} \tag{17}$$

If $k = 1$, then Equation (17) implies $(a+b)c_1 = 0$ for an arbitrary constant c_1, and thus $a + b = 0$. In this case, we see that the conclusion for Theorem 6 is true. If $k = 2$, then Equation (17) gives $(a+b)c_1 = 0$ and $(a+b)c_2 + acc_1 = 0$, and $ac \neq 0$ will show that $c_1 = 0$ and $a + b = 0$. Thus, the conclusion of Theorem 6 is valid.

If $k \geq 3$, then Equation (17) leads to the following relations:

$$\begin{cases} (a+b)c_1 = 0, \\ (k-1)acc_1 + (a+b)c_2 = 0, \\ \cdots\cdots \\ (k-1)ac^{k-2}c_1 + (k-2)ac^{k-3}c_2 + \cdots + 2acc_{k-2} + (a+b)c_{k-1} = 0, \\ ac^{k-1}c_1 + ac^{k-2}c_2 + \cdots + acc_{k-1} + (a+b)c_k = 0. \end{cases} \tag{18}$$

Trivially, it follows by Equation (18) and $ac \neq 0$ that $c_1 = \cdots = c_{k-1} = 0$ and $a + b = 0$. Thus, we have

$$f(z) = c_0 e^{\alpha z} + c_k,$$

where c_k is an arbitrary constant. The conclusion of Theorem 6 follows.

Using exactly the same method as above, Equation (16) still holds when $r(z) - s(z)$ or $s(z+c) - r(z)$ is constant. Thus, we see that Theorem 6 holds.

Subcase 2.2. Suppose that $r(z+c) - r(z)$ is a constant, such as B. Then, $r'(z)$ is a periodic function, and Lemma 3 yields $r'(z)$ as a constant, such as μ. Thus, $r(z) = \mu z + \nu$. In addition, Equation (12) becomes

$$ae^B - ae^{s(z+c)-r(z)} + b - p_2\omega_2 + (p_1\omega_1 - b)e^{s(z)-r(z)} \equiv 0. \tag{19}$$

Because of the discussion in **Subcase 2.1**, we can assume that $r(z+c) - s(z)$ is not constant. Now, by applying Lemma 3 to Equation (19), we deduce one of $s(z+c) - r(z)$, $s(z) - r(z)$ or $s(z+c) - s(z)$ must be a constant because $a \neq 0$.

First, if $s(z+c) - r(z)$ is a constant, then $s(z) = \mu z + \delta$, which is not inconsistent with our assumption that $r(z+c) - s(z)$ is constant. Therefore, this case cannot occur.

Secondly, if $s(z) - r(z)$ is a constant, then it follows by $r(z+c) - r(z) = B$ that $r(z+c) - s(z)$ is a constant, which is not inconsistent with our assumption that $r(z+c) - s(z)$ is constant. This case cannot occur.

Lastly, if $s(z+c) - s(z)$ is a constant, then Lemma 4 and $\rho(s) < 1$ mean that s' must be a constant. Consequently, we have $s(z) = \gamma z + \delta$ and $g(z) = \frac{(\gamma+\mu)z+\delta+\nu}{2} =: \alpha z + \beta$, where δ, μ and nu are constants. In this case, through Equation (11), we have

$$f(z) = c_0 e^{\mu z} + d_0 e^{\gamma z} + c_1 z^{k-1} + \cdots + c_{k-1} z + c_k, \tag{20}$$

where c_0, d_0, $c_1, \cdots, c_k$ are constants and $ae^\mu + b = \omega_2 \mu^k$, $ae^\gamma + b = \omega_1 \gamma^k$.

By substituting Equation (20) into the first expression in Equation (11), we have

$$\begin{aligned} &ac_0 e^{\mu(z+c)} + ad_0 e^{\gamma(z+c)} + ac_1(z+c)^{k-1} + \cdots + ac_{k-1}(z+c) + ac_k \\ &+bc_0 e^{\mu z} + bd_0 e^{\gamma z} + bc_1 z^{k-1} + \cdots + bc_{k-1} z + bc_k \equiv \frac{\omega_2 e^{\alpha z+\nu} - \omega_1 e^{\alpha z+\delta}}{\omega_2 - \omega_1}. \end{aligned} \tag{21}$$

Using the same method as that for **Subcase 2.1**, it follows by Equation (21) that

$$f(z) = c_0 e^{\mu z} + d_0 e^{\gamma z} + c_k,$$

where c_0, d_0 and c_k are constants and $a + b = 0$, and thus Theorem 6 follows.

Subcase 2.3. Suppose that $r(z+c) - s(z+c)$ is a constant, such as D. In this case, $p(z) = D/2$ is a constant. Now, using $r(z) = s(z) + D$, we change Equation (12) into

$$ae^D - a + [(b - p_2\omega_2)e^D + (p_1\omega_1 - b)]e^{s(z)-s(z+c)} \equiv 0. \tag{22}$$

In the following, according to **Subcase 2.1** and **Subcase 2.2**, we can assume that $r(z+c)-r(z)$ and $r(z+c)-s(z)$ are not constants. Clearly, by $r(z)=s(z)+D$, and since $r(z+c)-r(z)$ is not a constant, we know that $s(z)-s(z+c)$ is not a constant. Therefore, from Equation (22), we obtain $ae^D-a=0$ and $(b-p_2\omega_2)e^D+(p_1\omega_1-b)\equiv 0$. Since $a\neq 0$, then we obtain $e^D=1$ and

$$p_2\omega_2=p_1\omega_1. \tag{23}$$

Now, based on p being a constant, the expressions for p_1 and p_2 and Equation (23), we obtain

$$(\omega_2-\omega_1)(g')^k=p_{k-1}(g'), \tag{24}$$

where p_{k-1} denotes a differential polynomial of g' with $\deg p_{k-1}(g')\leq k-1$. It follows from Lemma 2 and Equation (24) that $g=\alpha z+\beta$ with constants $\alpha(\neq 0)$, β, and accordingly, it follows that both s and r are polynomials of one degree, which is, of course, impossible since $r(z+c)-s(z)$ is not a constant.

Subcase 2.4. Suppose that $s(z+c)-s(z)$ is a constant, such as ξ. In the following, we can assume that $r(z+c)-r(z)$, $r(z+c)-s(z)$ and $r(z+c)-s(z+c)$ are not constants. Now, we change Equation (12) into

$$ae^{r(z+c)-s(z)}-ae^{\xi}+(b-p_2\omega_2)e^{r(z)-s(z)}+p_1\omega_1-b\equiv 0. \tag{25}$$

Since $a\neq 0$, it is easy to see that by Lemma 3, Equation (25) does not hold. Hence, **Subcase 2.4** cannot occur.

Subcase 2.5. Suppose that $s(z+c)-r(z)$ is a constant, such as ξ. In the following, we can assume that $r(z+c)-r(z)$, $r(z+c)-s(z)$, $r(z+c)-s(z+c)$ and $s(z+c)-s(z)$ are not constants. In this case, Equation (12) can be rewritten as

$$ae^{r(z+c)-r(z)}-ae^{\xi}+(b-p_2\omega_2)+(p_1\omega_1-b)e^{s(z)-r(z)}\equiv 0. \tag{26}$$

Now, by resorting to Lemma 3 and $a\neq 0$, we can conclude that Equation (26) is invalid, and hence **Subcase 2.5** is ruled out.

Therefore, the proof of Theorem 6 is completed. □

4. Proof of Theorem 7

Proof. First of all, using the methods with which we proved Theorem 6, we have Equation (11). Clearly, p is not a constant because g is a constant and f is a transcendental entire function. On the other hand, by examining the proof of Theorem 6 carefully, we find that

$$ae^{p(z+c)}-ae^{-p(z+c)}+be^{p(z)}-be^{-p(z)}=\omega_2[e^{p(z)}]^{(k)}-\omega_1[e^{-p(z)}]^{(k)}. \tag{27}$$

It follows from Equation (27) that

$$ae^{p(z+c)}-ae^{-p(z+c)}+[b-\omega_2p_2]e^{p(z)}+[\omega_1p_1-b]e^{-p(z)}=0, \tag{28}$$

where $p_1=(-1)^k(p')^k+Q_1(p')$, $p_2=(p')^k+Q_2(p')$, and $Q_1(p')$ and $Q_2(p')$ denote differential polynomials of p' with $\deg Q_1(p')\leq k-1$ and $\deg Q_2(p')\leq k-1$.

If $a=0$, then Equation (28) yields $b-\omega_1p_1=0$ and $b-\omega_2p_2=0$. Thus, due to Lemma 2, p' must be a nonzero constant. However, using $b-\omega_1p_1=0$ and $b-\omega_2p_2=0$, we have $\omega_2+(-1)^k\omega_1=0$, which is impossible. Consequently, $a\neq 0$. Now, by applying Lemma 3 to Equation (28), we find that $p(z+c)-p(z)$ is a constant. Let $p=uz+v$ and $e^g=A$, and Equation (11) gives $f^{(k)}(z)=\frac{A}{\omega_2-\omega_1}(e^{uz+v}-e^{-uz-v})$. Thus, we have

$$f(z)=\frac{A}{\omega_2-\omega_1}\left[\frac{1}{u^k}e^{uz+v}-\frac{1}{(-u)^k}e^{-uz-v}\right]+c_1z^{k-1}+\cdots+c_{k-1}z+c_k, \tag{29}$$

in which $c_1, \cdots, c_k$ are arbitrary constants. Moreover, Equation (28) suggests that

$$ae^{uc} + b - \omega_2 u^k = 0, \quad ae^{-uc} + b - (-1)^k \omega_1 u^k = 0. \tag{30}$$

By substituting Equation (29) into the first expression of Equation (11) and using Equation (30), we have

$$\begin{aligned} &ac_1(z+c)^{k-1} + ac_2(z+c)^{k-2} + \cdots + ac_{k-1}(z+c) + ac_k \\ &+bc_1 z^{k-1} + bc_2 z^{k-2} + \cdots + bc_{k-1} z + bc_k \equiv 0. \end{aligned} \tag{31}$$

Trivially, using the same method as that in **Subcase 2.1**, it follows from Equation (31) and $ac \neq 0$ that $c_1 = \cdots = c_{k-1} = 0$ and $a + b = 0$. Therefore, we obtain

$$f(z) = \frac{A}{\omega_2 - \omega_1}\left[\frac{1}{u^k} e^{uz+v} - \frac{1}{(-u)^k} e^{-uz-v}\right] + c_k,$$

in which c_k is an arbitrary constant, and the conclusion of Theorem 7 follows.

Thus, the proof of Theorem 7 is finished. □

Finally, we would like to pose the following question:

Open question: Let $m\,(\geq 2)$, $n\,(\geq 2)$ and $k\,(\geq 1)$ be integers and $c(\neq 0)$ and $\omega\,(\neq 0, \pm 1)$ be constants. Suppose that g is a nonconstant entire function with $\rho(g) < 1$, where a and b are polynomials with $(a, b) \not\equiv (0, 0)$ and $L(f) = af(z+c) + bf(z)$. If the equation

$$[L(f)]^m + 2\omega Ł(f) f^{(k)}(z) + [f^{(k)}(z)]^n = e^{2g(z)}$$

has an entire solution f with $\rho_2(f) < 1$, can we find the concrete expression of f?

5. Conclusions

Using Nevanlinna theory, this paper provides two new results which extend and improve some related results. Bringing about our results from the more general hypotheses without complicated calculations will probably be the most interesting feature of this paper. Finally, one more general open question is posed in this paper for further study.

Author Contributions: All authors have made an equal contribution. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The authors would like to thank the referee for the valuable suggestions and comments.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Gol'dberg, A.A.; Ostrovskii, I.V. *Distribution of Values of Meromorphic Functions*; Nauka: Moscow, Russia, 1970. (In Russian)
2. Hayman, W.K. *Meromorphic Functions*; Clarendon Press: Oxford, UK, 1964.
3. Yang, C.C.; Yi, H.X. *Uniqueness Theory of Meromorphic Functions*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 2004; Volume 557.
4. Baker, I.N. On a class of meromorphic functions. *Proc. Am. Math. Soc.* **1966**, *17*, 819–822. [CrossRef]
5. Gross, F. On the equation $f^n + g^n = 1$. I. *Bull. Am. Math. Soc.* **1966**, *72*, 86–88. [CrossRef]
6. Gross, F. On the equation $f^n + g^n = 1$. II. *Bull. Am. Math. Soc.* **1966**, *72*, 647–648.
7. Gross, F. On the functional equation $f^n + g^n = h^n$. *Am. Math. Mon.* **1966**, *73*, 1093–1096. [CrossRef]
8. Montel, P. *Lecons Sur Les Familles Normales de Fonctions Analytiques et Leurs Appliciations*; Collect. Borel: Paris, France, 1927; pp. 135–136.
9. Yang, C.C.; Li, P. On the transcendental solutions of a certain type of nonlinear differential equations. *Arch. Math.* **2004**, *82*, 442–448. [CrossRef]

10. Liu, K.; Yang, L.Z. A note on meromorphic solutions of Fermat types equations. *An. Stiint. Univ. Al. I. Cuza Lasi Mat. (NS)* **2016**, *1*, 317–325.
11. Xu, H.Y.; Xuan, Z.X.; Luo, J.; Liu, S.M. On the entire solutions for several partial differential difference equations (systems) of Fermat type in $\mathbb{C}^2$. *AIMS Math.* **2021**, *6*, 2003–2017. [CrossRef]
12. Han, Q.; Lü, F. On the equation $f^n(z) + g^n(z) = e^{\alpha z+\beta}$. *J. Contemp. Math. Anal.* **2019**, *54*, 98–102. [CrossRef]
13. Luo, J.; Xu, H.Y.; Hu, F. Entire solutions for several general quadratic trinomial differential difference equations. *Open Math.* **2021**, *19*, 1018–1028. [CrossRef]
14. Hu, P.C.; Wang, Q. On meromorphic solutions of functional equations of Fermat type. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 2497–2515. [CrossRef]
15. Hu, P.C.; Wang, W.B.; Wu, L.L. Entire solutions of differential-difference equations of Fermat type. *Bull. Korean Math. Soc.* **2022**, *59*, 83–99.
16. Jiang, Y.Y.; Liao, Z.H.; Qiu, D. The existence of entire solutions of some systems of the Fermat type differential-difference equations. *AIMS Math.* **2022**, *7*, 17685–17698. [CrossRef]
17. Li, B.Q. On meromorphic solutions of $f^2 + g^2 = 1$. *Math. Z.* **2008**, *258*, 763–771. [CrossRef]
18. Liu, K.; Laine, I.; Yang, L. *Complex Delay-Differential Equations*; De Gruyter: Berlin, Germany, 2021.
19. Liu, K.; Ma, L.; Zhai, X. The generalized Fermat type difference equations. *Bull. Korean Math. Soc.* **2018**, *55*, 1845–1858. [CrossRef]
20. Liu, K.; Yang, L.Z.; Liu, X.L. Existence of entire solutions of nonlinear difference equations. *Czech. Math. J.* **2011**, *61*, 565–576. [CrossRef]
21. Yang, L.Z.; Zhang, J.L. Non-existence of meromorphic solutions of a Fermat type functional equation. *Aequationes Math.* **2008**, *76*, 140–150. [CrossRef]
22. Halburd, R.G.; Korhonen, R.J.; Tohge, K. Holomorphic curves with shift-invariant hyperplane preimages. *Trans. Am. Math. Soc.* **2014**, *366*, 4267–4298. [CrossRef]