axioms

Article

Resolution of an Isolated Case of a Quadratic Hypergeometric 2F1 Transformation

Mohamed Jalel Atia [1,2]

[1] Department of Mathematics, College of Science, Qassim University, Buraidah 51452, Saudi Arabia; m.attia@qu.edu.sa
[2] Laboratory LR17ES11, University of Gabes, Gabes 6072, Tunisia

Abstract: The identity ${}_2F_1(\alpha,\beta;2\alpha;z) = (1-\frac{z}{2})^{-\beta}\,{}_2F_1(\frac{\beta}{2},\frac{\beta+1}{2};\alpha+\frac{1}{2};(\frac{z}{2-z})^2)$ given, either by I.S. Gradshteyn and I.M. Ryzhik in Table of integrals series and products named 9.134 or in the handbook "mathematical functions with formulas, graphs and mathematical tables" done by Abramowitz-Stegun named 15.3.20 or in the book "special functions" done by G. Andrews, R. Askey and R. Roy named 3.1.7 page 127 with a slight modification is true provided that $\{2\alpha+1,\ \alpha+\frac{3}{2}\}$ are not natural numbers and $\alpha-\beta$ is not an integer (see Gradshteyn, Ryzhik, 9.130). In this manuscript we consider a case where $\alpha-\beta$ is an integer by taking $\beta=2a$, $\alpha=-n+1$. We give and prove the right identity for any positive integer a and for any any positive integer n.

Keywords: hypergeometric functions; quadratic transformation; hypergeometric series with finitely many terms and hypergeometric series with infinitely many terms

MSC: 33C05; 33D15

Citation: Atia, M.J. Resolution of an Isolated Case of a Quadratic Hypergeometric 2F1 Transformation. *Axioms* **2022**, *11*, 533. https://doi.org/10.3390/axioms11100533

Academic Editor: Gradimir V. Milovanović

Received: 6 September 2022
Accepted: 25 September 2022
Published: 5 October 2022

1. Introduction

The Gaussian hypergeometric function (GHF) ${}_2F_1(a,b;c;z)$ is a series defined by

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n z^n}{(c)_n n!} = 1 + \frac{abz}{c} + \frac{a(a+1)b(b+1)z^2}{2c(c+1)} + \ldots,\ |\,z\,| < 1, \quad (1) $$

with $c \neq 0,-1,-2,\ldots$ and where $(a)_n$ is the Pochhammer symbol (or shifted factorial) defined by

$$ (a)_n = a(a+1)(a+2)\ldots(a+n-1),\ n \geq 1,\ (a)_0 = 1. $$

First, we point out that the GHF has many interesting applications including but not limited to [1] where the authors used the GHF to develop a safe and secure Bank locker system for Banks.

Second, we also want to point out that the quadratic transformations which relate two hypergeometric functions (with the variable in one and a quadratic function of the variable in the other), are true under some condition. In fact, in [2], page 1008, 9.130 authors wrote "The series ${}_2F_1(\alpha,\beta;\gamma;z)$ defines an analytic function that, speaking generally, has singularities at the points $z=0,\ 1,$ and ∞. (In the general case, there are branch points). We make a cut in the $z-$plane along the real axis from $z=1$ to $z=\infty$; that is, we require that $|\ arg(-z)\ | < \pi$ for $|z|=1$. Then, the series ${}_2F_1(\alpha,\beta;\gamma;z)$ will, in the cut plane, yield a single-valued analytic continuation, which we can obtain by means of the formulas below (provided $\gamma+1$ is not a natural number and $\alpha-\beta$ and $\gamma-\alpha-\beta$ are not integers). These formulas make it possible to calculate the values of F in the given region, even in the case in which $|z|>1$. There are other closely related transformation formulas that can also be used to get the analytic continuation when the corresponding relationships hold between α,β,γ".

The identity

$$ {}_2F_1(\alpha,\beta;2\alpha;z) = (1-\frac{z}{2})^{-\beta}\, {}_2F_1(\frac{\beta}{2},\frac{\beta+1}{2};\alpha+\frac{1}{2};(\frac{z}{2-z})^2) \tag{2} $$

given in

- I.S. Gradshteyn and I.M. Ryzhik in Table of integrals series and products [2] named 9.134,
- the handbook "mathematical functions with formulas, graphs and mathematical tables" done by Abramowitz-Stegun [3] named 15.3.20,
- in the book "special functions" done by G. Andrews, R. Askey and R. Roy [4] named 3.1.7 page 127 with a slight modification

is true provided that $\{2\alpha+1,\ \alpha+\frac{3}{2}\}$ are not natural numbers and $\alpha-\beta$ is not an integer (see [2], 9.130, [5]). For generalized hypergeometric function see [6], page 312, (6.1).
Some people considered one of the cases where one of these conditions is not fulfilled for instance [7] where authors found an interesting result connected with the sum of $3F2$ $((16-17)$ page 78).
In this manuscript we consider a case where $\alpha-\beta$ is an integer by taking $\beta=2a$, $\alpha=-n+1$. Replacing $\dfrac{z}{2-z}$ by z and we prove that for any positive integer a the above identity (2) becomes

$$ {}_2F_1(a,a+\frac{1}{2};-n+\frac{3}{2};z^2) = \frac{1}{(1\pm z)^{2a}}\, {}_2F_1(2a,-n+1;-2n+2;\frac{\pm 2z}{1\pm z}). \tag{3} $$

and we prove that this identity (3) remains true for $n=0$ but for $n=1$ the above identity becomes

$$ {}_2F_1(a,a+\frac{1}{2};\frac{1}{2};z^2) = \frac{1}{(1\pm z)^{2a}} \mp \frac{2\sqrt{\pi}\Gamma(1+a)z^{2a-1}\, {}_2F_1(1-a,\frac{1}{2}-a;\frac{3}{2};\frac{1}{z^2})}{\Gamma(a)\Gamma(\frac{1}{2})(z^2-1)^{2a}}, $$

and should be written, for $n\geq 2$, as

$$ {}_2F_1(a,a+\frac{1}{2};-n+\frac{3}{2};z^2) = \frac{1}{(1\pm z)^{2a}}\, {}_2F_1(2a,-n+1;-2n+2;\frac{\pm 2z}{1\pm z}) \pm \frac{2\sqrt{\pi}\Gamma(n+a)z^{2n+2a-3}\, {}_2F_1(1-a,\frac{3}{2}-n-a;\frac{3}{2};\frac{1}{z^2})}{\Gamma(a)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-1}}. $$

Let us prove first that (3) is not true for any positive integer a and for any integer $n\geq 1$. The left hand side (LHS) of the identity (3) is well defined and is a series with infinitely many terms, whereas, in the right hand side RHS ${}_2F_1(2a,-n+1;-2n+2;\pm\frac{2z}{1\pm z})$ rises two **situations**:

- either the series

$$ {}_2F_1(2a,-n+1;-2n+2;\pm\frac{2z}{1\pm z}) $$

  is well-defined as it is a series with finitely many terms since the summation is only for $k=0,..,n-1$, and the fact that $-2n+2$ is also a negative integer does not do any harm,
- or

$$ {}_2F_1(2a,-n+1;-2n+2;\pm\frac{2z}{1\pm z}) $$

  is also a series with infinitely many terms by taking the limit as u tends to zero of

$$ {}_2F_1(2a,-n+1-u;-2n+2-u;\pm\frac{2z}{1\pm z}). $$

In this contribution we begin to prove that for $a = 1$ and $\alpha = -n + \frac{3}{2}$, $n \geq 2$ the identity (3) is not true in both situations then we prove that the identity (3) is not true for any positive integer a and $\alpha = -n + \frac{3}{2}$, $n \geq 1$ (n integer) and taking into account

$$u_n^{(a)} := \frac{2\sqrt{\pi}\Gamma(n+a)z^{2n+2a-3}\,{}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-1}} \tag{4}$$

the identity (3) should be written as

$${}_2F_1(a, a+\frac{1}{2}; -n+\frac{3}{2}; z^2) = \frac{{}_2F_1(2a, -n+1; -2n+2; \frac{\pm 2z}{1\pm z})}{(1\pm z)^{2a}} \pm u_n^{(a)},\ n \geq 2, \tag{5}$$

with, for $n = 0$ and for any positive integer a we have

$${}_2F_1(a, a+\frac{1}{2}; \frac{3}{2}; z^2) = \frac{1}{(1\pm z)^{2a}}\,{}_2F_1(2a, 1; 2; \frac{\pm 2z}{1\pm z}),$$

and for $n = 1$ and for any positive integer a we have

$${}_2F_1(a, a+\frac{1}{2}; \frac{1}{2}; z^2) = \frac{1}{(1\pm z)^{2a}} \mp u_1^{(a)},$$

please note the difference between $\pm$ and $\mp$.

Remark 1. *Throughout this manuscript we use the notation* (4).

2. The Case $a = 1$

Taking into account the value $a = 1$, the LHS of (3) becomes

$${}_2F_1(1, \frac{3}{2}; -n+\frac{3}{2}; z^2)$$

and the RHS of (3) as

$$\frac{1}{(1\pm z)^2}\,{}_2F_1(2, -n+1; -2n+2; \frac{\pm 2z}{1\pm z}).$$

Theorem 1. *For $a = 1$, the identity* (3) *remains true for $n = 0$ and should be written as*

$${}_2F_1(1, \frac{3}{2}; -n+\frac{3}{2}; z^2) = \frac{{}_2F_1(2, -n+1; -2n+2; \frac{\pm 2z}{1\pm z})}{(1\pm z)^2} \pm u_n^{(1)},\ n \geq 2 \tag{6}$$

where $u_n^{(1)} = \dfrac{n(2z)^{2n-1}}{\binom{2n-2}{n-1}(z^2-1)^{n+1}}$ *and should be written as*

$${}_2F_1(1, \frac{3}{2}; \frac{1}{2}; z^2) = \frac{1}{(1\pm z)^2} \mp u_1^{(1)} \text{ for } n = 1.$$

Proof. Let us consider the two term recurrence relation

$$(z^2-1)a_n = \frac{2n}{2n-3}z^2 a_{n-1},\ n \geq 3. \tag{7}$$

The sequence $u_n^{(1)}$ fulfils (7). Moreover, The sequence $u_n^{(1)}$ fulfils (7) with $n \geq 1$. In fact

$$(z^2-1)u_n^{(1)} = (z^2-1)\frac{n(2z)^{2n-1}}{\binom{2n-2}{n-1}(z^2-1)^{n+1}} = \frac{n(2z)^2(2z)^{2n-3}}{\frac{(2n-2)(2n-3)}{(n-1)^2}\binom{2n-4}{n-2}(z^2-1)^n} = \frac{2nz^2}{2n-3}u_{n-1}^{(1)},\ n \geq 3.$$

Now, we prove that the rational functions ${}_2F_1(1, \frac{3}{2}; -n + \frac{3}{2}; z^2)$ and $\dfrac{{}_2F_1(2, -n+1; -2n+2; \frac{\pm 2z}{1 \pm z})}{(1 \pm z)^2}$ fulfil the following two term recurrence relation (with a slight modification of (7))

$$(z^2-1)a_n = \frac{2n}{2n-3} z^2 a_{n-1} - 1,\ n \geq 3. \tag{8}$$

- The rational function ${}_2F_1(1, \frac{3}{2}; -n + \frac{3}{2}; z^2)$ fulfils (8). Moreover, this rational function ${}_2F_1(1, \frac{3}{2}; -n + \frac{3}{2}; z^2)$ fulfils (8) with $n \geq 1$, in fact

$$\begin{aligned}
&(z^2-1)\,{}_2F_1(1, \frac{3}{2}; -n + \frac{3}{2}; z^2) - \frac{2n}{2n-3} z^2\,{}_2F_1(1, \frac{3}{2}; -(n-1) + \frac{3}{2}; z^2) \\
&= (z^2-1) \sum_{k \geq 0} \frac{(\frac{3}{2})_k}{(-n+\frac{3}{2})_k} z^{2k} - \frac{2n}{2n-3} z^2 \sum_{k \geq 0} \frac{(\frac{3}{2})_k}{(-n+\frac{5}{2})_k} z^{2k} \\
&= \sum_{k \geq 0} \left(\frac{(\frac{3}{2})_k}{(-n+\frac{3}{2})_k} - \frac{2n(\frac{3}{2})_k}{(2n-3)(-n+\frac{5}{2})_k} \right) z^{2k+2} - \sum_{k \geq 0} \frac{(\frac{3}{2})_k}{(-n+\frac{3}{2})_k} z^{2k} \\
&= \sum_{k \geq 0} \left(\frac{(\frac{3}{2})_k}{(-n+\frac{3}{2})_k} - \frac{2n(\frac{3}{2})_k}{(2n-3)(-n+\frac{5}{2})_k} - \frac{(\frac{3}{2})_{k+1}}{(-n+\frac{3}{2})_{k+1}} \right) z^{2k+2} - 1 = -1,
\end{aligned}$$

in fact, using $(z+1)_n = \dfrac{(z+n)}{z}(z)_n$ and $(z)_{n+1} = (z+n)(z)_n$, we get

$$\begin{aligned}
&\frac{(\frac{3}{2})_k}{(-n+\frac{3}{2})_k} - \frac{2n(\frac{3}{2})_k}{(2n-3)(-n+\frac{5}{2})_k} - \frac{(\frac{3}{2})_{k+1}}{(-n+\frac{3}{2})_{k+1}} \\
&= \frac{(\frac{3}{2})_k}{(-n+\frac{3}{2})_k} - \frac{2n(\frac{3}{2})_k(-n+\frac{3}{2})}{(2n-3)(-n+\frac{3}{2}+k)(-n+\frac{3}{2})_k} - \frac{(\frac{3}{2}+k)(\frac{3}{2})_k}{(-n+\frac{3}{2}+k)(-n+\frac{3}{2})_k} = 0.
\end{aligned}$$

- Now, we prove that the rational function

$$\frac{{}_2F_1(2, -n+1; -2n+2; \frac{\pm 2z}{1 \pm z})}{(1 \pm z)^2}$$

fulfils (8). Note here that this proof is only true for $n \geq 3$. We begin by a change of variable, for the $+$ sign, we assume

$$\frac{1}{y} = \frac{2z}{z+1}, \tag{9}$$

whereas for the $-$ sign, we assume

$$\frac{1}{y} = \frac{-2z}{-z+1}.$$

For the $+$ sign, proving that the rational function

$$\frac{{}_2F_1(2, -n+1; -2n+2; \frac{2z}{1+z})}{(1+z)^2}$$

fulfils (8) proves

$$-\frac{y-1}{y}\,{}_2F_1(2, -n+1; -2n+2; \frac{1}{y}) - \frac{n}{2(2n-3)y^2}\,{}_2F_1(2, -n+2; -2n+4; \frac{1}{y}) = -1. \tag{10}$$

Let us expand the LHS of (10). Please note here that our hypergeometric is a series with finitely many terms. Some computations lead to

$$
\begin{aligned}
&-\left(1+\frac{2(-n+1)_1}{(-2n+2)_1 y}+\frac{3(-n+1)_2}{(-2n+2)_2 y^2}+\ldots+\frac{n(-n+1)_{n-1}}{(-2n+2)_{n-1}y^{n-1}}\right)\\
&+\left(\frac{1}{y}+\frac{2(-n+1)_1}{(-2n+2)_1 y^2}+\frac{3(-n+1)_2}{(-2n+2)_2 y^3}+\ldots+\frac{n(-n+1)_{n-1}}{(-2n+2)_{n-1}y^{n}}\right)\\
&-\frac{n}{2(2n-3)}\left(\frac{1}{y^2}+\frac{2(-n+2)_1}{(-2n+4)_1 y^3}+\frac{3(-n+2)_2}{(-2n+4)_2 y^4}+\ldots+\frac{(n-1)(-n+2)_{n-2}}{(-2n+4)_{n-2}y^{n}}\right).
\end{aligned}
$$

Thus, using

$$z(z+1)_{n-1}=(z)_n,\ (z)_{n+1}=(z+n)(z)_n,$$

the coefficients of $\frac{1}{y^{k+2}}$, $0\le k\le n-2$, are

$$
\begin{aligned}
&-\frac{n}{2(2n-3)}\frac{(k+1)(-n+2)_k}{(-2n+4)_k}-\frac{(k+3)(-n+1)_{k+2}}{(-2n+2)_{k+2}}+\frac{(k+2)(-n+1)_{k+1}}{(-2n+2)_{k+1}}\\
&=-\frac{n(k+1)(-2n+2)(-2n+3)(-n+1)_{k+1}}{2(-n+1)(2n-3)(-2n+k+3)(-2n+2)_{k+1}}-\frac{(k+3)(-n+k+2)(-n+1)_{k+1}}{(-2n+k+3)(-2n+2)_{k+1}}\\
&+\frac{(k+2)(-n+1)_{k+1}}{(-2n+2)_{k+1}}=\frac{(-n+1)_{k+1}}{(-2n+2)_{k+1}}\left(\frac{n(k+1)}{(-2n+k+3)}-\frac{(k+3)(-n+k+2)}{(-2n+k+3)}+(k+2)\right)
\end{aligned}
$$

which is identically zero. The remaining terms are

$$-1-\frac{2(-n+1)_1}{(-2n+2)_2 y}+\frac{1}{y}=-1$$

and

$$\frac{n(-n+1)_{n-1}}{(-2n+2)_{n-1}y^n}-\frac{n}{2(2n-3)}\frac{(n-1)(-n+2)_{n-2}}{(-2n+4)_{n-2}y^n}=0.$$

The same steps should be followed for the $-$ sign.

To finish the proof of the Theorem, it should be pointed out that, when the rational functions ${}_2F_1(1,\frac{3}{2};-n+\frac{3}{2};z^2)$ and $\frac{{}_2F_1(2,-n+1;-2n+2;\frac{\pm 2z}{1\pm z})}{(1\pm z)^2}$ fulfil (8), then subtracting the two quantities, we get

$${}_2F_1(1,\frac{3}{2};-n+\frac{3}{2};z^2)-\frac{{}_2F_1(2,-n+1;-2n+2;\frac{\pm 2z}{1\pm z})}{(1\pm z)^2}$$

fulfils (7) (the (-1) cancels) and

$$
\begin{aligned}
{}_2F_1(1,\frac{3}{2};-2+\frac{3}{2};z^2)&=\frac{3z^4+6z^2-1}{(z^2-1)^3},\\
\frac{{}_2F_1(2,-2+1;-4+2;\frac{2z}{1+z})}{(1+z)^2}&=\frac{3z+1}{(z+1)^3},\\
u_2^{(1)}=\frac{8z^3}{(z^2-1)^3}\ \textit{with}\ &\frac{3z^4+6z^2-1}{(z^2-1)^3}-\frac{3z+1}{(z+1)^3}-\frac{8z^3}{(z^2-1)^3}=0.
\end{aligned}
$$

With this achievement, we have proved that in **situation** one, where we considered the series

$${}_2F_1(2a,-n+1;-2n+2;\pm\frac{2z}{1\pm z})$$

is well-defined as it is a series with finitely many terms since the summation is only for $k = 0, \dots, n-1$, the fact that $-2n+2$ is also a negative integer does not do any harm as the identity (3) is not true. $\square$

2.1. The Case $a = 1$ and $n = 2$ for the Second Situation

The situation where

$$ {}_2F_1(2a, -n+1; -2n+2; \pm\frac{2\sqrt{z}}{1 \pm \sqrt{z}}) $$

is a series with infinitely many terms by taking the limit as u tends to zero of

$$ {}_2F_1(2a, -n+1-u; -2n+2-u; \pm\frac{2\sqrt{z}}{1 \pm \sqrt{z}}) $$

is, also wrong. In fact, with the $+$ sign (same steps for the $-$ sign), the identity (3) is false for $a = 1$ and $n = 2$. In fact

$$ {}_2F_1(1, 1+\frac{1}{2}; -\frac{1}{2}; z^2) = \frac{(3z^4 + 6z^2 - 1)}{(1+z)^3(z-1)^3}. $$

On the other side, we have

$$ (1+z)^{-2}\, {}_2F_1(2, -1-u; -2-u; \frac{2z}{1+z}) = \frac{u(z-1) + 2(3z-1)}{(2+u)(z-1)^3}. $$

In fact,

$$
\begin{aligned}
&(1+z)^{-2}\, {}_2F_1(2, -1-u; -2-u; \frac{2z}{1+z}) = (1+z)^{-2} \sum_{k\geq 0} \frac{(2)_k(-1-u)_k}{(-2-u)_k k!} (\frac{2z}{1+z})^k \\
&= (1+z)^{-2} \sum_{k\geq 0} \frac{(k+1)(-1-u)_k}{(-2-u)_k} (\frac{2z}{1+z})^k = (1+z)^{-2} \Bigg(1 + \frac{2(-1-u)}{(-2-u)} (\frac{2z}{1+z})^1 \\
&+ \frac{3(-1-u)(-u)}{(-2-u)(-1-u)} (\frac{2z}{1+z})^2 + \frac{4(-1-u)(-u)(-u+1)}{(-2-u)(-1-u)(-u)} (\frac{2z}{1+z})^3 \\
&+ \frac{5(-1-u)(-u)(-u+1)(-u+2)}{(-2-u)(-1-u)(-u)(-u+1)} (\frac{2z}{1+z})^4 \\
&+ \frac{6(-1-u)(-u)(-u+1)(-u+2)(-u+3)}{(-2-u)(-1-u)(-u)(-u+1)(-u+2)} (\frac{2z}{1+z})^5 + \dots\dots \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= (1+z)^{-2} \Bigg(1 + \frac{2(-1-u)}{(-2-u)} (\frac{2z}{1+z})^1 + \frac{3(-u)}{(-2-u)} (\frac{2z}{1+z})^2 + \frac{4(-u+1)}{(-2-u)} (\frac{2z}{1+z})^3 \\
&+ \frac{5(-u+2)}{(-2-u)} (\frac{2z}{1+z})^4 + \frac{6(-u+3)}{(-2-u)} (\frac{2z}{1+z})^5 + \dots\dots \Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(1+z)^2(-2-u)} \Bigg((-2-u) + 2(-1-u)(\frac{2z}{1+z})^1 + 3(-u)(\frac{2z}{1+z})^2 + 4(-u+1)(\frac{2z}{1+z})^3 \\
&+ 5(-u+2)(\frac{2z}{1+z})^4 + 6(-u+3)(\frac{2z}{1+z})^5 + \dots\dots \Bigg)
\end{aligned}
$$

$$ = \frac{1}{(1+z)^2(-2-u)} \sum_{k\geq 0} (k+1)(-2-u+k)(\frac{2z}{1+z})^k. \tag{11} $$

Using (for $\mid z \mid < 1$)

$$\sum_{k\geq 0} z^k = -\frac{1}{z-1}, \sum_{k\geq 0} kz^k = \frac{z}{(z-1)^2}, \sum_{k\geq 0} k^2 z^k = -\frac{z(z+1)}{(z-1)^3}$$

we have the desired result.

When u goes to 0, $\frac{u(z-1)+2(3z-1)}{(2+u)(z-1)^3}$ goes to $\frac{2(3z-1)}{2(z-1)^3}$ which is not equal to $\frac{(3z^4+6z^2-1)}{(1+z)^3(z-1)^3}$.

2.2. Appendix

Here is a Maple instruction for the case $a = 1$-theorem:

$>$ restart;
$> \ vn := n \to hypergeom([3/2, 1], [-n+3/2], x^2);$
$> \ wn := n \to 1/(1+x)^2 * hypergeom([2, -n+1], [-2*n+2], 2*x/(1+x));$
$> \ un := n \to n*(2*x)^{2*n-1}/(binomial(2*n-2, n-1)*(x^2-1)^{n+1});$
$>$ factor(simplify(vn(0)-wn(0)));
$>$ factor(simplify(vn(1)-wn(1)+un(1)));
$>$ factor(simplify(vn(2)-wn(2)-un(2)));
$>$ factor(simplify(vn(3)-wn(3)-un(3)));
$>$ factor(simplify(vn(4)-wn(4)-un(4)));

0
0
0
0
0

3. Resolution of an Isolated Case of the Identity for $a \in \mathbb{N}$

In the sequel, taking into account $a \in \mathbb{N}$, we write the LHS of (3) as

$${}_2F_1(a, a+\frac{1}{2}; -n+\frac{3}{2}; z^2)$$

and the RHS of (3) as

$$\frac{1}{(1\pm z)^2}\ {}_2F_1(2a, -n+1; -2n+2; \frac{\pm 2z}{1\pm z}).$$

Theorem 2. *For $a \in \mathbb{N}$ the identity (3) remains true for $n = 0$ and should be written as*

$${}_2F_1(a, a+\frac{1}{2}; -n+\frac{3}{2}; z^2) = \frac{{}_2F_1(2a, -n+1; -2n+2; \frac{\pm 2z}{1\pm z})}{(1\pm z)^{2a}} \pm u_n^{(a)},\ n \geq 2,$$

and should be written as

$${}_2F_1(a, a+\frac{1}{2}; \frac{1}{2}; z^2) = \frac{1}{(1\pm z)^{2a}} \mp u_1^{(a)} for\ n = 1.$$

Please note the difference between $\pm$ and $\mp$.

Proof. The following proof does not include the case where $a \in \mathbb{R}\backslash\mathbb{N}$ and this case remains an open problem. For a, any positive integer, we consider the following relation

$$(z^2-1)f(n,a) - \frac{2n+4a-4}{2n-3} z^2 f(n-1,a) = f(n,a-1),\ n \geq 2. \tag{12}$$

We prove that ${}_2F_1(a, a+\frac{1}{2}; -n+\frac{3}{2}; z^2)$, $\frac{{}_2F_1(2a,-n+1;-2n+2;\frac{\pm 2z}{1\pm z})}{(1\pm z)^{2a}}$ and $u_n^{(a)}$ fulfil this relation (12).

- Let us begin by proving that $u_n^{(a)}$ fulfil this relation (12).
  In fact, for $n \geq 2$, we have

$$
\begin{aligned}
&(z^2-1)u_n^{(a)} - \frac{2n+4a-4}{2n-3} z^2 u_{n-1}^{(a)} + u_n^{(a-1)} \\
&= (z^2-1)\frac{2\sqrt{\pi}\Gamma(n+a)z^{2n+2a-3}\, {}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-1}} \\
&-\frac{2n+4a-4}{2n-3} z^2 \frac{2\sqrt{\pi}\Gamma(n-1+a)z^{2n+2a-5}\, {}_2F_1(1-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a)\Gamma(n-\frac{3}{2})(z^2-1)^{n+2a-2}} \\
&+\frac{2\sqrt{\pi}\Gamma(n+a-1)z^{2n+2a-5}\, {}_2F_1(1-a+1, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a-1)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-3}},
\end{aligned}
$$

which becomes

$$
\begin{aligned}
&\frac{2\sqrt{\pi}\Gamma(n+a)z^{2n+2a-3}\, {}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-2}} \\
&-\frac{(2n+4a-4)}{(2n-3)} \frac{2\sqrt{\pi}\Gamma(n-1+a)z^{2n+2a-3}\, {}_2F_1(1-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a)\Gamma(n-\frac{3}{2})(z^2-1)^{n+2a-2}} \\
&+\frac{2\sqrt{\pi}\Gamma(n+a-1)z^{2n+2a-5}\, {}_2F_1(1-a+1, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2})}{\Gamma(a-1)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-3}},
\end{aligned}
$$

which becomes

$$
\begin{aligned}
&\frac{2\sqrt{\pi}\Gamma(n+a-1)z^{2n+2a-5}}{\Gamma(a-1)\Gamma(n-\frac{1}{2})(z^2-1)^{n+2a-3}} \Bigg(\frac{(n+a-1)z^2}{(a-1)(z^2-1)}\, {}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2}) \\
&-(n+2a-2)\frac{z^2}{(a-1)(z^2-1)}\, {}_2F_1(1-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) + {}_2F_1(2-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) \Bigg),
\end{aligned}
$$

to prove that this expression vanishes, it is equivalent to prove that

$$
\begin{aligned}
&\frac{(n+a-1)z^2}{(a-1)(z^2-1)}\, {}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2}) - \frac{(n+2a-2)z^2}{(a-1)(z^2-1)}\, {}_2F_1(1-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) \\
&+ {}_2F_1(2-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}),
\end{aligned}
$$

also vanishes. It is equivalent to prove that

$$
\begin{aligned}
&(n+a-1)z^2\, {}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2}) - (n+2a-2)z^2\, {}_2F_1(1-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) \\
&+(a-1)(z^2-1)\, {}_2F_1(2-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) = 0.
\end{aligned}
$$

Let us write this last expression as

$$
\begin{aligned}
&(n+a-1)z^2\, {}_2F_1(1-a, -n-a+\frac{3}{2}; \frac{3}{2}; \frac{1}{z^2}) - (n+2a-2)z^2\, {}_2F_1(1-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) \\
&+(a-1)z^2\, {}_2F_1(2-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}) - (a-1)\, {}_2F_1(2-a, -n-a+\frac{5}{2}; \frac{3}{2}; \frac{1}{z^2}).
\end{aligned}
$$

which becomes

$$\sum_{k=1}^{a-1} \frac{(1-a)_k(-n-a+\frac{5}{2})_k}{(\frac{3}{2})_k} \frac{1}{k!(z^2)^{k-1}} \left(\frac{(n+a-1)(-n-a+\frac{3}{2})}{(-n-a+k+\frac{3}{2})} - (n+2a-2) + (a-1)\frac{(1-a+k)}{(1-a)} \right) - (a-1)\,{}_2F_1(2-a,-n-a+\frac{5}{2};\frac{3}{2};\frac{1}{z^2}).$$

which becomes

$$\sum_{k=1}^{a-1} \frac{(1-a)_k(-n-a+\frac{5}{2})_k}{(\frac{3}{2})_k} \frac{1}{k!(z^2)^{k-1}} \left(\frac{k(2k+1)}{(2n+2a-2k-3)} \right) - (a-1)\,{}_2F_1(2-a,-n-a+\frac{5}{2};\frac{3}{2};\frac{1}{z^2}),$$

which is

$$\sum_{k=1}^{a-1} \frac{(1-a)_k(-n-a+\frac{5}{2})_k}{(\frac{3}{2})_k} \frac{1}{(k-1)!(z^2)^{k-1}} \left(\frac{(\frac{3}{2}+k-1)}{(n+a-k-\frac{3}{2})} \right) - (a-1)\,{}_2F_1(2-a,-n-a+\frac{5}{2};\frac{3}{2};\frac{1}{z^2}),$$

which is exactly zero.

- Second, let us prove that ${}_2F_1(a,a+\frac{1}{2};-n+\frac{3}{2};z^2)$ also fulfil the relation (12). In fact, for $n \geq 2$, let us prove that

$$(z^2-1)\,{}_2F_1(a,a+\frac{1}{2};-n+\frac{3}{2};z^2) - \frac{2n+4a-4}{2n-3}z^2\,{}_2F_1(a,a+\frac{1}{2};-n+\frac{5}{2};z^2) + {}_2F_1(a-1,a-\frac{1}{2};-n+\frac{3}{2};z^2) = 0,$$

equivalently, we prove that

$$(z^2-1)\sum_{k\geq 0} \frac{(a)_k(a+\frac{1}{2})_k}{(-n+\frac{3}{2})_k}\frac{x^{2k}}{k!} + \sum_{k\geq 0} \frac{(a-1)_k(a-\frac{1}{2})_k}{(-n+\frac{3}{2})_k}\frac{x^{2k}}{k!} = \frac{2n+4a-4}{2n-3}z^2 \sum_{k\geq 0} \frac{(a)_k(a+\frac{1}{2})_k}{(-n+\frac{5}{2})_k}\frac{x^{2k}}{k!}.$$

The left-hand side becomes

$$z^2 \sum_{k\geq 0} \frac{(a)_k(a+\frac{1}{2})_k}{(-n+\frac{3}{2})(-n+\frac{5}{2})_k} \left((-n+k+\frac{3}{2}) + \frac{(a-1)(a-\frac{1}{2})}{(k+1)} - \frac{(a+k)(a+k+\frac{1}{2})}{(k+1)} \right) \frac{x^{2k}}{k!}$$

which is exactly

$$z^2 \sum_{k\geq 0} \frac{(a)_k(a+\frac{1}{2})_k}{(-n+\frac{3}{2})(-n+\frac{5}{2})_k} \left(-n-2a+2 \right) \frac{x^{2k}}{k!}$$

which is exactly the right-hand side.

- Third, let us prove that the rational function

$$\frac{{}_2F_1(2a,-n+1;-2n+2;\frac{\pm 2z}{1\pm z})}{(1\pm z)^2}$$

fulfils (12). Using the same change of variable (9), and taking into account the $+$ sign, we claim that: proving that the rational function

$$\frac{{}_2F_1(2,-n+1;-2n+2;\frac{2z}{1+z})}{(1+z)^2}$$

fulfils (12) is equivalent to prove that

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}} \left((y-1)\,{}_2F_1(2a,-n+1;-2n+2;\frac{1}{y}) - y\,{}_2F_1(2a-2,-n+1;-2n+2;\frac{1}{y}) \right) \quad (13)$$

should be equal to

$$\frac{(2n+4a-4)(2y-1)^{2a-2}}{(2n-3)(2y)^{2a}}\ {}_2F_1(2a,-n+1;-2n+2;\frac{1}{y}). \tag{14}$$

The expression (13) becomes

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(-1+(y-1)\sum_{k=1}^{n-1}\frac{(2a)_k(-n+1)_k}{(-2n+2)_k}\frac{1}{k!y^k}-y\sum_{k=1}^{n-1}\frac{(2a-2)_k(-n+1)_k}{(-2n+2)_k}\frac{1}{k!y^k}\right)$$

this expression becomes

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\sum_{k=1}^{n-1}\left\{\frac{(2a)_k(-n+1)_k}{(-2n+2)_k}\frac{1}{(k-1)!y^{k-1}}(\frac{1}{k}-\frac{(2a-2)(2a-1)}{k(2a+k-2)(2a+k-1)})\right.\right.$$
$$\left.\left.-\frac{(2a)_{k-1}(-n+1)_{k-1}}{(-2n+2)_{k-1}}\frac{1}{(k-1)!y^{k-1}}\right\}-\frac{(2a)_{n-1}(-n+1)_{n-1}}{(-2n+2)_{n-1}}\frac{1}{(n-1)!y^{n-1}}\right)$$

some simplifications lead to

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\sum_{k=1}^{n-1}\frac{(2a)_k(-n+1)_k}{(-2n+2)_k}\frac{1}{(k-1)!y^{k-1}}(\frac{1}{k}-\frac{(2a-2)(2a-1)}{k(2a+k-2)(2a+k-1)}\right.$$
$$\left.-\frac{k(-2n+1+k)}{(2a+k-1)(-n+k)})-\frac{(2a)_{n-1}(-n+1)_{n-1}}{(-2n+2)_{n-1}}\frac{1}{(n-1)!y^{n-1}}\right)$$

which becomes

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\sum_{k=1}^{n-1}\frac{(2a)_k(-n+1)_k}{(-2n+2)_k}\frac{1}{k!y^{k-1}}\frac{k(k-1)(2a+n-2)}{(2a+k-1)(2a+k-2)(-n+k)}\right.$$
$$\left.-\frac{(2a)_{n-1}(-n+1)_{n-1}}{(-2n+2)_{n-1}}\frac{1}{(n-1)!y^{n-1}}\right)$$

this is equal to

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\frac{(2a+n-2)}{(-2n+3)}\sum_{k=1}^{n-2}\frac{(2a)_{k-1}(-n+1)_{k+1}}{(-2n+2)_{k+1}}\frac{1}{(k-1)!y^k}\frac{(-2n+3)}{(-n+k+1)}\right.$$
$$\left.-\frac{(2a)_{n-1}(-n+1)_{n-1}}{(-2n+2)_{n-1}}\frac{1}{(n-1)!y^{n-1}}\right)$$

this expression becomes

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\frac{(2a+n-2)}{(-2n+3)}\sum_{k=1}^{n-2}\frac{(2a)_{k-1}(-n+1)_k}{(-2n+2)_{k+1}}\frac{(-2n+3)}{(k-1)!y^k}\right.$$
$$\left.-\frac{(2a)_{n-1}(-n+1)_{n-1}}{(-2n+2)_{n-1}}\frac{1}{(n-1)!y^{n-1}}\right)$$

this gives

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\frac{(2a+n-2)}{(-2n+3)}\sum_{k=1}^{n-1}\frac{(2a)_{k-1}(-n+1)_k}{(-2n+2)(-2n+3)(-2n+4)_{k-1}}\frac{(-2n+3)}{(k-1)!y^k}\right)$$

finally it becomes

$$-\frac{(2y-1)^{2a-2}}{2^{2a-2}y^{2a-1}}\left(\frac{(2a+n-2)}{(-2n+3)}\sum_{k=1}^{n-1}\frac{(2a)_{k-1}(-n+1)_k}{(-2n+2)(-2n+4)_{k-1}}\frac{1}{(k-1)!y^k}\right)$$

which is exactly (14).

To conclude, we can easily see that $u_n^{(1)}$, $n \geq 2$ given by section 2 and $u_2^{(a)}$, $n \geq 2$ generate $u_n^{(a)}$, $n \geq 2$ and a positive integer. The same conclusion as for hypergeometric sums. □

Appendix

Here is a Maple instruction for the theorem:

$>$ restart;

$>$ hyper1 := (n, a) $\rightarrow hypergeom([a+1/2, a], [-n+3/2], x^2)$;

$>$ hyper2 := (n, a) $\rightarrow hypergeom([2*a, -n+1], [-2*n+2], 2*x/(1+x))/(1+x)^{2*a}$;

$>$ Una := (n, a) $\rightarrow 2*x^{2*a-3+2*n}*sqrt(Pi)*GAMMA(n+a)*hypergeom([1-a, 3/2-n-a], [3/2], 1/x^2)*(x^2-1)^{-n+1-2*a}/(GAMMA(n-1/2)*GAMMA(a))$;

$>$ simplify(hyper1(5, 3)-hyper2(5, 3)-Una(5, 3));

$>$ simplify(hyper1(1, 3)-hyper2(1, 3)+Una(1, 3));

$>$ simplify(hyper1(0, 3)-hyper2(0, 3));

$$0$$
$$0$$
$$0$$

4. Open Problem

For $a - \frac{1}{2} \in \mathbb{N}$ write the analogue of (5). We are working on this.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Acknowledgments: The author would like to thank the three referees for their valuable comments which helped to improve the manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Mandle, A.K.; Namdeo, V. Application of Gauss's hypergeometric functions to protect bank lockers. *Compliance Eng. J.* **2019**, *10*.
2. Gradshteyn, I.S.; Ryzhik, I.M. *Tables of Integrals, Series and Products*, 6th ed.; Academic Press: San Diego, CA, USA, 2000.
3. Abramowitz, M.; Stegun, I.A. (Eds.) *Handbook of Mathematical Functions*; Dover: New York, NY, USA, 1965.
4. Andrews, G.; Askey, R.; Roy, R. *Special Functions, Encyclopedia of Mathematics and Its Applications 71*; Cambridge University Press: Cambridge, UK, 1999.
5. DLMF: NIST Digital Library of Mathematical Functions. Available online: https://dlmf.nist.gov/ (accessed on 1 January 2020).
6. Miller, A.R.; Paris, R.B. Transformation formulas for the generalized hypergeometric function with integral parameter difference. *Rocky Mt. J. Math.* **2013**, *43*, 291–327. [CrossRef]
7. Qureshi, M.I.; Khan, M.K. Some Quadratic Transformations and Reduction Formulas associated with Hypergeometric Functions. *Appl. Appl. Math.* **2020**, *15*, 71–86.