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On One Approximate Method of a Boundary Value Problem for a One-Dimensional Advection–Diffusion Equation

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Abstract: This article discusses the author's version of the technology for solving a one-dimensional boundary value problem for a one-dimensional advection–diffusion equation based on the method of separation of variables, as well as the theory of eigenvalues and eigenfunctions when constructing a solution to a differential equation. This problem is solved in two stages. Firstly, we illustrate the technology of separating variables for equations with fractional derivatives, and then apply the theory of eigenvalues and eigenfunctions to obtain an exact solution in the form of an infinite series. Since this series converges very quickly, it is natural to replace it with the sum of the first few terms. The approximate solution obtained in this way is quite suitable for numerical calculations in practice. The article provides a listing of the program for performing calculations, as well as the results of calculations themselves.

Keywords: advection–diffusion; eigenvalue; eigenfunction; fractional derivative

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1. Introduction

It should be noted that fractional derivative equations are widely used in fields such as hydrodynamics, quantum science [1], medicine [2], mechanics of viscoelastic behavior of materials [3], modeling damping mechanisms [4], the study of biological processes [5], the study of viscoelastic material with partial damping [6], and the study of polymer concrete [7]. The article describes the development of the author's technology of separation of variables in solving differential variables with a fractional derivative, as well as the theory of eigenvalues and eigenfunctions by the example of studying the process of heat and mass transfer in various media with diffusion and subdifusion.

The problem of studying the patterns of formation of the radon environment is not new. The development of the mining industry (to study the regularities of the formation of the radon environment in mine workings, it was necessary to simulate the flux of radon density, which led to the construction of various models of radon transfer) became the main catalyst for in-depth research in this direction.

Note also that according to the radiation safety standards of the Russian Federation, the average annual equivalent equilibrium volumetric activity (concentration) of radon in the air of residential and public buildings should not exceed the established limit. To implement this decree, various models of mass transfer (radon) were built. Most of these models are based on the equation [8–12]:

$$d\frac{\partial^2 C(x)}{\partial x^2} - \lambda \cdot C(x) = 0, \tag{1}$$

where $C(x)$—distribution of radon volumetric activity in the sample, $\mathrm{Bq/m^3}$; d—diffusion coefficient of radon; λ—radon decay constant; λ—radon decay constant 2.09×10^{-6}.

Equation (1) is obtained under the following assumptions:

- Radon transfer occurs in one direction perpendicular to the sample cross section, while the influence of edge effects on its lateral surface is negligible;
- Barometric pressures at the boundaries of the sample are the same during the experiment;
- The emissions of radon in the sample material are negligible;
- There is no sorption of radon in the sample material.

Model (1) describes a stationary regime of mass transfer. Let us present the formulation and solution of the nonstationary problem. For this, we present laboratory studies of the diffusion radon permeability of materials in a non-stationary mode given in. There are two fundamentally different methods—"constant" and "instant" sources. In the well-known works of G. Zapalac [13], the theoretical basis of the non-stationary method of "constant source" is presented, as well as the scheme of the experimental setup and the results of determining the effective diffusion coefficient of radon in thin concrete samples. The mathematical formulation of the problem of radon mass transfer in the test sample corresponding to the experimental conditions is presented in the form of the equation:

$$d\frac{\partial^2 C(x,t)}{\partial x^2} = \frac{\partial C(x,t)}{\partial t}. \tag{2}$$

Equation (2) will be used by us in the future to model the process of radon transport in various environments. We also note the equation:

$$d\frac{\partial^2 C(x,t)}{\partial x^2} - \lambda \cdot C(x,t) = \frac{\partial C(x,t)}{\partial t},$$

which is widely used in the theory of heat and mass transfer. When it comes to abnormal diffusion, there are two main methods:

1. The first method is stochastic—in this case, diffusion is described using the process of a random walk of particles;
2. The second method is based on fractional calculus. Here, we are talking about models based on nonstationary fractional differential equations of the form:

$$d\frac{\partial^\alpha C(x,t)}{\partial x^\alpha} = \frac{\partial C(x,t)}{\partial t}, \tag{3}$$

where:

$$\frac{\partial^\alpha u(x,\, t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int\limits_0^x \frac{u(\tau,\, t)\, d\tau}{(x-\tau)^{\alpha-1}}$$

the derivative of the fractional (in the sense of Riemann–Liouville) order $1 < \alpha < 2$, which is widely used in modeling the process of radon transport.

In this paper, we carry out a detailed discussion and development of the method of separation of variables (Fourier method) when solving boundary value problems for Equation (3).

2. Research Method

2.1. Boundary Value Problem for an Inhomogeneous Fractional Differential Equation of Variance in a Local Setting

Consider the first boundary value problem for the inhomogeneous fractional dispersion equation:

$$\frac{\partial u(x,t)}{\partial t} = D \cdot \frac{\partial^\alpha u(x,t)}{\partial x^\alpha} + f(x,t), \tag{4}$$

$$u(1,t) = u(0,t) = 0, \tag{5}$$

$$u(x,0) = \varphi(x), \tag{6}$$

where $\frac{\partial^\alpha u(x,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int\limits_0^x \frac{u(\tau,t)\, d\tau}{(x-\tau)^{\alpha-1}}$ is the fractional derivative (in the sense of Riemann–Liouville) order $1 < \alpha < 2$ [14].

The following theorem holds.

Theorem 1. *The function:*

$$u(x,\, t) = \sum_{n=1}^{\infty} e^{\lambda_n D t}\left[\int_0^t f_n(t)\, e^{-\lambda_n D t} dt + \varphi_n\right] x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}) \tag{7}$$

is a solution to the boundary value problem (4)–(6). Here,

$$E_{\alpha,\alpha}(\lambda_n x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda_n x^{\alpha})^k}{\Gamma(\alpha + \alpha k)}$$

is the well-known Mittag–Leffler type function [14], φ_n is the expansion coefficient of functions $u(x,\, t)$ and $f(x,\, t)$ by the function basis $\omega_n(\lambda_n,\, x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha})$.

To prove Formula (7), we need the following lemma.

Lemma 1. *Function:*

$$u(x,\, t) = \sum_{n=1}^{\infty} \varphi_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}) \tag{8}$$

is a solution to the following problem:

$$\frac{\partial u(x,t)}{\partial t} = D_{0+}^{\alpha} u(x,t), \tag{9}$$

$$u(0+,t) = u(1,t) = 0, \tag{10}$$

$$u(x,0) = \phi(x), \tag{11}$$

where $D_{0+}^{\alpha} u(x,t) = \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int \frac{u(\tau,t)d\tau}{(x-\tau)^{\alpha-1}}$ is a fractional derivative (in the sense of Riemann–Liouville) of order $1 < \alpha < 2$.

Here,

$$E_{\alpha,\alpha}(\lambda_n x^{\alpha}) = \sum_{k=0}^{\infty} \frac{(\lambda_n x^{\alpha})^k}{\Gamma(\alpha + \alpha k)}$$

a well-known Mittag–Leffler type function, φ_n is the Fourier coefficient of the function expansion $\varphi(x)$ on a nonorthogonal basis of the system of functions $X_n(\lambda_n, x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha})$.

Proof of Lemma 1. We will look for a continuous nontrivial solution in a closed domain $(0 \le x \le 1,\, 0 \le t \le T)$ to the homogeneous fractional differential Equation (9) satisfying the boundary conditions (10) and the initial condition (11).

To solve this problem, consider, as is customary in the method of separation of variables [15], first the main auxiliary problem: find a solution to Equation (1) that is not identically zero, satisfying the homogeneous boundary conditions (2) and represented as a product:

$$u(x,\, t) = X(x)\, T(t), \tag{12}$$

where $X(x)$ is a function of only variable x, T is a function of only variable t. Let us use the author's approach [16], the method of separation of Fourier variables, for equations with fractional derivatives, which was successfully tested in [7,17].

Substituting the assumed form of solution (12) into Equation (1) and dividing both sides of the equality obtained as a result of this substitution by the product $X(x)\, T(t) \neq 0$, we have the equation:

$$\frac{T'(t)}{T(t)} = \frac{D_x^{\alpha} X(x)}{X(x)}. \tag{13}$$

In Equation (13), we can put:

$$\frac{T'(t)}{T(t)} = \frac{D_x^\alpha X(x)}{X(x)} = \lambda, \tag{14}$$

where $\lambda = const$ since the left side of Equation (13) depends only on t, and the right side only on x.

From (14) it follows that:

$$D_x^\alpha X(x) = \lambda X(x), \tag{15}$$

$$T'(t) = \lambda T(t). \tag{16}$$

Boundary conditions (10) gives:

$$X(0) = 0,\ X(1) = 0. \tag{17}$$

Thus, to determine the function $X(x)$, the eigenvalue problem was obtained (the Sturm–Liouville problem):

$$D_x^\alpha X(x) = \lambda X(x),\ X(0) = 0,\ X(1) = 0, \tag{18}$$

studied in [16,18,19]. In these works, it was shown that only for the eigenvalues λ_n, which are zeros of the function $E_{\alpha,\alpha}(\lambda)$, there exist eigenfunctions of the problem equal to:

$$X_n(\lambda_n,\ x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha). \tag{19}$$

These eigenvalues λ_n, obviously correspond to solutions of the Equation (16):

$$T_n(\lambda_n, t) = \varphi_n \exp\{\lambda_n t\},$$

where φ_n is the still undetermined coefficients.

Returning to the main auxiliary problem, we see that the functions:

$$u_n(x,\ t) = X_n(\lambda_n, t) \cdot T_n(\lambda_n, t) = \varphi_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha)$$

are particular solutions of Equation (1) that satisfy the zero boundary conditions.

Let us now turn to the solution of problem (9)–(11). We formally compose the series:

$$u(x,\ t) = \sum_{n=1}^{\infty} \varphi_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha). \tag{20}$$

The function $u(x,t)$ satisfies the boundary conditions since all members of the series satisfy them. Requiring the fulfillment of the initial conditions, we obtain:

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha). \tag{21}$$

In [20], it was shown that the system of functions $X_n(\lambda_n, t) = \{x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^\alpha)\}_{n=1}^{\infty}$ forms the basis in $L_2(0,1)$. Since the basis $X_n(\lambda_n t)$ is not orthogonal, then together with the system $X_n(\lambda_n, t)$ we will consider the system $z_n(\lambda_n, t) = \{(1-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda_n(1-x)^\alpha)\}_{n=1}^{\infty}$ to be biorthogonal to the $X_n(\lambda_n, t)$ [21]. Generally speaking, the system $z_n(\lambda_n, t) = \{(1-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda_n(1-x)^\alpha)\}_{n=1}^{\infty}$ is a system of eigenfunctions of the adjoint problem [22], which may be obtained the following way.

The spectral problem corresponding to (1) and (2) is:

$$D_x^\alpha X(x) = \lambda X(x), \quad X(0) = 0, \quad X(1) = 0. \tag{22}$$

The spectral problem was considered in [17,18] and the eigenfunctions of the spectral problem are:

$$\{X_n(x)\}_{n=1}^{\infty} = \left\{x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha})\right\}_{n=1}^{\infty} \tag{23}$$

corresponding to the eigenvalues λ_n, which are the zeros of the function $E_{\alpha,\alpha}(\lambda)$ with $Im(\lambda_n) > 0$.

The set $\{X_n(x)\}_{n=1}^{\infty}$ of eigenfunctions is complete but not orthogonal [23–26]. For the adjoint problem of the spectral problem (22), we have:

$$\langle cD_x^{\alpha}X(x), Y(x)\rangle = \left\langle \frac{d}{dx}J_x^{2-\alpha}\frac{d}{dx}X(x), Y(x)\right\rangle.$$

Integration by parts and taking $Y(0) = 0 = Y(1)$, we have:

$$\left\langle \frac{d}{dx}J_x^{2-\alpha}\frac{d}{dx}X(x), Y(x)\right\rangle = \left\langle J_x^{2-\alpha}\frac{d}{dx}X(x), \frac{d}{dx}Y(x)\right\rangle.$$

Thus,

$$\langle cD_x^{\alpha}X(x), Y(x)\rangle = \left\langle X(x), \frac{d}{dx}J_{1,x}^{2-\alpha}\frac{d}{dx}Y(x).\right\rangle$$

Hence, the adjoint problem of the spectral problem (22) is:

$$cD_{1,x}^{\alpha}Y(x) = \lambda Y(x),\ Y(0) = 0 = Y(1).$$

The adjoint problem has eigenfunctions $Y_n(x)$ corresponding to the same eigenvalues as that of spectral problem, where:

$$\{Y_n(x)\}_{n=1}^{\infty} = \left\{(1-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda_n(1-x)^{\alpha})\right\}_{n=1}^{\infty}. \tag{24}$$

The $\{X_n(x)\}_{n=1}^{\infty}$ sets and $\{Y_n(x)\}_{n=1}^{\infty}$ form a biorthogonal system of functions [20]. Let us provide some properties of the eigenvalues of the spectral problem.

Lemma 2.11: [20]. The eigenvalues λ_n, that are the zeros of the function $E_{\alpha,\alpha}$ with $Im(\lambda_n) > 0$, satisfy the following relations:

- $|\lambda_k| < |\lambda_{k+1}|$, for $k \geq 1$.
- For n large enough and $\arg(\lambda_n) > \frac{\alpha\pi}{2}$, we have $\left|e^{\lambda_n t}\right| < 1$ and $|\lambda_n| \sim O(n^{\alpha})$,$1 < \alpha < 2$.

Before we proceed further, notice that due to the properties of eigenvalues and the fact that $T > 0$, the Mittag–Leffler-type function $E_{\eta,1}(\lambda_n T^{\eta}) \neq 1$ (see [20]). Hence, we can find a positive constant C_2 independent of n such that:

$$\frac{1}{\left|E_{\eta,1}(\lambda_n T^{\eta}) - 1\right|} \leq C_2. \tag{25}$$

Let us mention that $E_{\gamma,1}(z) \neq 1$, only when $z = 0$.

To determine unknown coefficients φ_n, for both sides of equality (20), we multiply by the system of functions $z_n(\lambda_n t)$:

$$\varphi(x)z_n(\lambda_n, x) = \sum_{n=1}^{\infty}\varphi_n X_n(\lambda_n, x)z_n(\lambda_n, x). \tag{26}$$

It is known that for zeros $\{\lambda_n\}$ of the function $E_{\alpha,\beta}(\lambda)$ ($\alpha < 2$, β—arbitrary real number) such that $\mu \leq |\arg(\lambda)| \leq \pi$, where $\mu \in \left(\frac{\pi\alpha}{2}, \min\{\pi, \alpha\pi\}\right)$ the following estimate holds:

$$|E_{\alpha,\beta}(\lambda)| \leq \frac{C}{1 + |\lambda_n|}, \tag{27}$$

where C is an arbitrary real constant.

For sufficiently large (in absolute value) zeros $\{\lambda_n\}$ of the function $E_{\alpha,\beta}(\lambda)$, the following relation is also true:

$$\lambda_n^{\frac{1}{\alpha}} = 2n\pi i - (1+\alpha)\left[\ln(2n\pi) + \frac{\pi}{2}i\right] + \ln\frac{\alpha}{\Gamma(-\alpha)} + O(1) =$$

$$= \left[-(1+\alpha)\ln(2n\pi) + \ln\frac{\alpha}{\Gamma(-\alpha)} + O(1)\right] + \left[2n\pi - \frac{\pi}{2}(1+\alpha)\right]i,$$

from this we have:

$$|\lambda_n^{\frac{1}{\alpha}}| = \left[-(1+\alpha)\ln(2n\pi) + \ln\frac{\alpha}{\Gamma(-\alpha)} + O(1)\right]^2 + \left[2n\pi - \frac{\pi}{2}(1+\alpha)\right]^2 \sim$$

$$\sim [-(1+\alpha)\ln(2n\pi)]^2 + [2n\pi]^2 \sim O(n^2).$$

Thus, the following equivalence holds:

$$|\lambda_n| \sim O(n^\alpha). \tag{28}$$

Using estimate (27) and equivalence (28) for systems of functions $\{X_n\}$ and $\{z_n\}$, we obtain the following relations:

$$|X_n(x)| = |x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^\alpha)| \leq \frac{C_1 x^{\alpha-1}}{1+|\lambda_n x^\alpha|} \leq \frac{C_1}{|\lambda_n|x} \leq \frac{C_1}{n^\alpha x}, \quad C_1 = const; \tag{29}$$

$$|z_n(x)| = |(1-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda_n(1-x)^\alpha)| \leq$$

$$\leq \frac{C_2(1-x)^{\alpha-1}}{1+|\lambda_n(1-x)^\alpha|} \leq \frac{C_2}{|\lambda_n|(1-x)} \leq \frac{C_2}{n^\alpha(1-x)}, \quad C_2 = const. \tag{30}$$

From this, it follows that $\{X_n\}$ and $\{z_n\}$ in the right side of equality (26) are bounded; therefore, equality (26) can be term-by-term integrated over the segment $[0, 1]$:

$$\int_0^1 \varphi(x)z_n(x)dx = \sum_{n=1}^{\infty}\varphi_n \int_0^1 X_n(x)z_n(x)\,dx. \tag{31}$$

Equality (31) can be rewritten as:

$$(\varphi, z_n) = \sum_{n=1}^{\infty}\varphi_n(X_n, z_n), \tag{32}$$

where:

$$(\varphi, z_n) = \int_0^1 \varphi(x)z_n(x)dx, \quad (X_n, z_n) = \int_0^1 X_n(x)z_n(x)\,dx.$$

Due to the biorthogonality of the systems of functions $\{X_n(x)\}$ and $\{z_n(x)\}$, it follows from (31) that $(\varphi, z_n) = \varphi_n(X_n, z_n)$. From here:

$$\varphi_n = \frac{(\varphi, z_n)}{(X_n, z_n)}. \tag{33}$$

As the system of functions $\{X_n(x)\}$ and $\{z_n(x)\}$ are not only biorthogonal to each other, but also orthonormal, then:

$$(X_n, z_n) = \int\limits_0^1 X_n(x) z_n(x)\, dx = 1$$

and Formula (33), respectively, takes the form:

$$\varphi_n = (\varphi, z_n).$$

Thus, the Fourier coefficients φ_n in solution $u(x,\, t)$ of the boundary value problem (9)–(11) expressed as the dot product of functions $\varphi(x)$ and $z_n(x)$ in the form:

$$\varphi_n = (\varphi, z_n) = \int\limits_0^1 \varphi(x) z_n(x) dx. \tag{34}$$

Consider the series (20) with coefficients φ_n, defined by Formula (34) and show that this series satisfies all conditions of problem (4)–(6).

For this, it is necessary to prove that the function defined by the series is differentiable and satisfies Equation (1) in the domain $0 < x < 1$, $t > 0$ and is continuous at the points of the boundary of this region (for $t = 0$, $x = 0$, $x = 1$).

Since Equation (4) is linear, then, by virtue of the superposition principle, a series composed of particular solutions will also be a solution if it converges and can be differentiated term-by-term once by t and twice by x as:

$$(1 < \alpha < 2).$$

For zeros, λ_n of the Mittag–Leffler functions $E_{\alpha,\beta}(\lambda)$ for $n \to \infty$ holds [14] the asymptotic formula:

$$\lambda_n = e^{i\frac{\alpha\pi}{2}} (2\pi n)^\alpha \left[1 + O\left(\frac{\lg^2 n}{n^2}\right)\right] \tag{35}$$

and we may neglect the remainder term $O\left(\frac{\lg^2 n}{n^2}\right)$ since $\frac{\lg^2 n}{n^2} \to 0$ for $n \to \infty$. From (34), it follows that $\arg(\lambda_n) = \frac{\alpha\pi}{2} > \frac{\pi}{2}$ $(1 < \alpha < 2)$. From this, it follows that for the set of zeros $\ldots, \lambda_{-3}, \lambda_{-2}, \lambda_{-1}, \lambda_1, \lambda_2, \lambda_3, \ldots$ of the function $E_{\alpha,\beta}(\lambda)$, the argument $|\arg(\lambda_n)| > \frac{\pi}{2}$. Thus, $Re\,(\lambda_n) < 0$.

From the above it follows that:

$$\begin{aligned}
|\exp\{\lambda_n Dt\}| &= |\exp\{(Re(\lambda_n) + \mathrm{Im}(\lambda_n) \cdot i)Dt\}| = \\
&= |\exp\{(Re(\lambda_n)t\} \cdot \exp\{\mathrm{Im}(\lambda_n) \cdot i)Dt\}| = \\
&= |\exp\{Re(\lambda_n)Dt\}| \cdot |\exp\{\mathrm{Im}(\lambda_n)Dti\}| = \\
&= |\exp\{Re(\lambda_n)Dt\}| < 1 \quad (Re(\lambda_n) < 0).
\end{aligned}$$

Thus, $\exp\{\lambda_n Dt\}$ corresponds the estimation:

$$|\exp\{\lambda_n Dt\}| < 1. \tag{36}$$

Taking into account (26), (35) for the series, we obtain:

$$|\varphi_n \exp\{\lambda_n t x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^\alpha)| \le |\varphi_n| \cdot |E_{\alpha,\alpha}(\lambda_n x^\alpha)| \le |\varphi_n| \frac{1}{1 + |\lambda_n| x^\alpha} \le |\varphi_n| \frac{1}{|\lambda_n| x^\alpha}.$$

Consider now the majorizing series:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{|\lambda_n|}. \tag{37}$$

Using the equivalence (27), we rewrite the majorant (36) in the form:

$$\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}, \quad (1 < \alpha < 2). \tag{38}$$

As can be seen, the series (38) is a converging generalized harmonic series, which implies the absolute convergence of the series for any $0 \le x \le 1$ and $0 \le t \le T$.

Let us now show that for $t \ge \bar{t} \ge 0$ ($\bar{t}$—any auxiliary number) the series of derivatives,

$$\sum_{n=1}^{\infty} \frac{\partial u_n(x;t)}{\partial t} \text{and} \sum_{n=1}^{\infty} \frac{\partial u_n^2(x;t)}{\partial x^2}$$

converge uniformly. Let us formulate additional requirements that the function must satisfy $\varphi(x)$. Assume first that $\varphi(x)$ is bounded, $|\varphi(x)| < M$; then:

$$|\varphi_n| \ = 2\left|\int_0^1 \varphi(\xi) z_n(\xi) d\xi\right| < 2M,$$

from which it follows:

$$\left|\frac{\partial u_n(x,t)}{\partial t}\right| < 2M|\lambda_n \exp\{\lambda_n \bar{t}\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha})| <$$

$$< 2M|\lambda_n| \frac{1}{1+|\lambda_n|} < 2M|\lambda_n| \frac{1}{|\lambda_n|} < 2M$$

for $t \ge \bar{t}$.

Similarly, given that,

$$\left(\frac{d}{dz}\right)^m \left[z^{\beta-1} E_{\alpha,\beta}(z^{\alpha})\right] = z^{\beta-m-1} E_{\alpha,\beta-m}(z^{\alpha}),$$

$$\left|\frac{\partial^2 u_n(x,t)}{\partial x^2}\right| \le 2M\left|\left(\frac{\partial}{\partial x}\right)^2 \left[x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha})\right] \exp\{\lambda_n \bar{t}\}\right| \le$$

$$\le 2M|x^{\alpha-3} E_{\alpha,\alpha-2}(\lambda_n x^{\alpha})|\,|\exp\{\lambda_n \bar{t}\}| < 2M \text{ for } t \ge \bar{t}.$$

From this, it follows that for $t > 0$ the series is the function differentiable term-by-term once by t and twice by x, and so, having a derivative of order α since $1 < \alpha < 2$.

If the function $\varphi(x)$ is continuous, it has a piecewise continuous derivative, and satisfies the conditions $\varphi(0) = 0$ and $\varphi(1) = 0$, then the series defines a continuous function for $t \ge 0$.

Really, from the inequality:

$$|u(x,\, t)| \ < \ |\varphi_n| (\text{for} t \ge 0, 0 \le x \le 1)$$

it immediately follows the uniform convergence of the series for $t \ge 0$, $0 \le x \le 1$, which proves the validity of the above statement if we take into account that for a continuous and piecewise smooth function $\varphi(x)$, a series of moduli of the Fourier coefficients converges if $\varphi(0) = \varphi(1) = 0$.

Since it is established that:

$$u(x,\, t) = \sum_{n=1}^{\infty} \delta_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}).$$

We get:

$$u_{12}(x,t) = \sum_{n=1}^{12} \delta_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}), \quad u_{17}(x,t) = \sum_{n=1}^{17} \delta_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}).$$

From this, it follows:

$$\left| \sum_{n=1}^{17} \delta_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}) - \sum_{n=1}^{12} \delta_n \exp\{\lambda_n t\} x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha}) \right| \leq 0164$$

which speaks about the high accuracy of the stated approximate method. For implementation we proposed the algorithm, realized in Matlab R2017b. □

Consider first, the homogeneous fractional differential equation:

$$\frac{\partial u(x,\, t)}{\partial t} = D \cdot \frac{\partial^{\alpha} u(x,\, t)}{\partial x^{\alpha}} \tag{39}$$

corresponding to the fractional differential Equation (1). In order to find a nontrivial solution to the homogeneous fractional differential Equation (32), we use the Fourier method of separation of variables and represent the function $u(x,t)$ in the form:

$$u(x,\, t) = \omega(x)\rho(t), \tag{40}$$

where $\omega(x)$ is the function that depends only on a variable x, $\rho(t)$ is the function that depends only on a variable t. Substituting the assumed form of solution (40) into Equation (39), we obtain the following equality:

$$\omega(x)\rho'(t) = \rho(t) D_x^{\alpha} \omega(x). \tag{41}$$

Dividing both sides of equality (41) by the product $\omega(x)\rho(t)$, we obtain the equation:

$$\frac{\rho'(t)}{\rho(t)} = \frac{D_x^{\alpha}\omega(x)}{\omega(x)}. \tag{42}$$

The left side of Equation (42) depends only on the variable t, while the right side only on the variable x. Therefore, Equation (42) can be true only in the case:

$$\frac{\rho'(t)}{\rho(t)} = \frac{D_x^{\alpha}\omega(x)}{\omega(x)} = \lambda, \tag{43}$$

where λ is some real number.

Equation (42) implies that:

$$D_x^{\alpha}\omega(x) = \lambda\omega(x), \tag{44}$$

$$\rho'(t) = \lambda\rho(t). \tag{45}$$

The boundary conditions (2) in this case give:

$$\omega(0) = 0, \quad \omega(1) = 0.$$

Thus, to define the function $\omega(x)$, an eigenvalue problem is obtained (Sturm–Liouville problem):

$$\begin{cases} D_x^{\alpha}\omega(x) = \lambda\omega(x), \\ \quad \omega(0) = 0, \\ \quad \omega(1) = 0. \end{cases} \tag{46}$$

In this work, it was shown that only for eigenvalues $\{\lambda_k\}$, that are zeros of the function $E_{\alpha,\alpha}(\lambda)$, there exist eigenfunctions $\omega_n(\lambda_n, x)$, which are:

$$\omega_n(\lambda_n, x) = x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha}). \tag{47}$$

It was also proved that the system of functions $\omega_n(\lambda_n, x)$ forms a basis in a Hilbert space $L_2(0,1)$. Thus, the functions $u(x, t)$ and $f(x, t)$ can be expanded in a Fourier series on the basis of the system of functions (47):

$$u(x, t) = \sum_{n=1}^{\infty} v_n(t)\,\omega_n(\lambda_n, x), \tag{48}$$

$$f(x, t) = \sum_{n=1}^{\infty} f_n(t)\,\omega_n(\lambda_n, x). \tag{49}$$

In expressions (48) and (49), the functions $v_n(t)$ and $f_n(t)$ play the role of the Fourier coefficients of the function expansions $u(x,t)$ and $f(x,t)$ on the basis of the system of functions (47).

Now, substituting (48) and (49) into Equation (1), we obtain:

$$\sum_{n=0}^{\infty} v_n'(t)\,\omega(\lambda_n, x) = \sum_{n=0}^{\infty} v_n(t)\,D\,\frac{\partial^{\alpha}\omega(\lambda_n, x)}{\partial x^{\alpha}} + \sum_{n=0}^{\infty} f_n(t)\,\omega(\lambda_n, x). \tag{50}$$

Assuming the right-hand side of equality (50):

$$\frac{\partial^{\alpha}\omega(\lambda_n, x)}{\partial x^{\alpha}} = \lambda_n\omega(\lambda_n, x),$$

we rewrite Equation (50) as:

$$\sum_{n=0}^{\infty} v_n'(t)\,\omega(\lambda_n, x) = \sum_{n=0}^{\infty} \lambda_n D\,v_n(t)\,\omega(\lambda_n, x) + \sum_{n=0}^{\infty} f_n(t)\,\omega(\lambda_n, x). \tag{51}$$

Dividing both sides of equality (51) by $\omega_n(\lambda_n, x)$, we come to a linear differential equation of the first order:

$$v_n'(t) = \lambda_n D\,v_n(t) + f_n(t), \tag{52}$$

the solution that we find by the Lagrange method, i.e., by the method of variation of an arbitrary constant. First, as is customary with the Lagrange method, consider the homogeneous differential equation:

$$v_n'(t) = \lambda_n D\,v_n(t), \tag{53}$$

corresponding to the inhomogeneous differential Equation (18). Separating the variables in Equation (46), we obtain:

$$\frac{dv_n(t)}{v_n(t)} = \lambda_n D\,dt. \tag{54}$$

Integrating the resulting equation with separated variables (54), we obtain:

$$v_n(t) = \sigma_n e^{\lambda_n D\,t}, \tag{55}$$

where σ_n is the arbitrary constant. Further, in (55), as is customary in the Lagrange method, arbitrary constants σ_n considered as some functions of t, i.e., we assume that $\sigma_n = \varphi_n(t)$:

$$v_n(t) = \varphi_n(t) e^{\lambda_n D t}. \tag{56}$$

To find unknown functions $\varphi_n(t)$, we differentiate the Equation (56) by t:

$$v_n'(t) = \varphi_n'(t)\, e^{\lambda_n D t} + \lambda_n D\, \varphi_n(t)\, e^{\lambda_n D t}. \tag{57}$$

Now, substituting (56) and (57) into (52), we arrive at the equation:

$$\varphi_n'(t)\, e^{\lambda_n D t} + \lambda_n D\, \varphi_n(t)\, e^{\lambda_n D t} = \lambda_n D\, \varphi_n(t)\, e^{\lambda_n D t} + f_n(t),$$

solving which, we obtain:

$$\varphi_n(t) = \int_0^t f_n(t)\, e^{-\lambda_n D t} dt + \varphi_n, \tag{58}$$

where φ_n is an arbitrary real constant.

Thus, substituting (58) into (56), we find the value of the desired function $v_n(t)$:

$$v_n(t) = e^{\lambda_n D t} \left[\int_0^t f_n(t)\, e^{-\lambda_n D t} dt + \varphi_n \right]. \tag{59}$$

Finally, substituting expression (59) into equality (48), we obtain solution (7) of Equation (1), which satisfies the zero boundary conditions (2).

In order to determine the coefficients φ_n, on the right side of the series (7), we satisfy it to the initial condition (3): $u(x, 0) = \sum\limits_{n=1}^{\infty} \varphi_n x^{\alpha-1} E_{\alpha,\alpha}(\lambda_n x^{\alpha})$, i.e.,

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n \omega_n(\lambda_n, t). \tag{60}$$

Since the basis of the system of functions $\omega_n(\lambda_n, t)$ is not orthogonal, together with the system $\omega_n(\lambda_n, t)$, we introduce into consideration the system $z_n(\lambda_n, t) = (1-x)^{\alpha-1} E_{\alpha,\alpha}$ $(\lambda_n (1-x)^{\alpha})$, which is biorthogonal to it.

Let us multiply both sides of equality (60) by the system of functions $z_n(\lambda_n, t)$:

$$\varphi(x) z_n(\lambda_n, t) = \sum_{n=1}^{\infty} \varphi_n \omega_n(\lambda_n, t) z_n(\lambda_n, t). \tag{61}$$

It is known that for zeros $\{\lambda_n\}$ of the function $E_{\alpha,\beta}(\lambda)$ ($\alpha < 2$, β—is arbitrary real number) such that $\mu \le |\arg(\lambda)| \le \pi$, where $\mu \in \left(\frac{\pi\alpha}{2}, \min\{\pi, \alpha\pi\}\right)$ the following estimate holds:

$$|E_{\alpha,\beta}(\lambda)| \le \frac{C}{1 + |\lambda_n|}, \tag{62}$$

where C is an arbitrary real constant.

For sufficiently large (in absolute value) zeros $\{\lambda_n\}$ of the function $E_{\alpha,\beta}(\lambda)$, the following relation also holds:

$$\lambda_n^{\frac{1}{\alpha}} = 2n\pi i - (1+\alpha)\left[\ln(2n\pi) + \frac{\pi}{2} i\right] + \ln\frac{\alpha}{\Gamma(-\alpha)} + O(1) =$$

$$= \left[-(1+\alpha)\ln(2n\pi) + \ln\frac{\alpha}{\Gamma(-\alpha)} + O(1)\right] + \left[2n\pi - \frac{\pi}{2}(1+\alpha)\right] i,$$

from this it follows that:

$$|\lambda_n^{\frac{1}{\alpha}}| = \left[-(1+\alpha)\ln(2n\pi) + \ln\frac{\alpha}{\Gamma(-\alpha)} + O(1)\right]^2 + \left[2n\pi - \frac{\pi}{2}(1+\alpha)\right]^2$$

$$\sim [-(1+\alpha)\ln(2n\pi)]^2 + [2n\pi]^2 \sim O(n^2).$$

Thus, the following equivalence holds:

$$|\lambda_n| \sim O(n^\alpha). \tag{63}$$

Using the estimate (62) and the equivalence (63) for the systems $\{\omega_n\}$ and $\{z_n\}$ we obtain the following relations:

$$|\omega_n(x)| = |x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^\alpha)| \le \frac{C_1 x^{\alpha-1}}{1+|\lambda_n x^\alpha|} \le \frac{C_1}{|\lambda_n|x} \le \frac{C_1}{n^\alpha x}, \quad C_1 = conct; \tag{64}$$

$$|z_n(x)| = |(1-x)^{\alpha-1}E_{\alpha,\alpha}(\lambda_n(1-x)^\alpha)| \le \tag{65}$$

$$\le \frac{C_2(1-x)^{\alpha-1}}{1+|\lambda_n(1-x)^\alpha|} \le \frac{C_2}{|\lambda_n|(1-x)} \le \frac{C_2}{n^\alpha(1-x)}, \quad C_2 = const.$$

From this, it follow that the functions $\{\omega_n\}$ and $\{z_n\}$ in the right side of (61) are bounded, so for the equality (61), we may integrate it term-by-term over the segment $[0, 1]$:

$$\int_0^1 \varphi(x) z_n(x) dx = \sum_{n=1}^{\infty} \varphi_n \int_0^1 \omega_n(x) z_n(x)\, dx. \tag{66}$$

Let us rewrite (66) as follows:

$$(\varphi, z_n) = \sum_{n=1}^{\infty} \varphi_n(\omega_n, z_n), \tag{67}$$

where $(\varphi, z_n) = \int_0^1 \varphi(x) z_n(x) dx$, $(\omega_n, z_n) = \int_0^1 \omega_n(x) z_n(x)\, dx$.

Due to the biorthogonality of the systems of functions $\{\omega_n(x)\}$ and $\{z_n(x)\}$ from (67), it follows that $(\varphi, z_n) = \varphi_n(\omega_n, z_n)$. From this:

$$\varphi_n = \frac{(\varphi, z_n)}{(\omega_n, z_n)}. \tag{68}$$

Since the systems of the functions $\{\omega_n(x)\}$ and $\{z_n(x)\}$ are not only biorthogonal to each other, but also orthonormal, then $(\omega_n, z_n) = \int_0^1 \omega_n(x) z_n(x)\, dx = 1$ and formula (68), respectively, takes the form $\varphi_n = (\varphi, z_n)$.

Thus, the Fourier coefficients φ_n in the solution $u(x,t)$ boundary value problem (1)–(3) are expressed as the scalar product of functions $\varphi(x)$ and $z_n(x)$ in the form:

Hence, it follows that the real coefficients:

$$\varphi_n = (\varphi, z_n) = \int_0^1 \varphi(x) z_n(x) dx. \tag{69}$$

It follows that the real coefficients φ_n, can be defined as the coefficients of the expansion in the Fourier series of the function $\varphi(x)$ in the basis $\omega_n(\lambda_n, t)$:

$$\varphi_n = (\varphi(x),\, z_n(\lambda_n,\, x)).$$

Let us now determine the values of the functions $f_n(t)$. Let us now determine the values of the functions in the expression of the series (7). For this, we use equality (49). Since in expression (49), the function $f_n(t)$ plays the role of the coefficients of the expansion in the Fourier series of the function $f(x,\, t)$ in the basis $\omega_n(\lambda_n,\, t)$, then sought functions $f_n(t)$ can be defined as the scalar product:

$$f_n(t) = (f(x,\, t),\, z_n(\lambda_n,\, x)),$$

where, as noted earlier, $z_n(\lambda_n, t)$ is a system biorthogonal to the system $\omega_n(\lambda_n, t)$.

It is known that [8] for the zeros $\{\lambda_n\}$ of the Mittag–Leffler function $E_{\alpha,\,\beta}(\lambda)$ for $n \to \infty$ the asymptotic formula holds:

$$\lambda_n = e^{i\frac{\alpha\pi}{2}}(2\pi n)^{\alpha}\left[1 + O\left(\frac{\lg^2 n}{n^2}\right)\right], \tag{70}$$

with the remainder $O\left(\frac{\lg^2 n}{n^2}\right)$, which can be neglected since $\frac{\lg^2 n}{n^2} \to 0$ for $n \to \infty$. From Formula (57), it follows that $\arg(\lambda_n) = \frac{\alpha\pi}{2} > \frac{\pi}{2}$ $(1 < \alpha < 2)$. Hence, it follows that for the set of zeros, $\ldots,\, \lambda_{-3},\, \lambda_{-2},\, \lambda_{-1},\, \lambda_1,\, \lambda_2,\, \lambda_3,\, \ldots$ of the function $E_{\alpha,\,\beta}(\lambda)$, the argument is $|\arg(\lambda_n)| > \frac{\pi}{2}$. Thus, $Re\,(\lambda_n) < 0$.

From the above it follows that:

$$\begin{aligned} |\exp\{\lambda_n Dt\}| &= |\exp\{(Re(\lambda_n) + \mathrm{Im}(\lambda_n)\cdot i)Dt\}| = \\ &= |\exp\{(Re(\lambda_n)t\}\cdot \exp\{\mathrm{Im}(\lambda_n)\cdot i)Dt\}| = \\ &= |\exp\{Re(\lambda_n)Dt\}|\cdot|\exp\{\mathrm{Im}(\lambda_n)Dti\}| = \\ &= |\exp\{Re(\lambda_n)Dt\}| < 1 \quad (Re(\lambda_n) < 0). \end{aligned}$$

Thus, for $\exp\{\lambda_n Dt\}$ conforms to the estimation:

$$|\exp\{\lambda_n Dt\}| < 1. \tag{71}$$

Taking into account thee estimates (62) and (71), we obtain the following relation:

$$\left| e^{\lambda_n Dt}\left[\int_0^t f_n(t)\, e^{-\lambda_n Dt}dt + \varphi_n\right] x^{\alpha-1}E_{\alpha,\alpha}(\lambda_n x^{\alpha})\right| \le \left|\left[\int_0^t f_n(t)\, dt + \varphi_n\right]\frac{1}{1+|\lambda_n|x^{\alpha}}\right|.$$

Further, taking into account the equivalence (63), as well as the boundedness of the function $f_n(t)$ and the coefficients φ_n, we conclude about the convergence of the series determined by the right-hand side of the function (7).

Based on the theoretical results obtained, algorithms and programs have been compiled in Matlab R2017b to calculate the solution $u(x,\, t)$ (20) of the problem (9)–(11), $\varphi(x) = 1 - x$. In the first case, we limit ourselves in (20) to the number of terms $n = 12$ and in the second case $n = 17$. Below, there is one version of the program listing and two calculation options for comparing the accuracy of the result. The eigenvalues λ_n and the results of the values of the analytical approximate solution $u_{12}(x,\, t)$ and $u_{17}(x,\, t)$ are calculated.

Program Script

```
Clear
clc
global a
a=3/2;
global s0 s1 s2 s3 s4 s5 s6 s7 s8 s9 s10 s11 s12 s13 s14 s15 s16 s17
```

```
s0=1/(gamma(a+a*0));
s1=1/(gamma(a+a*1));
s2=1/(gamma(a+a*2));
s3=1/(gamma(a+a*3));
s4=1/(gamma(a+a*4));
s5=1/(gamma(a+a*5));
s6=1/(gamma(a+a*6));
s7=1/(gamma(a+a*7));
s8=1/(gamma(a+a*8));
s9=1/(gamma(a+a*9));
s10=1/(gamma(a+a*10));
s11=1/(gamma(a+a*11));
s12=1/(gamma(a+a*12));
s13=1/(gamma(a+a*13));
s14=1/(gamma(a+a*14));
s15=1/(gamma(a+a*15));
s16=1/(gamma(a+a*16));
s17=1/(gamma(a+a*17));
p=[s17 s16 s15 s14 s13 s12 s11 s10 s9 s8 s7 s6 s5 s4 s3 s2 s1 s0];
r=roots(p);
global l
l=sort(r)
for i=1:17
[I11,cnt11]=quad(@(x)2*x.*(1-x).^(a-1).*(s0+s1*(l(i)*(1-x).^a).^1+s2*(l(i)*(1-x).^a).^2+...
s3*(l(i)*(1-x).^a).^3+s4*(l(i)*(1-x).^a).^4+s5*(l(i)*(1-x).^a).^5+s6*(l(i)*(1-x).^a).^6+...
s7*(l(i)*(1-x).^a).^7+s8*(l(i)*(1-x).^a).^8+s9*(l(i)*(1-x).^a).^9+s10*(l(i)*(1-x).^a).^10+...
s11*(l(i)*(1-x).^a).^11+s12*(l(i)*(1-x).^a).^12+s13*(l(i)*(1-x).^a).^13+s14*(l(i)*(1-.^a).^
14+s15*(l(i)*(1-x).^a).^15+s16*(l(i)*(1-x).^a).^16+s17*(l(i)*(1-x).^a).^17),0,0.5);
[I12,cnt12]=quad(@(x)(4/3-2/3*x).*(1-x).^(a-1).*(s0+s1*(l(i)*(1-x).^a).^
1+s2*(l(i)*(1-x).^a).^2+...
s3*(l(i)*(1-x).^a).^3+s4*(l(i)*(1-x).^a).^4+s5*(l(i)*(1-x).^a).^5+s6*(l(i)*(1-x).^a).^6+...
s7*(l(i)*(1-x).^a).^7+s8*(l(i)*(1-x).^a).^8+s9*(l(i)*(1-x).^a).^9+s10*(l(i)*(1-x).^a).^10+...
s11*(l(i)*(1-x).^a).^11+s12*(l(i)*(1-x).^a).^12+s13*(l(i)*(1-x).^a).^13+
s14*(l(i)*(1-x).^a).^14+...
s15*(l(i)*(1-x).^a).^15+s16*(l(i)*(1-x).^a).^16+s17*(l(i)*(1-x).^a).^17),0,0.5);
ff(i)=I11+I12;
end
format('long','g')
ff
% syms x
% sol=0;
% for k=1:20
% sol=sol+ff(k)*x.^(a-1)
% end
```

3. Discussion of the Results

Based on the compiled algorithms and programs in Matlab R2017b, one of the listing options is presented in Section 2, the eigenvalues λ_n of the Sturm–Liouville problem and the results of the values of the analytical approximate solution for $n = 12$, $u_{12}(x, t)$, and for $n = 17$, $u_{17}(x, t)$ are calculated.

$1 =$
-5.0754300393759
-16.7908900478112
$-18.744700713431 - 5.27370466276291i$

$-18.744700713431 + 5.27370466276291i$
$-17.5935684739021 - 13.958778580314i$
$-17.5935684739021 + 13.958778580314i$
$-12.5037075124573 - 23.7208022342114i$
$-12.5037075124573 + 23.7208022342114i$
$-1.26795205683724 - 33.9468429922635i$
$-1.26795205683724 + 33.9468429922635i$
$22.0702143093482 - 42.7337132624382i$
$22.0702143093482 + 42.7337132624382i$
$ff =$
0.137401420534954
$l =$
-5.07543002954347
-17.4721082329839
-28.6069313666809
$-29.3861949229922 - 7.44053043356274i$
$-29.3861949229922 + 7.44053043356274i$
$-28.664909832168 - 16.9294907149687i$
$-28.664909832168 + 16.9294907149687i$
$-25.1399503163194 - 27.4228090665281i$
$-25.1399503163194 + 27.4228090665281i$
$-17.9295026344547 - 38.610702911213i$
$-17.9295026344547 + 38.610702911213i$
$-5.49276791576975 - 49.9819129876363i$
$-5.49276791576975 + 49.9819129876363i$
$15.1216190018407 - 60.4343903753214i$
$15.1216190018407 + 60.4343903753214i$
$51.7431493434345 - 66.9283288460732i$
$51.7431493434345 + 66.9283288460732i$
$ff =$
0.137401421100112

4. Conclusions

The paper describes the development of the author's approach to the technology of solving differential equations with fractional partial derivatives by the example of a one-dimensional boundary value problem for a one-dimensional advection–diffusion equation based on the method of separation of variables, as well as the theory of eigenvalues and eigenfunctions in constructing a solution to a differential equation. As the above list of references shows, this technology is successfully implemented in solving other boundary value problems and Cauchy problems. The article provides a listing of the program for performing calculations, as well as the results of calculations themselves.

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