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Goodness-of-Fit Tests for Weighted Generalized Quasi-Lindley Distribution Using SRS and RSS with Applications to Real Data

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Abstract: This paper deals with the problem of goodness-of-fit tests (GFTs) for the weighted generalized quasi-Lindley distribution (WGQLD) using ranked set sampling (RSS) and simple random sampling (SRS) techniques. The tests are based on the empirical distribution function and sample entropy. These tests include the Kullback–Leibler, Kolomogorov–Smirnov, Anderson–Darling, Cramér–von Mises, Zhang, Liao, and Shimokawa, and Watson tests. The critical values (CV) and power of each test are obtained based on a simulation study by using SRS and RSS methods considering various sample sizes and alternatives. A rain data set is used to investigate the effectiveness of the suggested GFTs. Based on the same number of measured units for the various alternatives taken into consideration in this study, it is discovered that the RSS tests are more effective than those of their rivals in SRS. Additionally, as the set size increases, the GFTs' power increases.

Keywords: entropy; goodness-of-fit tests; critical values; generalized quasi-Lindley distribution; maximum likelihood estimation; ranked set sample; Shannon entropy

MSC: 62G30; 46N30; 94A20

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1. Introduction

The RSS method was first introduced by [1] as a new strategy for collecting data for estimating of pasture and forage yields mean in Australia. It is shown that the population mean estimation using RSS is more efficient than its competitor in simple random sampling. The RSS design is useful when the process of ranking the sample units is easier and less expensive than measuring them. The RSS can be described as follows:

1. Draw a SRS of size k^2 from the study population. Then, partition them randomly into k sets each of size k, where k is the set size.
2. Rank the k units within each set from the smallest to largest relative to the variable under consideration based on any free cost method.
3. Obtain the ith ranked unit from the ith set, for $i = 1, 2, \ldots, k$).
4. The above Steps (1)–(3) can be repeated r times (cycles) if necessary to have an RSS of size $N = rm$.

The obtained RSS units are indicated by $\{X_{[i]j}, i = 1,2,\ldots,k; j = 1,2,\ldots,r\}$, where $X_{[i]j}$ is the ith smallest ranked unit at the jth cycle in a set of size k.

Due to the importance of the RSS scheme, some new variations of the RSS are developed in the literature, as the median ranked set design is proposed by [2,3] who introduced the double RSS design, the L RSS is suggested by [4,5] who suggested neoteric RSS, and the varied L RSS scheme is offered by [6].

In statical analysis and lifetime data, the GFT is an important process for choosing the best probability distribution function that can fit the observed data. Let X be a random variable that adheres to a distribution function $F(x)$, and assume the following hypothesis test:

$$H_0 : F(x) = F_0(x) \qquad \text{versus} \qquad H_1 : F(x) \neq F_0(x). \tag{1}$$

An efficient method for testing (1) is the goodness of fit test based on the empirical distribution function (EDF) which is defined for a SRS of size n, $X_1, X_2, \ldots, X_n$ as

$$F_n(x) = \frac{\text{number of observations} \leq x}{n} = \frac{1}{n}\sum_{i=1}^{n} \mathbb{I}(X_i \leq x), \tag{2}$$

where $\mathbb{I}$ is an indicator function. In practical cases, the parameters of the distribution function are unknown and hence may be estimated by using several techniques, including maximum likelihood and moments methods. The EDF-based tests are useful in cases of unknown population parameters, and hence they can be estimated from the data, because they can provide good power in some scenarios. For more details on the goodness-of-fit and entropy, one can refer to [7] for an entropy estimate by random sampling and some developed measures of sample entropy of continuous random variables are proposed by [8]. For goodness of-fit-tests (GFTs) for Laplace distribution in RSS, see [9,10]. For GFTs for logistic distribution based on phi divergence, see reference [11] for measures of sample entropy, and [12] for functionals' estimation by class of statistics based on spacings. See reference [13] for GFT and entropy estimation for the inverse Gaussian and Laplace distributions using pair RSS, and [14] for Bayesian estimation of GFT and extropy.

In the current paper, we investigate some well-known GFTs based on RSS and SRS schemes for the weighted generalized quasi-Lindley distribution with unknown parameters. To the best of the authors' knowledge, this problem has been not considered before. The rest of this paper is structured as follows. In Section 2, a description of the WGQLD and some of its main properties are presented. The maximum likelihood estimators (MLE) based on RSS and SRS methods are included in Section 3. In Section 4, some GFTs based on SRS and RSS designs are presented theoretically in detail. In Section 5, a simulation study is done to look at the efficacy of the suggested tests. Section 6 provides an application of actual data. Section 7 provides a few conclusions and recommendations for the future.

2. The WGQLD Model Description

In the last few decades, many researchers suggested various probability distributions based on different methods to fit real data in various fields. Some of them studied the concept of the weighted distributions, which is useful in modeling lifetime data and survival analysis. The WGQLD is a two-parameter continuous probability distribution proposed by [15]. Let X be a random variable that follows the WGQLD with parameters θ and α. The probability density function (pdf) of X is given by

$$f_{WGQLD}(x;\theta,\alpha) = \frac{\theta^3}{2(\alpha^2+3\alpha+2)}\left(\frac{\theta^2x^4}{6} + \alpha\theta x^3 + \alpha^2x^2\right)\mathrm{e}^{-\theta x};\ x>0,\ \alpha>0,\ \theta>0, \tag{3}$$

and its corresponding cumulative distribution function (cdf) is

$$F_{WGQLD}(x;\theta,\alpha) = 1 - \frac{\begin{array}{c}24 + 6\alpha^2[2 + x\theta(2 + x\theta)] \\ +6\alpha[6 + x\theta(6 + x\theta(3 + x\theta))] \\ +x\theta[24 + x\theta(12 + x\theta(4 + x\theta))]\end{array}}{12(1+\alpha)(2+\alpha)}\mathrm{e}^{-\theta x}. \tag{4}$$

Figure 1 displays the WGQLD pdf plots for various values of the distribution parameters. It is clear that the WGQLD pdf is right-skewed in general, and the shape of the distribution depends on the parameters values with coefficient of skewness ([15]) given by

$$SK_{WGQLD} = \frac{2\left(3\alpha^6 + 36\alpha^5 + 150\alpha^4 + 306\alpha^3 + 330\alpha^2 + 180\alpha + 40\right)(\alpha+1)(\alpha+2)}{(\alpha+1)(\alpha+2)(3\alpha^4 + 24\alpha^3 + 60\alpha^2 + 60\alpha + 20)^{\frac{3}{2}}}. \quad (5)$$

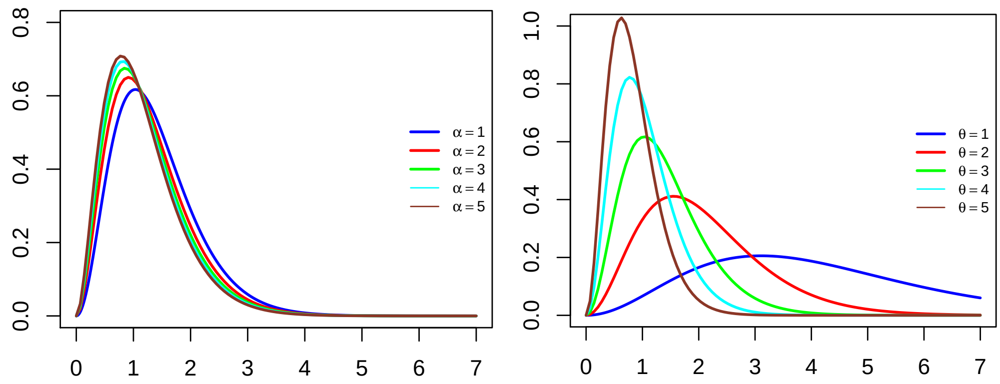

Figure 1. Plots of the WGQLD PDF with $\theta = 3$ (**left**) and $\alpha = 1$ (**right**) for some parameters' values.

3. Maximum Likelihood Estimation

3.1. Under SRS

Assume that $X_1, X_2, \ldots, X_n$ are random variables of size n follow the WGQLD. The likelihood function based on this sample is

$$L(x;\theta,\ \alpha) = \prod_{i=1}^{n} f(x_i,\ \theta,\ \alpha) = \left(\frac{\theta^3}{2(\alpha^2+3\alpha+2)}\right)^n \prod_{i=1}^{n}\left(\alpha^2 x_i^2 + \alpha\theta x_i^3 + \theta^2\frac{x_i^4}{6}\right) \mathrm{e}^{-\theta x_i}. \quad (6)$$

The log of the likelihood function, say, $\Xi = \log L\ (x;\theta,\ \alpha)$, is

$$\Xi = n\log\left(\frac{\theta^3}{2(\alpha^2+3\alpha+2)}\right) + \sum_{i=1}^{n}\log\left(\alpha^2 x_i^2 + \alpha\theta x_i^3 + \theta^2\frac{x_i^4}{6}\right) - \theta\sum_{i=1}^{n} x_i. \quad (7)$$

Taking the derivatives of Ξ relative to θ and α yields

$$\frac{d\Xi}{d\theta} = \frac{3n}{\theta} + \sum_{i=1}^{n}\frac{2x_i(x_i\theta + 3\alpha)}{x_i^2\theta^2 + 6\alpha x_i\theta + 6\alpha^2} - \sum_{i=1}^{n} x_i, \quad (8)$$

$$\frac{d\Xi}{d\alpha} = -\frac{n(2\alpha+3)}{\alpha^2+3\alpha+2} + \sum_{i=1}^{n}\frac{12\alpha + 6\theta x_i}{6\alpha^2 + 6\theta x_i\alpha + \theta^2 x_i^2}. \quad (9)$$

Because these equations have no closed forms, then the MLEs $\hat{\theta}_{SRS}$ and $\hat{\alpha}_{SRS}$ of θ and α, respectively, can be obtained by using any numerical method.

3.2. Under RSS

Let $\{X_{[i]j}, i = 1,2,\ldots,k; j = 1,2,\ldots,r\}$ be RSS data for X with a sample size $n = kr$, denoted by the ith-order statistics from the ith set of size k at the jth cycle assuming that the ranking is perfect. Similarly, the likelihood function using RSS is given by

$$L_{RSS}(\theta,\alpha) = \prod_{j=1}^{r}\prod_{i=1}^{k} f_{i:k}(x_{[i]j},\theta,\alpha), \quad (10)$$

where

$$f_{i:k}(x_{[i]j},\theta,\alpha) = \frac{k!}{(i-1)!(k-i)!}[F(x_{[i]j})]^{i-1}[1-F(x_{[i]j})]^{k-i} f(x_{[i]j}). \quad (11)$$

The log-likelihood function $\Xi_{RSS} = \ln L_{RSS}(\theta, \alpha)$ is

$$\begin{aligned} \Xi_{RSS} &= \sum_{j}^{r}\sum_{i}^{k}\log\left\{f_{i:k}(x_{[i]j}, \theta, \alpha)\right\} \\ &= \sum_{j}^{r}\sum_{i}^{k}\log\left(\frac{k!}{(i-1)!(k-i)!}\right) + \sum_{j}^{r}\sum_{i}^{k}(i-1)\log(F(x_{[i]j})) \\ &+ \sum_{j}^{r}\sum_{i}^{k}(k-i)\log(1-F(x_{[i]j})) + \sum_{j}^{r}\sum_{i}^{k}\log f(x_{[i]j}). \end{aligned} \tag{12}$$

We take the derivatives of (10) with respect to θ and α, respectively, and equate the resulting quantities to zero and then solve numerically to find the MLE $\hat{\theta}_{RSS}$ and $\hat{\alpha}_{RSS}$ of θ and α, respectively.

4. Goodness-of-Fit Tests

Here, the GFT is considered to investigate whether a specific data set is consistent with a hypothesized null distribution. Hence, different EDF tests are considered to study the null hypothesis H_0. The random sample $X_1, X_2, \ldots, X_n$ follows the WGQLD with unknown parameters. The GFT for different distributions with unknown parameters has been studied by many researchers. For example about these studies, the GFTs for generalized Rayleigh distribution by [16,17] studied GFT of generalized exponential distribution, and reference [18] investigated Cramér–von Mises, Kolmogorov–Smirnov, and Watson and Liao and Shimokawa statistics, and Anderson–Darling who study GFTs for the generalized Frechet distribution. Here, the suggested GFTs are investigated based on both sampling methods.

4.1. GFT Using SRS

Let $X_1, X_2, \ldots, X_n$ be a random sample from WGQLD with PDF $f(.;\theta,\alpha)$ and CDF $F(.;\theta,\alpha)$. In addition, let $\hat{\theta}_{SRS}$ and $\hat{\alpha}_{SRS}$ be the MLE of θ and α, respectively. We proposed the following test statistics:

- The Kullback–Leibler distance (KL) between $f(x)$ and $f(x;\theta,\alpha)$ suggested by Kullback and Leibler [19] as

$$\begin{aligned} KL(f(x), f(x;\theta,\alpha)) &= \int_{-\infty}^{\infty} f(x)\log\left(\frac{f(x)}{f(x;\theta,\alpha)}\right)dx, \\ &= -H(f) - \int_{-\infty}^{\infty} f(x)\log[f(x;\theta,\alpha)]dx, \end{aligned}$$

where $H(f(x))$ is the entropy defined by [20] as

$$H(f) = -\int_{-\infty}^{\infty} f(x)\log(f(x))dx,$$

which is estimated later by [21] by

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left(\frac{n}{2m}(x_{(i+m)} - x_{(i-m)})\right),$$

where m is an integer less than $n/2$, known as window size, $x_{(i)} = x_{(n)}$ if $i > n$ and $x_{(i)} = x_{(1)}$ if $i < 1$. The estimator HV_{mn} converges in probability to $H(f)$ as $n, m \to \infty$ and $\frac{m}{n} \to 0$. Therefore, the KL test is defined by [22] is

$$KL_{mn} = -HV_{mn} - \frac{1}{n}\sum_{i=1}^{n}\log(f(x_i, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})).$$

- The Kolmogorov–Smirnov test statistic (KS) is offered by [23,24] with the form

$$KS = \max\left\{ \max_{1\le i\le n}\left(\frac{i}{n} - F\,(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})\right), \max_{1\le i\le n}\left(F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS}) - \frac{i-1}{n}\right)\right\}.$$

- The Anderson–Darling test statistic (AD) which is suggested by [25] by

$$A^2 = -2\sum_{i=1}^{n}\left\{\left(i-\frac{1}{2}\right)\log\left(F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})\right) + \left(n-i+\frac{1}{2}\right)\log\left[(1-F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})\right)\right\} - n.$$

- The Cramér–von Mises (CV) test statistic [26] and [27] is given by

$$W^2 = \sum_{i=1}^{n}\left(F(x_{(i)}, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS}) - \frac{2i-1}{2n}\right)^2 + \frac{1}{12n}.$$

- The Zhang test statistic [28] is defined as

$$Z_K = \max_{1\le i\le n}\left\{\left(i-\frac{1}{2}\right)\log\left(\frac{i-\frac{1}{2}}{n(x_{(i)}, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})}\right) + \left(n-i+\frac{1}{2}\right)\log\left(\frac{n-i-\frac{1}{2}}{n\left[1-F(x_{(i)}, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})\right]}\right)\right\},$$

$$Z_A = -\sum_{i=1}^{n}\left\{\frac{\log\left(F(x_{(i)}, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})\right)}{n-i+\frac{1}{2}} + \frac{\log\left(1-F(x_{(i)}, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})\right)}{i-\frac{1}{2}}\right\},$$

$$Z_C = \sum_{i=1}^{n}\left(\log\left(\frac{F(x_{(i)}, \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})^{-1}-1}{\frac{(n-\frac{1}{2})}{(i-\frac{3}{4})^{-1}}}\right)\right)^2.$$

- The Liao and Shimokawa (LS) test statistic [29] has the form

$$L_n = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{\max\left(\frac{i}{n} - F\,(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS}), F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS}) - \frac{i-1}{n}\right)}{\sqrt{F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})[1-F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})]}}.$$

- The Watson test statistic [30,31] is defined by

$$U_n^2 = W^2 - n\left(\frac{\sum_{i=1}^{n}F(x_{(i)},\ \hat{\theta}_{SRS}, \hat{\alpha}_{SRS})}{n} - \frac{1}{2}\right)^2.$$

4.2. Using RSS

Let $\{X_{[i]j}, i = 1, 2, \ldots, k; j = 1, \ldots, r\}$ be an RSS of size $N = kr$ from the WGQLD with corresponding ordered values $z_{(1)} \le z_{(2)} \le \ldots \le z_{(N)}$. Moreover, let $\hat{\theta}_{RSS}$ and $\hat{\alpha}_{RSS}$ be the RSS maximum likelihood estimators of θ and α, respectively. Thus, the abovementioned GFT based on the SRS method can be studied using RSS and described as follows.

- The test based on RSS KL is defined as

$$KL_{mN}^{RSS} = -HV_{mN} - \frac{1}{N}\sum_{i=1}^{N}\log\left(f(z_i, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})\right).$$

- The Kolmogorov–Smirnov test statistic in RSS is

$$KS = \max\left\{ \max_{1\le i\le N}\left(\frac{i}{N} - F\left(z_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS}\right)\right), \max_{1\le i\le N}\left(F(z_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS}) - \frac{i-1}{N}\right)\right\}.$$

- The Anderson–Darling test statistic based on RSS is

$$A^2 = -2\sum_{i=1}^{N}\left\{\left(i-\frac{1}{2}\right)\log\left(F(z_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})\right) + \left(N-i+\frac{1}{2}\right)\log\left(1-F(z_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})\right)\right\} - N.$$

- The Cramér–von Mises test statistic in terms of RSS is

$$W^2 = \sum_{i=1}^{N}\left(F(z_{(i)}, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS}) - \frac{2i-1}{2N}\right)^2 + \frac{1}{12N}.$$

- The Zhang [28] test statistic using RSS is

$$Z_K = \max_{1\le i\le N}\left\{\left(i-\frac{1}{2}\right)\log\left(\frac{i-\frac{1}{2}}{NF(z_{(i)}, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})}\right) + \left(N-i+\frac{1}{2}\right)\log\left(\frac{N-i-\frac{1}{2}}{N\left[1-F(z_{(i)}, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})\right]}\right)\right\},$$

$$Z_A = -\sum_{i=1}^{N}\left\{\frac{\log\left[F(z_{(i)}, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})\right]}{N-i+\frac{1}{2}} + \frac{\log\left[1-F(z_{(i)}, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})\right]}{i-\frac{1}{2}}\right\},$$

$$Z_C = \sum_{i=1}^{N}\left[\log\left(\frac{F(z_{(i)}, \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})^{-1}-1}{\frac{(N-\frac{1}{2})}{(i-\frac{3}{4})^{-1}}}\right)\right]^2.$$

- The Liao and Shimokawa (LS) test statistic with RSS is

$$L_n = \frac{1}{\sqrt{N}}\sum_{i=1}^{N}\frac{\max\left(\frac{i}{N} - F\left(x_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS}\right), F(x_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS}) - \frac{i-1}{N}\right)}{\sqrt{F(x_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})[1-F(x_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})]}}.$$

- The Watson test statistic in RSS is defined as

$$U_n^2 = W^2 - N\left(\frac{\sum_{i=1}^{N} F(x_{(i)},\ \hat{\theta}_{RSS}, \hat{\alpha}_{RSS})}{N} - \frac{1}{2}\right)^2,$$

where $HV_{nN}^{RSS} = \frac{1}{N}\sum_{i=1}^{N}\log\left(\frac{N}{2m}(z_{(i+m)} - z_{(i-m)})\right)$ and $F(.,\theta,\alpha)$ is the distribution function of WGQLD distribution.

5. Simulation Study

To determine how well the aforementioned GFT performed, a simulation study is conducted. A total of 100,000 samples are generated from the WGQLD with various sample sizes by using SRS and RSS methods. The powers of tests based on KL_{mn} and KL_{mN}^{RSS} depend on the window size m. Therefore, we use the method of [32] by Vexler and Gurevich (2010) for choosing the optimal m and we proposed the following modifications:

$$KL = \min_{1 \leq m \leq [\sqrt{n}]} KL_{mn},$$

and

$$KL^{RSS} = \min_{1 \leq m \leq [\sqrt{N}]} KL^{RSS}_{mN}.$$

5.1. Critical Values

By generating 100,000 samples from the WGQLD of parameters $\theta = 1$ and $\alpha = 1$, we compute the critical values of the tests for the WGQLD in RSS design for significance level of $\gamma = 0.01, 0.05, 0.1$. The values of n and k are chosen to be $n = 2, 3, ..., 9$ and the set size $k = 2, 3, ..., 5$. Tables 1–3 present the obtained results of critical values.

5.2. Power Comparison

In order to assess the powers of various GFT in SRS and RSS for the WGQLD, a simulation study for the power of each test is conducted. The probability of rejecting the null hypothesis correctly with a true alternative hypothesis is known as the power of a hypothesis test. We calculated the power of the tests for the WGQLD by simulating data from the alternative distributions in (H_1) and computing the MLE estimators of θ and α; then we applied our test. The process is repeated 10,000 times by using SRS and RSS for $N = 10, 20,$ and 40. In order to discuss the effects of increasing the sample size for a given set size and subsequently the effects of increasing set size with a fixed sample size, the values of the set size k based on RSS are 2 and 5. Tables 4–6 present the findings for significance level of $\gamma = 0.05$ for the following alternative distributions:

- The generalized quasi-Lindley distribution with scale parameter (SP) 1 and shape parameter (ShP) 1 denoted by GQLD(1,1).
- The generalized quasi-Lindley distribution with SP 1 and ShP 2 denoted by GQLD(1,2).
- The log-logistic distribution with SP 3 and ShP 1 denoted by Llogis(3,1).
- The log-logistic distribution with SP 2 and ShP 1 denoted by Llogis(2,1).
- The Pareto distribution with SP 1.6 and ShP 2 denoted by Pareto(1.6,2).
- The Pareto distribution with SP 1 and ShP 2 denoted by Pareto(1,2).
- The Weibull distribution with SP 3 and ShP 1 denoted by Weibull(3,1).
- The Weibull distribution with SP 4 and ShP 1 denoted by Weibull(4,1).
- The power Lindley distribution with SP 3 and ShP 3 denoted by Plindley(3,3).
- The power Lindley distribution with SP 1 and ShP 2 denoted by Plindley(1,2).
- The generalized Rayleigh distribution with SP 1 and ShP 2 denoted by Genray(1,2).

Table 1. The CV values of different tests of WGQLD for some values of (n, k) in RSS, at significance level of $\gamma = 0.01$.

n	k	KL	KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
2	2	1.896	0.551	18.604	0.285	1.944	4.481	14.662	3.657	0.208
2	3	1.282	0.454	39.730	0.276	2.482	4.286	20.129	3.410	0.196
2	4	1.031	0.392	68.688	0.266	2.752	4.117	22.785	3.034	0.185
2	5	0.806	0.349	105.637	0.255	2.973	4.004	24.661	2.744	0.177
3	2	1.333	0.479	40.426	0.317	2.490	4.319	18.024	3.121	0.213
3	3	0.909	0.390	87.615	0.302	3.025	4.110	23.327	2.848	0.202
3	4	0.736	0.335	152.415	0.290	3.288	3.955	25.371	2.502	0.190
3	5	0.618	0.298	235.527	0.280	3.494	3.845	26.876	2.252	0.178

Table 1. *Cont.*

n	k	KL	KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
4	2	1.093	0.428	70.606	0.338	2.915	4.211	20.512	2.825	0.214
4	3	0.756	0.347	153.828	0.322	3.394	3.991	24.532	2.451	0.204
4	4	0.589	0.296	269.028	0.309	3.606	3.852	27.349	2.215	0.191
4	5	0.492	0.262	415.204	0.295	3.782	3.758	28.651	2.000	0.182
5	2	0.870	0.393	109.169	0.358	3.230	4.112	22.121	2.552	0.217
5	3	0.648	0.319	239.030	0.345	3.665	3.912	26.215	2.241	0.206
5	4	0.505	0.272	418.138	0.328	3.868	3.781	28.631	2.016	0.195
5	5	0.419	0.240	645.603	0.306	4.013	3.694	30.000	1.890	0.186
6	2	0.774	0.368	155.961	0.380	3.506	4.045	23.453	2.371	0.222
6	3	0.554	0.295	342.687	0.362	3.910	3.849	27.379	2.093	0.209
6	4	0.443	0.251	598.775	0.334	4.116	3.729	29.798	1.929	0.197
6	5	0.367	0.222	926.643	0.316	4.246	3.647	31.244	1.799	0.188
7	2	0.694	0.344	211.148	0.389	3.671	3.981	24.602	2.245	0.223
7	3	0.498	0.275	464.071	0.368	4.048	3.792	28.768	2.017	0.210
7	4	0.391	0.235	811.851	0.339	4.27	3.686	30.741	1.856	0.198
7	5	0.329	0.209	1257.269	0.327	4.404	3.614	31.471	1.729	0.190
8	2	0.616	0.326	274.281	0.402	3.830	3.927	25.735	2.147	0.224
8	3	0.453	0.260	603.344	0.374	4.205	3.751	29.650	1.940	0.210
8	4	0.354	0.223	1057.019	0.352	4.420	3.651	31.458	1.797	0.198
8	5	0.297	0.198	1639.578	0.338	4.563	3.586	32.604	1.692	0.190
9	2	0.565	0.309	345.894	0.412	4.001	3.886	26.393	2.059	0.223
9	3	0.409	0.248	761.379	0.386	4.386	3.715	30.492	1.916	0.212
9	4	0.324	0.212	1335.103	0.361	4.508	3.623	32.389	1.768	0.203
9	5	0.274	0.189	2071.215	0.347	4.718	3.563	33.868	1.669	0.191

Table 2. The CV values of different tests of WGQLD for some values of (n, k) in RSS, at significance level of $\gamma = 0.05$.

n	k	KL	KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
2	2	1.374	0.477	16.799	0.206	1.369	4.092	9.647	2.318	0.149
2	3	1.004	0.392	37.231	0.195	1.727	3.965	11.791	2.139	0.141
2	4	0.828	0.340	65.667	0.190	1.936	3.861	13.09	1.969	0.135
2	5	0.660	0.303	101.955	0.184	2.066	3.777	13.869	1.816	0.128
3	2	1.042	0.412	37.726	0.221	1.793	3.993	11.841	2.072	0.152
3	3	0.738	0.336	83.791	0.214	2.150	3.856	13.916	1.894	0.144
3	4	0.601	0.289	147.741	0.206	2.341	3.754	15.139	1.751	0.136
3	5	0.506	0.255	228.762	0.195	2.441	3.678	15.913	1.633	0.129
4	2	0.869	0.368	66.888	0.235	2.113	3.915	13.490	1.929	0.153
4	3	0.615	0.298	148.716	0.226	2.425	3.774	15.512	1.765	0.146
4	4	0.485	0.256	261.646	0.215	2.564	3.678	16.497	1.639	0.138
4	5	0.407	0.227	405.883	0.205	2.677	3.613	17.058	1.532	0.131
5	2	0.710	0.337	104.285	0.247	2.339	3.853	14.659	1.817	0.155
5	3	0.526	0.271	231.625	0.234	2.617	3.716	16.656	1.685	0.146
5	4	0.415	0.233	407.661	0.223	2.764	3.627	17.490	1.564	0.139
5	5	0.346	0.207	633.39	0.215	2.868	3.570	18.092	1.471	0.132
6	2	0.628	0.314	150.018	0.258	2.511	3.797	15.550	1.745	0.156
6	3	0.456	0.252	332.568	0.242	2.789	3.670	17.559	1.636	0.147
6	4	0.367	0.215	585.995	0.229	2.930	3.590	18.413	1.510	0.140
6	5	0.306	0.192	911.54	0.224	3.028	3.540	19.209	1.440	0.133

Table 2. *Cont.*

n	k	KL	KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
7	2	0.562	0.293	203.748	0.266	2.684	3.755	16.399	1.705	0.156
7	3	0.409	0.236	451.471	0.248	2.920	3.634	18.136	1.585	0.150
7	4	0.325	0.202	796.530	0.237	3.057	3.560	19.114	1.476	0.141
7	5	0.275	0.180	1239.977	0.232	3.171	3.516	19.942	1.417	0.135
8	2	0.505	0.278	265.646	0.273	2.812	3.719	17.114	1.685	0.158
8	3	0.372	0.222	588.408	0.252	3.026	3.603	18.812	1.548	0.149
8	4	0.297	0.191	1039.454	0.243	3.194	3.538	19.793	1.454	0.142
8	5	0.250	0.172	1618.991	0.242	3.305	3.499	20.828	1.400	0.136
9	2	0.461	0.265	335.389	0.280	2.907	3.689	17.789	1.672	0.157
9	3	0.339	0.211	743.971	0.259	3.143	3.581	19.490	1.524	0.150
9	4	0.272	0.182	1314.732	0.251	3.286	3.522	20.675	1.443	0.143
9	5	0.231	0.164	2048.216	0.249	3.420	3.482	21.452	1.394	0.138

Table 3. The CV values of different tests of WGQLD for some values of (n,k) in RSS, at significance level of $\gamma = 0.1$.

n	k	KL	KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
2	2	1.167	0.441	16.081	0.170	1.127	3.919	8.045	1.910	0.125
2	3	0.884	0.362	36.212	0.162	1.417	3.824	9.550	1.783	0.117
2	4	0.732	0.313	64.307	0.157	1.594	3.744	10.448	1.663	0.112
2	5	0.594	0.279	100.105	0.152	1.692	3.676	10.927	1.557	0.106
3	2	0.911	0.379	36.586	0.181	1.487	3.850	9.762	1.754	0.125
3	3	0.664	0.309	82.109	0.176	1.772	3.743	11.360	1.629	0.119
3	4	0.538	0.265	145.162	0.168	1.927	3.660	12.143	1.528	0.113
3	5	0.453	0.235	225.796	0.160	2.013	3.600	12.609	1.435	0.107
4	2	0.769	0.339	65.345	0.194	1.745	3.789	11.082	1.656	0.127
4	3	0.551	0.273	146.065	0.183	1.993	3.679	12.578	1.554	0.121
4	4	0.437	0.235	257.979	0.175	2.127	3.601	13.207	1.444	0.114
4	5	0.367	0.208	401.803	0.169	2.221	3.550	13.753	1.362	0.109
5	2	0.635	0.310	102.264	0.202	1.959	3.742	12.119	1.602	0.128
5	3	0.472	0.249	227.92	0.189	2.171	3.632	13.506	1.495	0.121
5	4	0.373	0.214	403.087	0.180	2.295	3.561	14.117	1.389	0.115
5	5	0.314	0.190	628.17	0.177	2.390	3.517	14.642	1.320	0.110
6	2	0.560	0.287	147.185	0.209	2.100	3.699	12.918	1.569	0.128
6	3	0.410	0.230	328.007	0.195	2.321	3.596	14.272	1.448	0.122
6	4	0.331	0.197	580.262	0.186	2.430	3.532	14.964	1.353	0.116
6	5	0.278	0.177	904.988	0.184	2.532	3.493	15.516	1.298	0.111
7	2	0.503	0.269	200.171	0.214	2.235	3.663	13.591	1.552	0.129
7	3	0.369	0.216	446.024	0.200	2.435	3.567	14.809	1.408	0.123
7	4	0.296	0.185	789.856	0.192	2.543	3.509	15.511	1.326	0.116
7	5	0.251	0.166	1232.089	0.191	2.656	3.473	16.211	1.283	0.112
8	2	0.454	0.254	261.22	0.219	2.346	3.638	14.288	1.522	0.129
8	3	0.336	0.203	582	0.201	2.519	3.543	15.343	1.380	0.123
8	4	0.270	0.176	1031.543	0.197	2.665	3.491	16.179	1.311	0.117
8	5	0.228	0.158	1609.654	0.200	2.773	3.459	16.933	1.275	0.113
9	2	0.416	0.241	330.236	0.224	2.427	3.612	14.725	1.491	0.129
9	3	0.307	0.193	736.492	0.206	2.614	3.525	15.893	1.362	0.123
9	4	0.248	0.167	1305.608	0.203	2.755	3.478	16.785	1.305	0.118
9	5	0.211	0.151	2037.527	0.206	2.882	3.447	17.501	1.271	0.114

Table 4. Power estimates of different GFT in SRS and RSS for $N = 10$ and γ = 0.05. Rounded numbers of four digits greater than zero.

Sampling Scheme	Alternative Distribution	Test KL	Statistics KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
SRS	GQLD(1,1)	0.041	0.017	0.030	0.012	0.048	0.036	0.064	0.099	0.052
	GQLD(1,2)	0.038	0.009	0.022	0.004	0.039	0.030	0.057	0.090	0.049
	Llogis(3,1)	0.074	0.040	0.040	0.031	0.042	0.059	0.077	0.086	0.081
	Llogis(2,1)	0.233	0.190	0.291	0.185	0.274	0.298	0.356	0.447	0.246
	Pareto(1.6,2)	0.622	0.327	0.194	0.219	0.358	0.392	0.307	0.173	0.525
	Pareto(1,2)	0.610	0.290	0.161	0.180	0.322	0.377	0.302	0.161	0.498
	Weibull(3,1)	0.271	0.081	0.026	0.049	0.076	0.155	0.122	0.002	0.257
	Weibull(4,1)	0.610	0.250	0.138	0.211	0.241	0.460	0.410	<1e-03	0.608
	Plindley(3,3)	0.299	0.093	0.033	0.058	0.086	0.175	0.141	0.002	0.285
	Plindley(1,2)	0.091	0.021	0.006	0.008	0.026	0.038	0.043	0.026	0.083
	Genray(1,2)	0.236	0.068	0.020	0.038	0.070	0.139	0.102	<1e-03	0.221
RSS (k = 2)	GQLD(1,1)	0.049	0.023	0.064	0.022	0.075	0.067	0.090	0.127	0.062
	GQLD(1,2)	0.059	0.027	0.095	0.028	0.104	0.093	0.122	0.181	0.078
	Llogis(3,1)	0.090	0.060	0.071	0.061	0.062	0.093	0.105	0.104	0.093
	Llogis(2,1)	0.272	0.268	0.408	0.285	0.371	0.390	0.419	0.506	0.263
	Pareto(1.6,2)	0.672	0.381	0.281	0.311	0.420	0.551	0.422	0.206	0.504
	Pareto(1,2)	0.664	0.313	0.263	0.299	0.317	0.484	0.392	0.194	0.521
	Weibull(3,1)	0.293	0.049	0.063	0.099	0.048	0.219	0.188	0.002	0.273
	Weibull(4,1)	0.647	0.190	0.270	0.353	0.189	0.586	0.536	<1e-03	0.643
	Plindley(3,3)	0.327	0.058	0.077	0.117	0.060	0.253	0.216	0.003	0.306
	Plindley(1,2)	0.102	0.013	0.020	0.021	0.031	0.064	0.069	0.034	0.083
	Genray(1,2)	0.255	0.047	0.053	0.085	0.044	0.192	0.161	<1e-03	0.240
RSS (k = 5)	GQLD(1,1)	0.057	0.041	0.117	0.051	0.117	0.105	0.116	0.154	0.074
	GQLD(1,2)	0.068	0.047	0.171	0.067	0.160	0.147	0.159	0.217	0.097
	Llogis(3,1)	0.110	0.192	0.145	0.186	0.110	0.141	0.135	0.124	0.127
	Llogis(2,1)	0.331	0.431	0.592	0.478	0.501	0.510	0.498	0.591	0.326
	Pareto(1.6,2)	0.788	0.572	0.497	0.516	0.605	0.760	0.550	0.256	0.588
	Pareto(1,2)	0.775	0.432	0.478	0.576	0.332	0.614	0.493	0.228	0.662
	Weibull(3,1)	0.386	0.423	0.303	0.467	0.276	0.472	0.361	0.005	0.401
	Weibull(4,1)	0.785	0.822	0.759	0.880	0.684	0.890	0.808	0.001	0.842
	Plindley(3,3)	0.470	0.636	0.493	0.695	0.451	0.641	0.486	0.006	0.479
	Plindley(1,2)	0.129	0.096	0.072	0.093	0.084	0.137	0.113	0.046	0.097
	Genray(1,2)	0.323	0.254	0.196	0.316	0.146	0.334	0.277	0.001	0.343

Table 5. Power estimates of different GFT in RSS and SRS for $N = 20$ and γ = 0.05.

Sampling Scheme	Alternative Distribution	Test KL	Statistics KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
SRS	GQLD(1,1)	0.051	0.014	0.034	0.010	0.061	0.044	0.089	0.103	0.059
	GQLD(1,2)	0.051	0.012	0.031	0.007	0.056	0.042	0.084	0.099	0.061
	Llogis(3,1)	0.099	0.061	0.063	0.053	0.092	0.124	0.169	0.139	0.101
	Llogis(2,1)	0.467	0.371	0.515	0.389	0.508	0.548	0.627	0.676	0.428
	Pareto(1.6,2)	0.974	0.645	0.512	0.496	0.831	0.935	0.777	0.345	0.852
	Pareto(1,2)	0.973	0.620	0.492	0.468	0.814	0.940	0.779	0.335	0.840
	Weibull(3,1)	0.469	0.261	0.228	0.261	0.244	0.462	0.418	0.009	0.496
	Weibull(4,1)	0.915	0.693	0.770	0.790	0.673	0.920	0.906	0.118	0.926
	Plindley(3,3)	0.526	0.296	0.272	0.309	0.273	0.514	0.471	0.012	0.551
	Plindley(1,2)	0.118	0.051	0.024	0.032	0.049	0.082	0.084	0.024	0.120
	Genray(1,2)	0.397	0.217	0.180	0.202	0.245	0.445	0.386	0.004	0.429

Table 5. *Cont.*

Sampling Scheme	Alternative Distribution	Test KL	Statistics KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
RSS (k = 2)	GQLD(1,1)	0.067	0.030	0.085	0.032	0.102	0.080	0.119	0.161	0.078
	GQLD(1,2)	0.093	0.043	0.148	0.054	0.157	0.131	0.180	0.250	0.114
	Llogis(3,1)	0.114	0.111	0.123	0.119	0.117	0.164	0.188	0.172	0.129
	Llogis(2,1)	0.506	0.506	0.665	0.556	0.611	0.632	0.669	0.750	0.457
	Pareto(1.6,2)	0.990	0.822	0.764	0.713	0.963	0.988	0.911	0.429	0.865
	Pareto(1,2)	0.989	0.770	0.747	0.735	0.924	0.984	0.868	0.420	0.890
	Weibull(3,1)	0.513	0.170	0.310	0.308	0.272	0.603	0.511	0.017	0.524
	Weibull(4,1)	0.939	0.624	0.849	0.835	0.779	0.973	0.947	0.181	0.945
	Plindley(3,3)	0.586	0.214	0.368	0.361	0.357	0.684	0.582	0.022	0.587
	Plindley(1,2)	0.135	0.025	0.044	0.041	0.059	0.136	0.121	0.035	0.115
	Genray(1,2)	0.432	0.150	0.265	0.264	0.218	0.535	0.467	0.011	0.470
RSS (k = 5)	GQLD(1,1)	0.074	0.058	0.153	0.074	0.140	0.118	0.145	0.210	0.097
	GQLD(1,2)	0.106	0.066	0.240	0.108	0.211	0.185	0.215	0.315	0.146
	Llogis(3,1)	0.133	0.364	0.290	0.426	0.171	0.223	0.215	0.216	0.205
	Llogis(2,1)	0.564	0.707	0.850	0.808	0.713	0.730	0.727	0.851	0.554
	Pareto(1.6,2)	0.999	0.979	0.964	0.929	0.998	1.000	0.985	0.516	0.943
	Pareto(1,2)	0.999	0.930	0.945	0.952	0.989	0.999	0.945	0.522	0.973
	Weibull(3,1)	0.692	0.820	0.861	0.906	0.762	0.930	0.834	0.088	0.741
	Weibull(4,1)	0.989	0.997	0.999	1.000	0.994	1.000	0.999	0.528	0.995
	Plindley(3,3)	0.806	0.955	0.986	0.996	0.915	0.987	0.946	0.203	0.833
	Plindley(1,2)	0.189	0.191	0.188	0.201	0.193	0.347	0.229	0.062	0.146
	Genray(1,2)	0.549	0.566	0.648	0.713	0.481	0.773	0.684	0.042	0.662

Table 6. Power estimates of different GFT in RSS and SRS for $N = 40$ and $\gamma = 0.05$.

Sampling Scheme	Alternative Distribution	Test KL	Statistics KS	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
SRS	GQLD(1,1)	0.081	0.017	0.050	0.011	0.094	0.068	0.130	0.120	0.088
	GQLD(1,2)	0.080	0.016	0.048	0.011	0.092	0.067	0.129	0.119	0.088
	Llogis(3,1)	0.146	0.089	0.095	0.080	0.196	0.242	0.316	0.217	0.145
	Llogis(2,1)	0.768	0.643	0.793	0.679	0.805	0.839	0.884	0.890	0.702
	Pareto(1.6,2)	1.000	0.969	0.957	0.898	0.999	1.000	0.999	0.733	0.997
	Pareto(1,2)	1.000	0.962	0.953	0.891	1.000	1.000	1.000	0.732	0.995
	Weibull(3,1)	0.813	0.662	0.752	0.751	0.620	0.864	0.835	0.429	0.838
	Weibull(4,1)	0.999	0.986	0.999	0.998	0.986	1.000	1.000	0.979	0.999
	Plindley(3,3)	0.868	0.719	0.816	0.815	0.677	0.905	0.883	0.520	0.886
	Plindley(1,2)	0.209	0.140	0.096	0.114	0.099	0.168	0.162	0.053	0.223
	Genray(1,2)	0.711	0.585	0.666	0.641	0.627	0.837	0.808	0.293	0.769
RSS (k = 2)	GQLD(1,1)	0.097	0.042	0.133	0.057	0.146	0.112	0.170	0.221	0.116
	GQLD(1,2)	0.167	0.068	0.249	0.104	0.250	0.214	0.284	0.369	0.193
	Llogis(3,1)	0.158	0.208	0.229	0.239	0.210	0.277	0.322	0.297	0.215
	Llogis(2,1)	0.782	0.798	0.907	0.859	0.856	0.878	0.898	0.942	0.739
	Pareto(1.6,2)	1.000	0.998	0.999	0.990	1.000	1.000	1.000	0.886	0.998
	Pareto(1,2)	1.000	0.997	0.998	0.992	1.000	1.000	1.000	0.910	0.999
	Weibull(3,1)	0.848	0.479	0.777	0.702	0.758	0.951	0.902	0.427	0.854
	Weibull(4,1)	1.000	0.977	0.999	0.998	0.999	1.000	1.000	0.975	1.000
	Plindley(3,3)	0.905	0.596	0.847	0.776	0.863	0.979	0.945	0.498	0.902
	Plindley(1,2)	0.226	0.044	0.104	0.084	0.140	0.296	0.226	0.060	0.184
	Genray(1,2)	0.737	0.400	0.714	0.637	0.600	0.890	0.854	0.366	0.803

Table 6. *Cont.*

Sampling Scheme	Alternative Distribution	Test Statistics								
		KL	**KS**	$\mathbf{A^2}$	$\mathbf{W^2}$	$\mathbf{Z_K}$	$\mathbf{Z_A}$	$\mathbf{Z_C}$	$\mathbf{L_n}$	$\mathbf{U_n}$
RSS (k = 5)	GQLD(1,1)	0.109	0.075	0.220	0.111	0.187	0.148	0.195	0.305	0.152
	GQLD(1,2)	0.183	0.087	0.357	0.167	0.301	0.264	0.316	0.463	0.250
	Llogis(3,1)	0.181	0.628	0.651	0.833	0.289	0.344	0.342	0.492	0.392
	Llogis(2,1)	0.831	0.933	0.989	0.987	0.916	0.929	0.929	0.990	0.840
	Pareto(1.6,2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.948	1.000
	Pareto(1,2)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0.986	1.000
	Weibull(3,1)	0.966	0.993	1.000	1.000	0.994	1.000	0.999	0.863	0.976
	Weibull(4,1)	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
	Plindley(3,3)	0.993	1.000	1.000	1.000	1.000	1.000	1.000	0.999	0.993
	Plindley(1,2)	0.351	0.375	0.552	0.480	0.463	0.699	0.523	0.182	0.250
	Genray(1,2)	0.864	0.905	0.983	0.981	0.896	0.986	0.971	0.705	0.948

With reference to the results in Tables 1–6, the following remarks can be made.

- The power values for the suggested tests based on RSS and SRS techniques are greater than zero for all cases considered in this section.
- In general, based on the SRS and RSS techniques the power reaches 1 for the alternatives Pareto, Weibull, and Lindley distributions with samples of size 40 or greater.
- The suggested RSS GFT for the WGQLD are more powerful than their SRS rivals for most cases investigated in this study. As an example, consider the case when $N = 10, k = 2$ and the Pareto(1.6, 2) as an alternative, the power values of the tests KS, A^2, W^2 using RSS are 0.381, 0.281, and 0.311 compared to 0.327, 0.194, and 0.219 using SRS, respectively.
- The power of the GFT increases as the set size k increases. As an illustration, when $N = 40$ for the generalized Rayleigh distribution, it is observed that $Z_K = 0.218$, $Z_A = 0.535$, and $Z_C = 0.467$ for $k = 2$, whereas for $k = 5$, they are $Z_K = 0.481$, $Z_A = 0.773$, and $Z_C = 0.684$.
- By using RSS, the power of the GFT increases as the sample size increases. As an example, with $k = 5$ for the power of the Lindley distribution (1,2), the power values of the Watson test are 0.097, 0.146, and 0.250 for $N = 10$, 20, and 40, respectively.
- For the fixed test, the power values of the suggested GFT depend on the distribution parameters values assuming a comparable sample size. As an example, the power of the Anderson–Darling test are 0.074 and 0.108, for generalized quasi-Lindley distribution with parameters (1,1) and (1,2), respectively, when $n = 20$ and $k = 5$.

6. Real Data Example

In this section, three real datasets are used to show the importance of the WGQLD in modeling real-life data. The parameters of the considered models have been estimated through the MLE method. We consider the Akaike information criterion (AIC), Bayesian information criterion (BIC), consistent Akaike information criterion (AICc), Kolmogorov–Smirnov (KS), where AIC = $-2L + k$, CAIC = $-2L + 2\frac{2kn}{n-k-1}$, HQIC = $2\log\log(n)[k - 2L]$, BIC = $-2L + k\log(n)$, K-S = $\sup|F_n(x) - F(x)|$, and the p-value of the corresponding K-S test, where $F_n(x) = \frac{1}{n}\sum_i^n 1_{x_i \leq x}$, k is the number of parameters, n is the sample size, and L is the value of maximum log-likelihood function.

The best model is the one with the smallest values of the AIC, BIC, HQIC, CAIC, and K-S, and the greatest value for the p-value of the K-S test. Now, based on the real datasets, the null and alternative hypothesis will be

$$H_0 : F(x) = F_{WGQLD}(x;\theta,\alpha) \qquad \text{versus} \qquad H_1 : F(x) \neq F_{WGQLD}(x;\theta,\alpha).$$

The first data is from the rain dataset considered by [33]. The data consists of the annual maximum precipitation (inches) for one rain gauge in Fort Collins, Colorado from

1900 to 1999 and are given by 239, 232, 434, 85, 302, 174, 170, 121, 193, 168, 148, 116, 132, 132, 144, 183, 223, 96, 298, 97, 116, 146, 84, 230, 138, 170, 117, 115, 132, 125, 156, 124, 189, 193, 71, 176, 105, 93, 354, 60, 151, 160, 219, 142, 117, 87, 223, 215, 108, 354, 213, 306, 169, 184, 71, 98, 96, 218, 176, 121, 161, 321, 102, 269, 98, 271, 95, 212, 151, 136, 240, 162, 71, 110, 285, 215, 103, 443, 185, 199, 115, 134, 297, 187, 203, 146, 94, 129, 162, 112, 348, 95, 249, 103, 181, 152, 135, 463, 183, 241.

The second set of data represents the lifetime data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by [34], and it is given by 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2.

The third set of data on failure times for a particular model of windshield are given in [35]. These data were recently studied by [36]. The failure times of 84 aircraft windshields are 0.040, 1.866, 2.385, 3.443, 0.301, 1.876, 2.481, 3.467, 0.309, 1.899, 2.610, 3.478, 0.557, 1.911, 2.625, 3.578, 0.943, 1.912, 2.632, 3.595, 1.070, 1.914, 2.646, 3.699, 1.124, 1.981, 2.661, 3.779,1.248, 2.010, 2.688, 3.924, 1.281, 2.038, 2.82,3, 4.035, 1.281, 2.085, 2.890, 4.121, 1.303, 2.089, 2.902, 4.167, 1.432, 2.097, 2.934, 4.240, 1.480, 2.135, 2.962, 4.255, 1.505, 2.154, 2.964, 4.278, 1.506, 2.190, 3.000, 4.305, 1.568, 2.194, 3.103, 4.376, 1.615, 2.223, 3.114, 4.449, 1.619, 2.224, 3.117, 4.485, 1.652, 2.229, 3.166, 4.570, 1.652, 2.300, 3.344, 4.602, 1.757, 2.324, 3.376, 4.663.

To show the potential of the WGQLD with three other well-known competitive models, including the quasi-Lindley distribution (QLD) proposed by [37], the Pareto distribution (PD), and the two-parameter Sujatha distribution (TSPD) given by [38].

The PDFs of these competitive models are as follows.

- The PDF of the QLD distribution is

$$f(x) = \frac{\theta(\alpha + x\theta)}{\alpha + 1} e^{-\theta x}, x > 0, \alpha > -1, \theta > 0.$$

- The PDF of the PD distribution is

$$f(x) = \frac{\alpha\theta^{\alpha}}{x^{\alpha+1}}, x \geq \theta, \alpha > 0, \theta > 0.$$

- The PDF of the TSPD distribution is

$$f(x) = \frac{\theta^3(x^2 + x + \alpha)e^{-\theta x}}{\alpha\theta^2 + \theta + 2}, x > 0, \alpha \geq 0, \theta > 0.$$

The values of the AIC, AICc, BIC, HQIC, and K-S with the corresponding p-value based on the rain data, patient data, and aircraft windshield data are reported in Tables 7–9, respectively. Moreover, Figures 2–4 show the plots of estimated probability density functions and cumulative distribution functions for the three datasets, respectively.

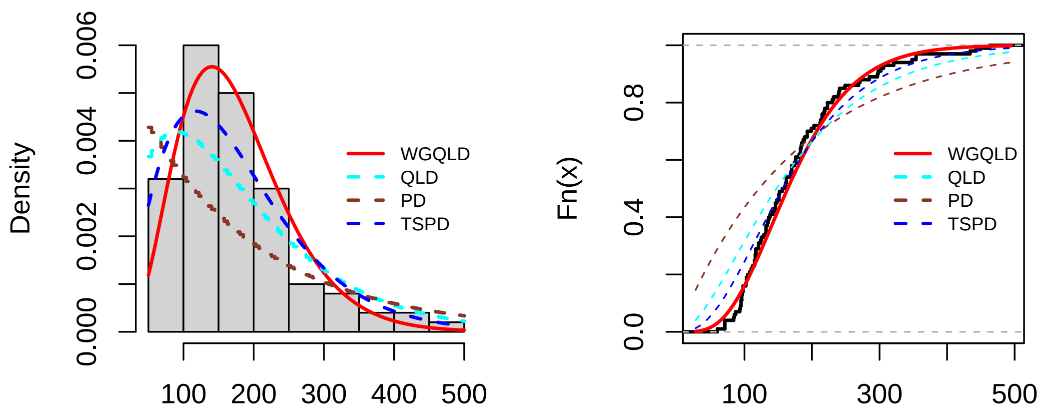

Figure 2. Plots of estimated probability density functions and cumulative distribution functions for rain data.

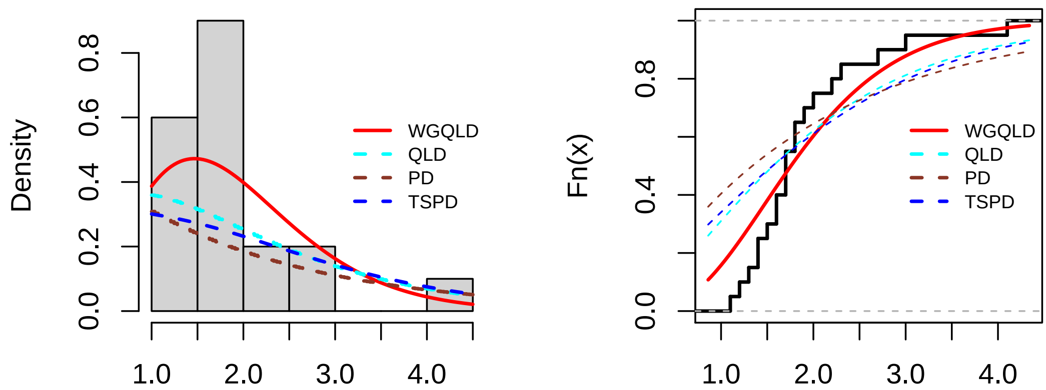

Figure 3. Plots of estimated probability density functions and cumulative distribution functions for patient data.

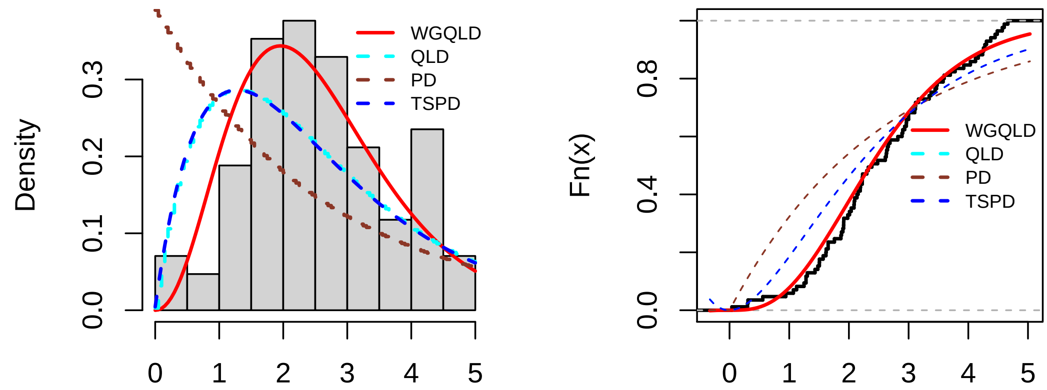

Figure 4. Plots of estimated probability density functions and cumulative distribution functions for aircraft windshield data.

Table 7. The measures AIC, AICc, BIC, HQIC, and K-S with the corresponding *p*-value for rain data.

Model	AIC	AICc	BIC	HQIC	K-S	*p*-Value
WGQLD	1142.145	1142.268	1147.355	1144.253	0.060	0.866
QLD	1180.179	1180.303	1185.389	1182.288	0.216	0.001
PD	1237.721	1237.845	1242.932	1239.83	0.341	<1e-4
TSPD	1156.301	1156.425	1161.511	1158.410	0.144	0.032

Table 8. The measures AIC, AICc, BIC, HQIC, and K-S with the corresponding *p*-value for patient data.

Model	AIC	AICc	BIC	HQIC	K-S	*p*-Value
WGQLD	45.543	46.250	47.534	45.932	0.198	0.411
QLD	59.141	59.847	61.133	59.530	0.346	0.017
PD	69.674	70.380	71.666	70.063	0.440	0.001
TSPD	56.327	57.033	58.319	56.716	0.322	0.032

Table 9. The measures AIC, AICc, BIC, HQIC, and K-S with the corresponding *p*-value for aircraft windshield data.

Model	AIC	AICc	BIC	HQIC	K-S	*p*-Value
WGQLD	276.666	276.813	281.552	278.631	0.084	0.581
QLD	293.513	293.660	298.398	295.478	0.181	0.008
PD	333.975	334.122	338.861	335.940	0.303	<1e-4
TSPD	293.507	293.654	298.393	295.472	0.181	0.0081

With reference to Tables 7–9, we observe that the WGQLD have the lowest values for all AIC, AICc, BIC, HQIC, and K-S with greatest p-values $p = 0.866$ for the rain data, $p = 0.411$ for the patients data, and $p = 0.581$ for the aircraft data among all other competitive models, and Figures 2–4 support our claim. Therefore, the WGQLD outperformed the other models for the three real datasets.

Now, we used the rain dataset to examine the applicability of the proposed SRS and RSS goodness of fit tests. The P-P and Q-Q plots for the WGQLD for the rain data are given in Figure 5. The TTT, box, and density plots for the data are presented in Figure 6.

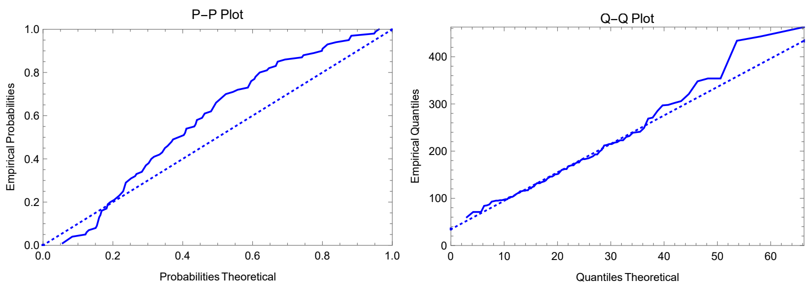

Figure 5. P-P and Q-Q plots of the WGQLD for the rain data.

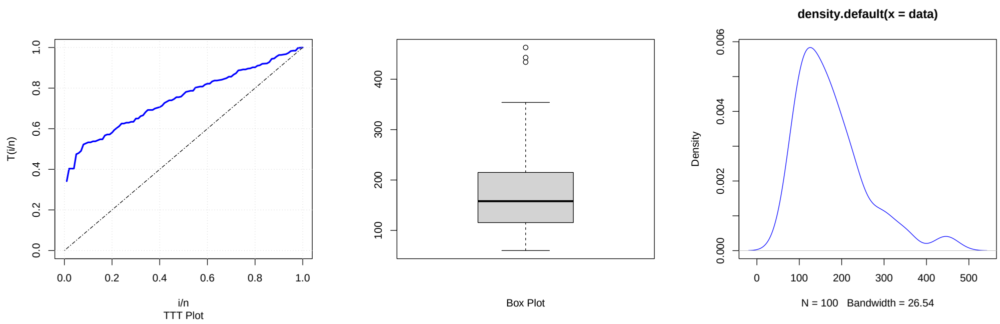

Figure 6. The TTT, box, and density plots of the WGQLD for the rain data.

We extracted a ranked set sample of size $N = 10$ using a set size $k = 5$ and $n = 2$ from the data assuming that the ranking is perfect and applied the different GFT considered in this study. The sampled values based on RSS are shown in Table 10.

Table 10. The sample values using RSS.

Cycle 1	103	117	212	219	354
Cycle 2	95	213	176	239	138

The MLE of the parameters are $\hat{\theta} = 0.0338$ and $\hat{\alpha} = -0.0513$. The values of the test statistics given above are computed and provided in Table 11.

Table 11. Values of different test statistics in RSS.

	KL	K − S	A^2	W^2	Z_K	Z_A	Z_C	L_n	U_n
Test statistics	0.449	0.335	105.682	0.263	2.209	3.810	12.329	1.745	0.065
Critical values	0.660	0.303	101.955	0.184	2.066	3.777	13.869	1.816	0.128

By comparing the values of different test statistics in RSS with the corresponding critical values, it is found that the null hypothesis that assumed that the data is from the WGQLD is not rejected by the KL, Z_C, L_n, and U_n at $\gamma = 0.05$ of significance level.

7. Conclusions

In this paper, various GFTs are presented for the WGQLD using SRS and RSS methods. The suggested tests are compared based on a simulation study for different sample sizes and alternative distributions to study their applicability. The empirical distribution function and sample entropy are considered for the tests involving the Kullback–Leibler, Kolomogorov–Sminrov, Anderson–Darling, Cramér–von Mises, Zhang, Liao, and Shimokawa, and Watson tests. Applications to three real datasets are considered to illustrate the flexibility of the proposed GFT for the WGQLD. It is found that the offered GFT tests based on SRS and RSS perform well, and the tests using the RSS are more efficient than their SRS competitors based on the same number of quantified units. For future research, the authors may investigate the same GFTs proposed in this study to other probability distributions based on perfect and imperfect ranking by using some variations of the RSS as the median RSS design, L RSS, and neoteric RSS methods.

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