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Some New Refinements of Trapezium-Type Integral Inequalities in Connection with Generalized Fractional Integrals

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Abstract: The main objective of this article is to introduce a new notion of convexity, i.e., modified exponential type convex function, and establish related fractional inequalities. To strengthen the argument of the paper, we introduce two new lemmas as auxiliary results and discuss some algebraic properties of the proposed notion. Considering a generalized fractional integral operator and differentiable mappings, whose initial absolute derivative at a given power is a modified exponential type convex, various improvements of the Hermite–Hadamard inequality are presented. Thanks to the main results, some generalizations about the earlier findings in the literature are recovered.

Keywords: convex function; Hölder's inequality; power-mean integral inequality; m–type convexity; exponential convex function

MSC: 26A51; 26A33; 26D07; 26D10; 26D15

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1. Introduction

Convexity theory has had a substantial and crucial influence on the development of numerous disciplines such as economics [1], financial mathematics [2], engineering [3], and optimization [4] in modern mathematics. This theory gives a fantastic framework for initiating and developing numerical tools for tackling and studying complex mathematical problems.

In the current decade, many mathematicians have been merging new ideas with fractional analysis to bring new dimensions with different features to the field of mathematical analysis. Fractional analysis has many applications in modeling [5,6], epidemiology [7], fluid flow [8], nanotechnology [9], mathematical biology [10], and control systems [11]. It is particularly crucial while studying optimization problems because it has a variety of useful inequalities. This explains why convex functions and convex sets have such a robust theoretical foundation. There are numerous practical uses for convex functions in optimization, circuit design, controller design, modeling, etc. Because it has gained so much attention, the concept of "convexity" has developed into a fertile area of research and inspiration.

The theory of inequalities has been expanded and generalized during the past few decades, and this has been made possible by the concept of convex analysis. Inequalities theory and the theory of convexity are strongly related to one another. Many mathematicians and research scientists have made considerable efforts and contributions to the study of this inequality over the last few decades. Some authors have also studied dynamic inequalities [12–17] to further strengthen the theory of convexity and inequality. As a result, there is a rich and insightful literature on convexity and inequalities; for further information, see the references at [18–21].

Many mathematicians and scientists in a wide range of applied and scientific areas have been fascinated and inspired by fractional calculus. Because of its ability to interpolate between operators of integer order, fractional integrals and derivatives have a rich history and are used frequently in practical situations. Given its wide range of applications in the mathematical modeling of numerous complicated and nonlocal nonlinear systems, fractional calculus has become a crucial topic of research. The nonlocal nature of fractional-order operators, which explains the hereditary characteristics of the underlying phenomena, is an important property of these operators. A macroscopic stress–strain relation expressed in terms of fractional differential operators results from the interactions between macromolecules in damping phenomena. Its appeal in modeling different transport characteristics in complicated heterogeneous and disordered media is largely due to the fact that it offers a suitable context for describing processes with memory and is fractal or multi-fractal in origin.

We organized the study in the following manner in light of the aforementioned findings and literature on inequality theory: We review some well-known concepts and definitions in Section 2. We describe the idea and algebraic characteristics of modified exponential type convex functions in Section 3. The H–H inequality, whose first derivatives in absolute value at a given power is of the modified exponential type convex, and additional extensions of it are developed in Section 4. Finally, we provide a brief conclusion in Section 5.

2. Preliminaries

Because there are so many theorems and definitions in the preliminary section, it will be advisable to examine and investigate it for the sake of thoroughness. We will review a few well-known terms, definitions, and findings in this section that we will be required for our inquiry in subsequent sections. Convex functions, Hermite–Hadamard type inequality, m-convex functions, and exponential type convex functions are introduced first. We recall here the Riemann–Liouville fractional integral operator, its k-generalization, and certain crucial functions, such as the incomplete gamma function and gamma function, which will be needed in our investigations.

Definition 1 ([22])**.** *If* $\mathsf{G} : \mathbb{X} \subset \mathbb{R} \to \mathbb{R}$, *then an inequality of the form*

$$\mathsf{G}(g_1\varrho + (1-\varrho)g_2) \leq \varrho\mathsf{G}(g_1) + (1-\varrho)\mathsf{G}(g_2), \tag{1}$$

is said to be convex if for all $g_1, g_2 \in \mathbb{X}$ *and* $\varrho \in [0,1]$.

The well-known Hermite–Hadamard inequality must be mentioned in any paper on Hermite inequalities. This inequality claims that, if $\mathsf{G} : \mathbb{X} \subset \mathbb{R} \to \mathbb{R}$ is convex in $\mathbb{X}$ for $g_1, g_2 \in \mathbb{X}$ and $g_1 < g_2$, then

$$\mathsf{G}\left(\frac{g_1+g_2}{2}\right) \leq \frac{1}{g_2-g_1}\int_{g_1}^{g_2}\mathsf{G}(\chi)d\chi \leq \frac{\mathsf{G}(g_1)+\mathsf{G}(g_2)}{2}. \tag{2}$$

Interested readers can refer to [23–26].

In 1985, the famous mathematician G. Toader [27] first considered and examined the new version of convexity, namely the m-convex function.

Definition 2 ([27]). *Let* $\mathsf{G} : [0, b] \to \mathbb{R}$, $b > 0$ *and* $m \in (0, 1]$. *An inequality of the form*

$$\mathsf{G}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) \leq \varrho \mathsf{G}(\mathrm{g}_1) + m(1-\varrho)\mathsf{G}(\mathrm{g}_2), \tag{3}$$

is then said to be m*–convex if* $\forall\ \mathrm{g}_1, \mathrm{g}_2 \in [0, b]$ *and* $\varrho \in [0, 1]$. *Otherwise,* G *is* m*–concave if* $(-\mathsf{G})$ *is* m*–convex.*

Definition 3 ([28]). *Let* G *be a nonnegative function.* $\mathsf{G} : \mathbb{X} \to \mathbb{R}$, *is then said to be a exponential type convex if*

$$\mathsf{G}(\varrho \mathrm{g}_1 + (1-\varrho)\mathrm{g}_2) \leq (e^{\varrho} - 1)\mathsf{G}(\mathrm{g}_1) + \left(e^{(1-\varrho)} - 1\right)\mathsf{G}(\mathrm{g}_2) \tag{4}$$

holds $\forall\ \mathrm{g}_1, \mathrm{g}_2 \in \mathbb{X}$, *and* $\varrho \in [0, 1]$.

Definition 4 (Hölder Integral Inequality [29]). *If* G *and* H *be two integrable functions, then the Hölder inequality is given by*

$$\int_0^1 |\mathsf{G}(\nu)\mathsf{H}(\nu)| d\nu \leq \left(\int_0^1 |\mathsf{G}(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |\mathsf{H}(x)|^q dx\right)^{\frac{1}{q}}. \tag{5}$$

Definition 5 (Power-mean integral inequality [30]). *If* G *and* H *be two integrable functions, then power mean inequality is given by*

$$\int_0^1 |\mathsf{G}(\nu)\mathsf{H}(\nu)| d\nu \leq \left(\int_0^1 |\mathsf{G}(x)| dx\right)^{1-\frac{1}{q}} \left(\int_0^1 |\mathsf{G}(x)| dx \int_0^1 |\mathsf{H}(x)|^q dx\right)^{\frac{1}{q}}. \tag{6}$$

The concept of fractional integral inequalities have many applications in applied sciences. Such types of inequalities have always been established and have managed the uniqueness of solutions to some fractional partial differential equations. Additionally, they offer upper and lower bounds for the solutions to the fractional boundary value problems. In order to study specific extensions and generalizations, scholars in the subject of integral inequalities have used fractional calculus operators; for further information, see [31–34].

Let $\mathsf{G} \in L[\mathrm{g}_1, \mathrm{g}_2]$. Riemann–Liouville fractional integrals of order $\alpha > 0$ with $\mathrm{g}_1 \geq 0$ are then defined as follows:

$$J^{\alpha}_{\mathrm{g}_1^+}\mathsf{G}(x) = \frac{1}{\Gamma(\alpha)} \int_{\mathrm{g}_1}^{x} (x-\chi)^{\alpha-1}\, \mathsf{G}(\chi)\, d\chi, \quad x > \mathrm{g}_1$$

and

$$J^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\mathrm{g}_2} (\chi-x)^{\alpha-1}\, \mathsf{G}(\chi)\, d\chi, \quad x < \mathrm{g}_2.$$

For further details, one may see [35–40].

In [41,42], there is a given definition of k—fractional Riemann–Liouville integrals. Let $\mathsf{G} \in L[\mathrm{g}_1, \mathrm{g}_2]$. $k-$fractional integrals of order $\alpha, k > 0$ with $\mathrm{g}_1 \geq 0$ are then defined as follows:

$${}^{k}J^{\alpha}_{\mathrm{g}_1^+}\mathsf{G}(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{\mathrm{g}_1}^{x} (x-\chi)^{\frac{\alpha}{k}-1}\, \mathsf{G}(\chi)\, d\chi \quad x > \mathrm{g}_1,$$

and

$${}^{k}J^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(x) = \frac{1}{k\Gamma_k(\alpha)} \int_{x}^{\mathrm{g}_2} (\chi-x)^{\frac{\alpha}{k}-1}\, \mathsf{G}(\chi)\, d\chi, \quad x < \mathrm{g}_2,$$

where $\Gamma_k(\alpha)$ is the k—Gamma function defined as

$$\Gamma_k(\alpha) = \int_0^{+\infty} \chi^{\alpha-1} e^{-\frac{\chi^k}{k}} d\chi.$$

We can notice that

$$\Gamma_k(\alpha + k) = \alpha \Gamma_k(\alpha)$$

and

$${}^1J^0_{g_1^+}\psi(x) = {}^1J^0_{g_2^-}\psi(x) \quad = \psi(x).$$

By choosing $k = 1$, the above k—fractional integrals yield Riemann–Liouville integrals. The incomplete gamma function $\gamma(\vartheta, \varrho)$ is defined for $\vartheta > 0$ and $\varrho \geq 0$ by integral

$$\gamma(\vartheta, \varrho) = \int_0^{\varrho} e^{-\mu}\, \mu^{\vartheta-1}\, d\mu.$$

The gamma function $\Gamma(\vartheta)$ is defined for $\vartheta > 0$ by integral

$$\Gamma(\vartheta) = \int_0^{+\infty} e^{-\mu}\, \mu^{\vartheta-1}\, d\mu.$$

3. The Modified Exponential Type Convex Function and Its Associated Algebraic Properties

There has recently been a rise in interest in information theory involving exponentially convex functions because of the substantial and valuable research on big data analysis and extended learning. As a result, other mathematicians, including Antczak (2001), Pecaric (2013), Dragomir (2015), Pal (2017), Alirezaei (2018), Awan (2018), Saima (2019), Noor (2019), and Kadakal (2020), worked on the idea of exponential type convexity in various ways and made contributions to the field of analysis.

The main attention of this section is to present a new definition of modified exponential type convex function and its associated properties.

Definition 6. *Let* G *be a nonnegative function.* $\mathsf{G} : \mathbb{X} \to \mathbb{R}$*, is then said to be a modified exponential type convex if*

$$\mathsf{G}(\varrho g_1 + m(1-\varrho)g_2) \leq (e^{\varrho} - 1)\mathsf{G}(g_1) + m\left(e^{1-\varrho} - 1\right)\mathsf{G}(g_2), \tag{7}$$

holds $\forall\ g_1, g_2 \in \mathbb{X}$, $m \in [0,1]$, *and* $\varrho \in [0,1]$.

We will denote by MEXPC($\mathbb{X}$) the class of modified exponential type convex functions on interval $\mathbb{X}$.

Remark 1. *For* $m = 1$*, we attain exponential type convexity, which is explored by İşcan in* [28].

Remark 2. *The range of the MEXP convex functions for* $m \in [0,1]$ *is* $[0, +\infty)$.

Proof. The proof is obvious. □

We explore some relations between the class of MEXPC functions and other classes of generalized convex functions.

Lemma 1. *The following inequalities* $(e^{\varrho} - 1) \geq \varrho$ *and* $(e^{1-\varrho} - 1) \geq (1-\varrho)$ *hold* $\forall \varrho \in [0,1]$.

Proof. The proof is obvious, so omitted. □

Proposition 1. *If $m \in [0,1]$, then every nonnegative m–convex function is an MEXPC function.*

Proof. Since $m \in [0,1]$, by using Lemma 1, we have

$$\begin{aligned} & \mathsf{G}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) \le \varrho \mathsf{G}(\mathrm{g}_1) + m(1-\varrho)\mathsf{G}(\mathrm{g}_2) \\ \le \; & (e^{\varrho}-1)\mathsf{G}(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)\mathsf{G}(\mathrm{g}_2). \end{aligned}$$

□

Theorem 1. *The sum of two MEXPC functions is an MEXPC function.*

Proof. Let G and P be MEXPC functions. It follows that

$$\begin{aligned} & (\mathsf{G}+\mathsf{P})\left[(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2)\right] \\ = \; & \mathsf{G}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) + \mathsf{P}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) \\ \le \; & (e^{\varrho}-1)\mathsf{G}(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)\mathsf{G}(\mathrm{g}_2) \\ & +(e^{\varrho}-1)\mathsf{P}(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)\mathsf{P}(\mathrm{g}_2) \\ = \; & (\mathrm{e}^{\varrho}-1)[\mathsf{G}(\mathrm{g}_1)+\mathsf{P}(\mathrm{g}_1)] + m\left(e^{1-\varrho}-1\right)[\mathsf{G}(\mathrm{g}_2)+\mathsf{P}(\mathrm{g}_2)] \\ = \; & (e^{\varrho}-1)(\mathsf{G}+\mathsf{P})(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)(\mathsf{G}+\mathsf{P})(\mathrm{g}_2), \end{aligned}$$

which implies that $\mathsf{G}+\mathsf{P}$ is an MEXP convex function. □

Theorem 2. *Scalar multiplication of the MEXPC function is also an MEXPC function.*

Proof. Let G be an MEXPC function. It follows that

$$\begin{aligned} & (c\mathsf{G})\left[(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2)\right] \\ & = c\left[\mathsf{G}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2)\right] \\ & \le c\left[(e^{\varrho}-1)\mathsf{G}(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)\mathsf{G}(\mathrm{g}_2)\right] \\ & = (e^{\varrho}-1)c\mathsf{G}(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)c\mathsf{G}(\mathrm{g}_2) \\ & = (e^{\varrho}-1)(c\mathsf{G})(\mathrm{g}_1) + m\left(e^{1-\varrho}-1\right)(c\mathsf{G})(\mathrm{g}_2), \end{aligned}$$

which implies that $c\mathsf{G}$ is an MEXPC function. □

Theorem 3. *Let $\mathsf{P} : [0,b] \to J$ be an m–convex function for $b > 0$ and $m \in [0,1]$, and $\mathsf{G} : \mathbb{X} \to \mathbb{R}$ is non-decreasing and an MEXPC function. It follows that the function $\mathsf{G} \circ \mathsf{P} : [0,b] \to \mathbb{R}$ is an MEXPC function.*

Proof. $\forall \mathrm{g}_1, \mathrm{g}_2 \in [0, b]$, $m \in [0,1]$, and $\varrho \in [0,1]$, we have

$$\begin{aligned}
&(\mathsf{G} \circ \mathsf{P})(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) \\
&= \mathsf{G}(\mathsf{P}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2)) \\
&\leq \mathsf{G}(\varrho \mathsf{P}(\mathrm{g}_1) + m(1-\varrho)\mathsf{P}(\mathrm{g}_2)) \\
&\leq (e^{\varrho} - 1)\mathsf{G}(\mathsf{P})(\mathrm{g}_1) + m\left(e^{1-\varrho} - 1\right)\mathsf{G}(\mathsf{P})(\mathrm{g}_2) \\
&= (e^{\varrho} - 1)(\mathsf{G} \circ \mathsf{P})(\mathrm{g}_1) + m\left(e^{1-\varrho} - 1\right)(\mathsf{G} \circ \mathsf{P})(\mathrm{g}_2),
\end{aligned}$$

which implies that $\mathsf{G} \circ \mathsf{P}$ is an MEXPC function. □

Theorem 4. *Let $\mathsf{G}_i : [\mathrm{g}_1, \mathrm{g}_2] \to \mathbb{R}$ be a class of MEXP convex functions for $m \in [0,1]$ and let $\mathsf{G}(\mathrm{g}) = \sup_i \mathsf{G}_i(\mathrm{g})$. If $E = \{\mathrm{g} \in [\mathrm{g}_1, \mathrm{g}_2] : \mathsf{G}(\mathrm{g}) < +\infty\} \neq \emptyset$, then E is an interval, and G is an MEXP convex function on E.*

Proof. For all $\mathrm{g}_1, \mathrm{g}_2 \in E$, $m \in [0,1]$, and $\varrho \in [0,1]$, we have

$$\begin{aligned}
& \mathsf{G}(\varrho \mathrm{g}_1 + m(1-\mathsf{G})\mathrm{g}_2) = \sup_i \mathsf{G}_i(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) \\
\leq \quad & \sup_i \left[(e^{\varrho} - 1)\mathsf{G}_i(\mathrm{g}_1) + m\left(e^{1-\varrho} - 1\right)\mathsf{G}_i(\mathrm{g}_2)\right] \\
\leq \quad & (e^{\varrho} - 1)\sup_i \mathsf{G}_i(\mathrm{g}_1) + m\left(e^{1-\varrho} - 1\right)\sup_i \mathsf{G}_i(\mathrm{g}_2) \\
= \quad & (e^{\varrho} - 1)\mathsf{G}(\mathrm{g}_1) + m\left(e^{1-\varrho} - 1\right)\mathsf{G}(\mathrm{g}_2) < +\infty.
\end{aligned}$$

□

Theorem 5. *If the function $\mathsf{G} : [\mathrm{g}_1, \mathrm{g}_2] \to \mathbb{R}$ is an MEXPC function for $m \in [0,1]$, then G is bounded on $[\mathrm{g}_1, m\mathrm{g}_2]$.*

Proof. Suppose $x \in [\mathrm{g}_1, \mathrm{g}_2]$ is a point, $m \in [0,1]$, and $L = \max\{\mathsf{G}(\mathrm{g}_1), m\mathsf{G}(\mathrm{g}_2)\}$. It follows that $\exists\ \varrho \in [0,1]$ such that $x = \varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2$. Thus, since $e^{\varrho} \leq e$ and $e^{1-\varrho} \leq e$, we have

$$\begin{aligned}
\mathsf{G}(x) &= \mathsf{G}(\varrho \mathrm{g}_1 + m(1-\varrho)\mathrm{g}_2) \\
&\leq (e^{\varrho} - 1)\mathsf{G}(\mathrm{g}_1) + m\left(e^{1-\varrho} - 1\right)\mathsf{G}(\mathrm{g}_2) \\
&\leq (e-1)L + m(e-1)L = L(m+1)(e-1) = M.
\end{aligned}$$

□

4. Refinements of (H–H) Type Inequality for the k-Fractional Integral

Numerous academics across a wide range of fields have been studying fractional calculus and its applications in depth for a very long time, and interest in this topic has increased significantly. The notion of fractional derivatives and integrals has been used to propose numerous extensions of them, and authors have obtained new perspectives in a variety of fields, including engineering, physics, economics, biology, and statistics. Here, the term "Riemann–Liouville fractional integral" and its k-generalization are used, as well as some of the theorems that will be mentioned in this section.

Here, we first introduce and demonstrate two new lemmas. We achieve certain improvements of the trapezium type inequality for functions whose first derivative in absolute value at a specific power is an MEXPC function based on these new lemmas.

Lemma 2. *Let $0 < w \leq 1$, and $\mathsf{G} : [mw\mathrm{g}_1, \mathrm{g}_2] \to \mathbb{R}$ is a differentiable mapping on $(mw\mathrm{g}_1, \mathrm{g}_2)$ with $0 < m\mathrm{g}_1 < \mathrm{g}_2$ and $m \in (0,1]$. If $\mathsf{G}' \in L_1[mw\mathrm{g}_1, \mathrm{g}_2]$, then the following equality for k—fractional integral holds true:*

$$\begin{aligned} &\frac{\mathsf{G}(mw\mathrm{g}_1) + \frac{\alpha}{k}\mathsf{G}(\mathrm{g}_2)}{\frac{\alpha}{k}+1} - \frac{\Gamma_k(\alpha+k)}{(\mathrm{g}_2 - mw\mathrm{g}_1)^{\frac{\alpha}{k}}}\, {}^kJ^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(mw\mathrm{g}_1) \\ = &\left(\frac{\mathrm{g}_2 - mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\int_0^1 \left[\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}} - 1\right] \mathsf{G}'(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)\, d\varrho, \end{aligned} \tag{8}$$

where $\alpha, k > 0$ and $\Gamma(\cdot)$ is the Euler Gamma function.

Proof. Applying integrating by parts, we have

$$\begin{aligned} &\left(\frac{\mathrm{g}_2 - mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\int_0^1 \left[\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}} - 1\right] \mathsf{G}'(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)\, d\varrho \\ = &\left(\frac{\mathrm{g}_2 - mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\left\{\int_0^1 \left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)\, d\varrho \right. \\ &\left. - \int_0^1 \mathsf{G}'(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)\, d\varrho\right\} \\ = &\left(\frac{\mathrm{g}_2 - mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\left[\left(\frac{\alpha}{k}+1\right)\left\{\frac{\varrho^{\frac{\alpha}{k}}\mathsf{G}(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)}{\mathrm{g}_2 - mw\mathrm{g}_1}\Bigg|_0^1 \right.\right. \\ &\left.\left. - \int_0^1 \frac{\mathsf{G}(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)}{\mathrm{g}_2 - mw\mathrm{g}_1}\frac{\alpha}{k}\varrho^{\frac{\alpha}{k}-1}\, d\varrho\right\} - \frac{\mathsf{G}(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)}{\mathrm{g}_2 - mw\mathrm{g}_1}\Bigg|_0^1\right] \\ = &\left(\frac{\mathrm{g}_2 - mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\left[\left(\frac{\alpha}{k}+1\right)\right. \\ &\times\left\{\frac{\mathsf{G}(\mathrm{g}_2)}{\mathrm{g}_2 - mwa_1} - \frac{\alpha}{k(\mathrm{g}_2 - mw\mathrm{g}_1)}\int_0^1 \varrho^{\frac{\alpha}{k}-1}\mathsf{G}(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)\, d\varrho\right\} \\ &\left. - \frac{\mathsf{G}(\mathrm{g}_2) - \mathsf{G}(mw\mathrm{g}_1)}{\mathrm{g}_2 - mw\mathrm{g}_1}\right] \\ = &\frac{\mathsf{G}(mw\mathrm{g}_1) + \frac{\alpha}{k}\mathsf{G}(\mathrm{g}_2)}{\frac{\alpha}{k}+1} - \frac{\Gamma_k(\alpha+k)}{(\mathrm{g}_2 - mw\mathrm{g}_1)^{\frac{\alpha}{k}}}\, {}^kJ^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(mw\mathrm{g}_1), \end{aligned}$$

which completes the proof. □

Lemma 3. *Let $0 < w \leq 1$, and $\mathsf{G} : [mw\mathrm{g}_1, \mathrm{g}_2] \to \mathbb{R}$ is a differentiable mapping on $(mw\mathrm{g}_1, \mathrm{g}_2)$ with $0 < m\mathrm{g}_1 < \mathrm{g}_2$ and $m \in (0,1]$. If $\mathsf{G}' \in L_1[mw\mathrm{g}_1, \mathrm{g}_2]$, then the following equality for k—fractional integral holds true:*

$$\begin{aligned} &\frac{\mathsf{G}(mw\mathrm{g}_1) + \mathsf{G}(\mathrm{g}_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(\mathrm{g}_2 - mw\mathrm{g}_1)^{\frac{\alpha}{k}}}\left\{{}^kJ^{\alpha}_{\mathrm{g}_1^+}\mathsf{G}(\mathrm{g}_2) + {}^kJ^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(mw\mathrm{g}_1)\right\} \\ = &\left(\frac{\mathrm{g}_2 - mw\mathrm{g}_1}{w+1}\right)\int_0^1 \left[\varrho^{\frac{\alpha}{k}} - (1-\varrho)^{\frac{\alpha}{k}}\right]\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1 + \varrho\mathrm{g}_2)d\varrho. \end{aligned} \tag{9}$$

Proof. Applying integrating by parts, we have

$$\begin{aligned}
&\left(\frac{g_2 - mwg_1}{w+1}\right)\int_0^1 \left[\varrho^{\frac{\alpha}{k}} - (1-\varrho)^{\frac{\alpha}{k}}\right] G'(mw(1-\varrho)g_1 + \varrho g_2)d\varrho \\
=& \left(\frac{g_2 - mwg_1}{w+1}\right)\left[\int_0^1 \varrho^{\frac{\alpha}{k}} G'(mw(1-\varrho)g_1 + \varrho g_2)d\varrho\right. \\
&\left. - \int_0^1 (1-\varrho)^{\frac{\alpha}{k}} G'(mw(1-\varrho)g_1 + \varrho g_2)d\varrho\right] \\
=& \left(\frac{g_2 - mwg_1}{w+1}\right)[I_1 - I_2], \qquad (10)
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= \int_0^1 \varrho^{\frac{\alpha}{k}} G'(mw(1-\varrho)g_1 + \varrho g_2)d\varrho \\
&= \frac{\varrho^{\frac{\alpha}{k}} G(mw(1-\varrho)g_1 + \varrho g_2)}{g_2 - mwg_1}\Bigg|_0^1 - \int_0^1 \frac{G(mw(1-\varrho)g_1 + \varrho g_2)}{g_2 - mwg_1}\,\frac{\alpha}{k}\varrho^{\frac{\alpha}{k}-1}\,d\varrho \\
&= \frac{G(g_2)}{g_2 - mwg_1} - \frac{\alpha}{k(g_2 - mwg_1)}\int_0^1 \varrho^{\frac{\alpha}{k}-1} G(mw(1-\varrho)g_1 + \varrho g_2)\,d\varrho \\
&= \frac{G(g_2)}{g_2 - mwg_1} - \frac{\Gamma_k(\alpha+k)}{(g_2 - mwg_1)^{\frac{\alpha}{k}+1}}\ {}^{k}J^{\alpha}_{g_2^-} G(mwg_1) \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
I_2 &= \int_0^1 (1-\varrho)^{\frac{\alpha}{k}}\ G'(mw(1-\varrho)g_1 + \varrho g_2)d\varrho \\
&= \frac{(1-\varrho)^{\frac{\alpha}{k}} G(mw(1-\varrho)g_1 + \varrho g_2)}{g_2 - mwg_1}\Bigg|_0^1 \\
&\quad - \int_0^1 \frac{G(mw(1-\varrho)g_1 + \varrho g_2)}{g_2 - mwg_1}\,\frac{\alpha}{k}(1-\varrho)^{\frac{\alpha}{k}-1}\,(-1)\,d\varrho \\
&= -\frac{G(mwg_1)}{g_2 - mwg_1} + \frac{\alpha}{k(g_2 - mwg_1)}\int_0^1 (1-\varrho)^{\frac{\alpha}{k}-1}\ G(mw(1-\varrho)g_1 + \varrho g_2)\,d\varrho \\
&= -\frac{G(mwg_1)}{g_2 - mwg_1} + \frac{\Gamma_k(\alpha+k)}{(g_2 - mwg_1)^{\frac{\alpha}{k}+1}}\ {}^{k}J^{\alpha}_{g_1^+} G(g_2). \qquad (12)
\end{aligned}$$

Combining Equations (11) and (12) in (10) and multiplying it by $\frac{g_2 - wg_1}{w+1}$, we obtain (9), which completes the proof. □

Theorem 6. *Let $0 < w \leq 1$, and $G : \left(0, \frac{g_2}{mw}\right] \to \mathbb{R}$ is a differentiable mapping on $\left(0, \frac{g_2}{mw}\right)$ with $0 < g_1 < g_2$. If $|G'|^q$ is an MEXPC function on $\left(0, \frac{g_2}{mw}\right]$ for $q > 1$ and $q^{-1} + p^{-1} = 1$, then for some fixed $m \in (0,1]$ the following inequality for k—fractional integral holds true:*

$$\begin{aligned}
&\left|\frac{G(mwg_1) + \frac{\alpha}{k}G(g_2)}{\frac{\alpha}{k}+1} - \frac{\Gamma_k(\alpha+k)}{(g_2 - mwg_1)^{\frac{\alpha}{k}}}\ {}^{k}J^{\alpha}_{g_2^-} G(mwg_1)\right| \\
\leq& \left(\frac{g_2 - mwg_1}{\frac{\alpha}{k}+1}\right)[U_1(\alpha,k,p) + U_2(\alpha,k,p)]^{\frac{1}{p}}\left[(e-2)\left(m|G'(wg_1)|^q + |G'(g_2)|^q\right)\right]^{\frac{1}{q}}, \qquad (13)
\end{aligned}$$

where

$$
\begin{aligned}
U_1(\alpha,k,p) &= \int_0^{\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}\left(1-\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}\right)^p d\varrho, \\
U_2(\alpha,k,p) &= \int_{\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}^1 \left(\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right)^p d\varrho.
\end{aligned}
$$

Proof. Using Lemma 2, with the help of Hölder's inequality and the MEXPC function of $|\mathsf{G}'|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathrm{g}_1)+\frac{\alpha}{k}\mathsf{G}(\mathrm{g}_2)}{\frac{\alpha}{k}+1} - \frac{\Gamma_k(\alpha+k)}{(\mathrm{g}_2-mw\mathrm{g}_1)^{\frac{\alpha}{k}}}\,{}^kJ^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(mw\mathrm{g}_1)\right| \\
\leq\;& \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|\,\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|\,d\varrho \\
\leq\;& \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|^p d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|^q d\varrho\right)^{\frac{1}{q}} \\
\leq\;& \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|^p d\varrho\right)^{\frac{1}{p}} \\
&\times\left(\int_0^1\left[m\left(e^{1-\varrho}-1\right)\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q+(e^{\varrho}-1)\left|\mathsf{G}'(\mathrm{g}_2)\right|^q\right]d\varrho\right)^{\frac{1}{q}} \\
=\;& \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)[U_1(\alpha,k,p)+U_2(\alpha,k,p)]^{\frac{1}{p}}\left[(e-2)\left(m\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q+\left|\mathsf{G}'(\mathrm{g}_2)\right|^q\right)\right]^{\frac{1}{q}},
\end{aligned}
$$

which completes the proof. □

Theorem 7. *Let $0 < w \leq 1$, and $\mathsf{G} : \left(0, \frac{\mathrm{g}_2}{mw}\right] \to \mathbb{R}$ is a differentiable mapping on $\left(0, \frac{\mathrm{g}_2}{mw}\right)$ with $0 < \mathrm{g}_1 < \mathrm{g}_2$. If $|\mathsf{G}'|^q$ is an MEXPC function on $\left(0, \frac{\mathrm{g}_2}{mw}\right]$ for $q \geq 1$, then for some fixed $m \in (0,1]$ the following inequality for k—fractional integral holds true:*

$$
\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathrm{g}_1)+\frac{\alpha}{k}\mathsf{G}(\mathrm{g}_2)}{\frac{\alpha}{k}+1} - \frac{\Gamma_k(\alpha+k)}{(\mathrm{g}_2-mw\mathrm{g}_1)^{\frac{\alpha}{k}}}\,{}^kJ^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(mw\mathrm{g}_1)\right| \\
\leq\;& \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{\frac{\alpha}{k}+1}\right)\left(\frac{2\alpha}{k\left(\frac{\alpha}{k}+1\right)^{\frac{k}{\alpha}+1}}\right)^{1-\frac{1}{q}} \\
&\times\left[m\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q\left\{-\frac{2\alpha}{k\left(\frac{\alpha}{k}+1\right)^{\frac{k}{\alpha}+1}}-2e^{\left(1-\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)}\right.\right. \\
&-\left(\frac{\alpha}{k}+1\right)e\,\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right) \\
&\left.+\left(\frac{\alpha}{k}+1\right)e\gamma_{1-\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)+1\right\} \\
&+\left|\mathsf{G}'(\mathrm{g}_2)\right|\left\{2e^{\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}-\frac{2\alpha}{k\left(\frac{\alpha}{k}+1\right)^{\frac{k}{\alpha}+1}}+\left(\frac{\alpha}{k}+1\right)\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)\right. \\
&\left.\left.-\left(\frac{\alpha}{k}+1\right)\gamma_{1-\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)-e\right\}\right]^{\frac{1}{q}}.
\end{aligned}
\tag{14}
$$

Proof. Using Lemma 2, with the help of power mean inequality and the MEXPC function of $|\mathsf{G}'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathsf{g}_1)+\frac{\alpha}{k}\mathsf{G}(\mathsf{g}_2)}{\frac{\alpha}{k}+1}-\frac{\Gamma_k(\alpha+k)}{(\mathsf{g}_2-mw\mathsf{g}_1)^{\frac{\alpha}{k}}}\,{}^kJ^{\alpha}_{\mathsf{g}_2^-}\mathsf{G}(mw\mathsf{g}_1)\right|\\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{\frac{\alpha}{k}+1}\right)\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|\,|\mathsf{G}'(mw(1-\varrho)\mathsf{g}_1+\varrho\mathsf{g}_2)|\,d\varrho\\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{\frac{\alpha}{k}+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|d\varrho\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|\left|\mathsf{G}'(mw(1-\varrho)\mathsf{g}_1+\varrho\mathsf{g}_2)\right|^q d\varrho\right)^{\frac{1}{q}}\\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{\frac{\alpha}{k}+1}\right)\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|d\varrho\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\left(\frac{\alpha}{k}+1\right)\varrho^{\frac{\alpha}{k}}-1\right|\left[m\left(e^{1-\varrho}-1\right)|\mathsf{G}'(w\mathsf{g}_1)|^q+(e^{\varrho}-1)|\mathsf{G}'(\mathsf{g}_2)|^q\right]d\varrho\right)^{\frac{1}{q}}\\
=\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{\frac{\alpha}{k}+1}\right)\left(\frac{2\alpha}{k\left(\frac{\alpha}{k}+1\right)^{\frac{k}{\alpha}+1}}\right)^{1-\frac{1}{q}}\left[m|\mathsf{G}'(wa_1)|^q\left\{-\frac{2\alpha}{k\left(\frac{\alpha}{k}+1\right)^{\frac{k}{\alpha}+1}}-2e^{\left(1-\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)}\right.\right.\\
&-\left(\frac{\alpha}{k}+1\right)e\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)+\left(\frac{\alpha}{k}+1\right)e\gamma_{1-\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)+1\Bigg\}\\
&+|\mathsf{G}'(\mathsf{g}_2)|\left\{2e^{\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}-\frac{2\alpha}{k\left(\frac{\alpha}{k}+1\right)^{\frac{k}{\alpha}+1}}+\left(\frac{\alpha}{k}+1\right)\gamma\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)\right.\\
&\left.\left.-\left(\frac{\alpha}{k}+1\right)\gamma_{1-\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}}\left(\frac{\alpha}{k}+1,\frac{1}{\sqrt[\alpha]{\left(\frac{\alpha}{k}+1\right)^k}}\right)-e\right\}\right]^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 8. *Let $0<w\leq 1$, and $\mathsf{G}:\left(0,\frac{\mathsf{g}_2}{m}\right]\to\mathbb{R}$ is a differentiable mapping on $\left(0,\frac{g_2}{m}\right)$ with $0<\mathsf{g}_1<\mathsf{g}_2$. If $|\mathsf{G}'|^q$ is an MEXPC function on $\left(0,\frac{g_2}{m}\right]$ for $q>1$ and $q^{-1}+p^{-1}=1$, then for some fixed $m\in(0,1]$ the following inequality for k—fractional integral holds true:*

$$\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathsf{g}_1)+\mathsf{G}(\mathsf{g}_2)}{w+1}-\frac{\Gamma_k(\alpha+k)}{(w+1)(\mathsf{g}_2-mw\mathsf{g}_1)^{\frac{\alpha}{k}}}\left\{{}^kJ^{\alpha}_{\mathsf{g}_1^+}\mathsf{G}(\mathsf{g}_2)+{}^kJ^{\alpha}_{\mathsf{g}_2^-}\mathsf{G}(mw\mathsf{g}_1)\right\}\right|\\
\leq\ &\frac{2(\mathsf{g}_2-mw\mathsf{g}_1)}{w+1}\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left[(e-2)\left(m|\mathsf{G}'(w\mathsf{g}_1)|^q+|\mathsf{G}'(\mathsf{g}_2)|^q\right)\right]^{\frac{1}{q}}.
\end{aligned}\tag{15}$$

Proof. Using Lemma 3, with the help of Hölder's inequality and the MEXPC function of $|\mathsf{G}'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathsf{g}_1)+\mathsf{G}(\mathsf{g}_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(\mathsf{g}_2-mw\mathsf{g}_1)^{\frac{\alpha}{k}}}\left\{{}^kJ^{\alpha}_{\mathsf{g}_1^+}\mathsf{G}(\mathsf{g}_2)+{}^kJ^{\alpha}_{\mathsf{g}_2^-}\mathsf{G}(mw\mathsf{g}_1)\right\}\right| \\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{w+1}\right)\int_0^1\left|\varrho^{\frac{\alpha}{k}}-(1-\varrho)^{\frac{\alpha}{k}}\right|\left|\mathsf{G}'(mw(1-\varrho)\mathsf{g}_1+\varrho\mathsf{g}_2)\right|d\varrho \\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{w+1}\right)\left[\int_0^1\varrho^{\frac{\alpha}{k}}\left|\mathsf{G}'(mw(1-\varrho)\mathsf{g}_1+\varrho\mathsf{g}_2)\right|d\varrho\right. \\
&\left.+\int_0^1(1-\varrho)^{\frac{\alpha}{k}}\left|\mathsf{G}'(mw(1-\varrho)\mathsf{g}_1+\varrho\mathsf{g}_2)\right|d\varrho\right] \\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{w+1}\right)\left[\left(\int_0^1\varrho^{\frac{\alpha}{k}p}d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathsf{G}'(mw(1-\varrho)\mathsf{g}_1+\varrho\mathsf{g}_2)\right|^qd\varrho\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1(1-\varrho)^{\frac{\alpha}{k}p}d\varrho\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathsf{G}'(mw(1-\varrho)a_1+\varrho\mathsf{g}_2)\right|^qd\varrho\right)^{\frac{1}{q}}\right] \\
\leq\ &\left(\frac{\mathsf{g}_2-mw\mathsf{g}_1}{w+1}\right)\left[\left(\int_0^1\varrho^{\frac{\alpha}{k}p}d\varrho\right)^{\frac{1}{p}}\right. \\
&\times\left(\int_0^1\left[m\left(e^{1-\varrho}-1\right)\left|\mathsf{G}'(w\mathsf{g}_1)\right|^q+(e^{\varrho}-1)\left|\mathsf{G}'(\mathsf{g}_2)\right|^q\right]d\varrho\right)^{\frac{1}{q}}+\left(\int_0^1(1-\varrho)^{\frac{\alpha}{k}p}d\varrho\right)^{\frac{1}{p}} \\
&\left.\times\left(\int_0^1\left[m\left(e^{1-\varrho}-1\right)\left|\mathsf{G}'(w\mathsf{g}_1)\right|^q+(e^{\varrho}-1)\left|\mathsf{G}'(\mathsf{g}_2)\right|^q\right]d\varrho\right)^{\frac{1}{q}}\right] \\
=\ &\frac{2(\mathsf{g}_2-mw\mathsf{g}_1)}{w+1}\left(\frac{k}{\alpha p+k}\right)^{\frac{1}{p}}\left[(e-2)\left(m\left|\mathsf{G}'(w\mathsf{g}_1)\right|^q+\left|\mathsf{G}'(\mathsf{g}_2)\right|^q\right)\right]^{\frac{1}{q}},
\end{aligned}$$

which completes the proof. □

Theorem 9. *Let $0 < w \leq 1$, and $\mathsf{G} : \left(0, \frac{a_2}{m}\right] \to \mathbb{R}$ is a differentiable mapping on $\left(0, \frac{\mathsf{g}_2}{m}\right)$ with $0 < \mathsf{g}_1 < \mathsf{g}_2$. If $|\mathsf{G}'|^q$ is an MEXPC function on $\left(0, \frac{\mathsf{g}_2}{m}\right]$ for $q \geq 1$, then for some fixed $m \in (0,1]$ the following inequality for k—fractional integral holds true:*

$$\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathsf{g}_1)+\mathsf{G}(\mathsf{g}_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(\mathsf{g}_2-mw\mathsf{g}_1)^{\frac{\alpha}{k}}}\left\{{}^kJ^{\alpha}_{\mathsf{g}_1^+}\mathsf{G}(\mathsf{g}_2)+{}^kJ^{\alpha}_{\mathsf{g}_2^-}\mathsf{G}(mw\mathsf{g}_1)\right\}\right| \\
\leq\ &\left(\frac{a_2-mw\mathsf{g}_1}{w+1}\right)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}}\left[\left\{m\left|\mathsf{G}'(w\mathsf{g}_1)\right|^q\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,1\right)e-\frac{1}{\frac{\alpha}{k}+1}\right)\right.\right. \\
&\left.+\left|\mathsf{G}'(\mathsf{g}_2)\right|^q\left(\Gamma\left(\frac{\alpha}{k}+1,-1\right)-\Gamma\left(\frac{\alpha}{k}+1\right)-\frac{1}{\frac{\alpha}{k}+1}\right)\right\}^{\frac{1}{q}} \\
&+\left\{m\left|\mathsf{G}'(w\mathsf{g}_1)\right|^q\left((-1)^{\frac{\alpha}{k}-1}\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,-1\right)\right)-\frac{1}{\frac{\alpha}{k}+1}\right)\right. \\
&\left.\left.+\left|\mathsf{G}'(\mathsf{g}_2)\right|^q\left(\frac{\left(\frac{\alpha}{k}+1\right)ek\left(\Gamma\left(\frac{\alpha}{k}+1,1\right)-\Gamma\left(\frac{\alpha}{k}+1\right)\right)}{k+\alpha}-\frac{1}{\frac{\alpha}{k}+1}\right)\right\}^{\frac{1}{q}}\right].
\end{aligned} \tag{16}$$

Proof. Using Lemma 3 with the help of power mean inequality and the MEXPC function of $|\mathsf{G}'|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{\mathsf{G}(mw\mathrm{g}_1)+\mathsf{G}(\mathrm{g}_2)}{w+1} - \frac{\Gamma_k(\alpha+k)}{(w+1)(\mathrm{g}_2-mw\mathrm{g}_1)^{\frac{\alpha}{k}}}\left\{ {}^kJ^{\alpha}_{\mathrm{g}_1^+}\mathsf{G}(\mathrm{g}_2) + {}^kJ^{\alpha}_{\mathrm{g}_2^-}\mathsf{G}(mw\mathrm{g}_1)\right\}\right| \\
\leq\ & \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{w+1}\right)\int_0^1 \left|\varrho^{\frac{\alpha}{k}}-(1-\varrho)^{\frac{\alpha}{k}}\right|\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|d\varrho \\
\leq\ & \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{w+1}\right) \\
&\times\left[\int_0^1 \varrho^{\frac{\alpha}{k}}\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|d\varrho + \int_0^1 (1-\varrho)^{\frac{\alpha}{k}}\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|d\varrho\right] \\
\leq\ & \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{w+1}\right)\left[\left(\int_0^1 \varrho^{\frac{\alpha}{k}}d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho^{\frac{\alpha}{k}}\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|^q d\varrho\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 (1-\varrho)^{\frac{\alpha}{k}}d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 (1-\varrho)^{\frac{\alpha}{k}}\left|\mathsf{G}'(mw(1-\varrho)\mathrm{g}_1+\varrho\mathrm{g}_2)\right|^q d\varrho\right)^{\frac{1}{q}}\right] \\
\leq\ & \left(\frac{\mathrm{g}_2-mw\mathrm{g}_1}{w+1}\right)\left(\int_0^1 \varrho^{\frac{\alpha}{k}}d\varrho\right)^{1-\frac{1}{q}} \\
&\times\left[\left(\int_0^1 \varrho^{\frac{\alpha}{k}}\left[m\left(e^{1-\varrho}-1\right)\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q+(e^{\varrho}-1)\left|\mathsf{G}'(\mathrm{g}_2)\right|^q\right]d\varrho\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 (1-\varrho)^{\frac{\alpha}{k}}\left[m\left(e^{1-\varrho}-1\right)\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q+(e^{\varrho}-1)\left|\mathsf{G}'(\mathrm{g}_2)\right|^q\right]d\varrho\right)^{\frac{1}{q}}\right] \\
=\ & \left(\frac{a_2-mw\mathrm{g}_1}{w+1}\right)\left(\frac{k}{\alpha+k}\right)^{1-\frac{1}{q}} \\
&\times\left[\left\{m\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,1\right)e-\frac{1}{\frac{\alpha}{k}+1}\right)\right.\right. \\
&\left.+\left|\mathsf{G}'(\mathrm{g}_2)\right|^q\left(\Gamma\left(\frac{\alpha}{k}+1,-1\right)-\Gamma\left(\frac{\alpha}{k}+1\right)-\frac{1}{\frac{\alpha}{k}+1}\right)\right\}^{\frac{1}{q}} \\
&+\left\{m\left|\mathsf{G}'(w\mathrm{g}_1)\right|^q\left((-1)^{\frac{\alpha}{k}-1}\left(\Gamma\left(\frac{\alpha}{k}+1\right)-\Gamma\left(\frac{\alpha}{k}+1,-1\right)\right)-\frac{1}{\frac{\alpha}{k}+1}\right)\right. \\
&\left.\left.+\left|\mathsf{G}'(\mathrm{g}_2)\right|^q\left(\left(\frac{\alpha}{k}+1\right)ek\left(\Gamma\left(\frac{\alpha}{k}+1,1\right)-\Gamma\left(\frac{\alpha}{k}+1\right)\right)k+\alpha-\frac{1}{\frac{\alpha}{k}+1}\right)\right\}^{\frac{1}{q}}\right],
\end{aligned}
$$

which completes the proof. □

5. Conclusions

In this study, some fresh evaluations of the $(\mathrm{H}-\mathrm{H})$ type inequality for a new generalized convex function are presented. Recently, many mathematicians have worked on the inequality hypothesis to provide a new dimension to mathematical analysis. To proceed in this direction, we have generalized a new definition and have established related inequalities. Since it is simple and convenient to move forward by application of the expectation, we contend that the novel mathematical thoughts, concepts, and strategies we have introduced here are more natural than those currently presented in the literature. In future, we intend to work on concepts such as interval valued analysis, time scale calculus, and quantum calculus for this new convexity and improve inequalities, including the Opial, Simpson, Bullen, Newton, Fejé, Mercer, and Ostrowski types.

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