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Generalized Hypergeometric Function ${}_3F_2$ Ratios and Branched Continued Fraction Expansions

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Abstract: The paper is related to the classical problem of the rational approximation of analytic functions of one or several variables, particulary the issues that arise in the construction and studying of continued fraction expansions and their multidimensional generalizations—branched continued fraction expansions. We used combinations of three- and four-term recurrence relations of the generalized hypergeometric function ${}_3F_2$ to construct the branched continued fraction expansions of the ratios of this function. We also used the concept of correspondence and the research method to extend convergence, already known for a small region, to a larger region. As a result, we have established some convergence criteria for the expansions mentioned above. It is proved that the branched continued fraction expansions converges to the functions that are an analytic continuation of the ratios mentioned above in some region. The constructed expansions can approximate the solutions of certain differential equations and analytic functions, which are represented by generalized hypergeometric function ${}_3F_2$. To illustrate this, we have given a few numerical experiments at the end.

Keywords: generalized hypergeometric function; branched continued fraction; convergence; rational approximation

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1. Introduction

Questions that arise in economics, physics, biology, etc., lead to mathematical models, which are often formulated in the form of functional equations of various types, in particular, differential, integro-differential and difference equations (see, for example, [1–7]). One of the fundamental problems in approaches to finding solutions of such equations is the reconstruction of functions of one or several variables, as well as problems that arise in the development and implementation of effective methods and algorithms for representing and approximating the functions of one or several variables. There are many various tools for representing and approximating the above-mentioned functions, among which, perhaps, one of the most effective are continued fractions [8–17], and their multidimensional generalizations—branched continued fractions [18–28].

The concept of constructing continued fraction expansions of the ratios of hypergeometric functions was first introduced by C.F. Gauss in 1812 [29] as a composition of their three-term recurrence relations. This contributed to the construction and study of continued fraction expansions of many special functions, including those that are solutions of various differential equations (see more examples in ([8], Part III: Special Functions)).

The first branched continued fraction expansion for the Appell's hypergeometric function F_1 proposed by N.S. Dronyuk in 1966 (see, ([21], pp. 244-252)). In a similarly way, using three- and four-term recurrence relations of the function F_2, its branched continued fraction expansions were constructed in [30]. Here also pointed out which three- and four-term recurrent relations give similar expansions for Appell's hypergeometric function F_4. Finally, the branched continued fraction expansions for function F_3 can be found in [31]. Note that the above-mentioned expansions have the same structure. The expansions of other structures were studied in [32] for Appell's hypergeometric function F_4, in [33,34] for Lauricella's hypergeometric function F_D, in [35] for Lauricella-Saran's hypergeometric function F_S, in [36] for Horn's hypergeometric function H_3. In the work [26], the branched continued fraction expansions were found for some ratios of generalized hypergeometric function ${}_rF_s$ and these results were generalized to the basic hypergeometric function ${}_r\phi_s$.

We consider a generalized hypergeometric function

$$ {}_3F_2(\mathbf{a};\mathbf{b};z) = {}_3F_2(a_1,a_2,a_3;b_1,b_2;z) = \sum_{n=0}^{\infty} \frac{(a_1)_n(a_2)_n(a_3)_n}{(b_1)_n(b_2)_n}\frac{z^n}{n!}, \tag{1} $$

where a_1, a_2, a_3, b_1, and b_2 are complex constants, b_1, b_2 are not equal to a non-positive integer, $z \in \mathbb{C}$, $(\cdot)_k$ is the Pochhammer symbol defined for any complex number α and non-negative integer n by $(\alpha)_0 = 1$ and $(\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1)$, which is the solution of differential equation (see, for example, ([37], p. 8))

$$ z^3(1-z)\frac{d^3u}{dz^3} + z(a_2z-b_2)\frac{d^2u}{dz^2} + (a_1z-b_1)\frac{du}{dz} + a_3u = 0, \tag{2} $$

where $u = u(z)$ is an unknown function of z. It is know that the series (1) converges for all $|z| < 1$ and that for $|z| = 1$ its convergence depends on the parameters a_1, a_2, a_3, b_1, and b_2 as follows:

(a) if $\mathrm{Re}(b_1+b_2-a_1-a_1-a_3) > 1$, the series (1) converges for all $|z| = 1$;
(b) if $0 < \mathrm{Re}(b_1+b_2-a_1-a_1-a_3) \le 1$, the series (1) converges for $|z| = 1$ and $z \neq 1$;
(c) if $\mathrm{Re}(b_1+b_2-a_1-a_1-a_3) \le 0$, the series (1) diverges for all $|z| = 1$.

Note that in [26] the authors found, in particular, the branched continued expansions of three different types of ratios of generalized hypergeometric function:

$$ \frac{{}_3F_2(\mathbf{a};\mathbf{b};z)}{{}_3F_2(a_1,a_2,a_3+1;b_1,b_2+1;z)}, \tag{3} $$
$$ \frac{{}_3F_2(\mathbf{a};\mathbf{b};z)}{{}_3F_2(a_1,a_2,a_3+1;\mathbf{b};z)}, \quad \frac{{}_3F_2(\mathbf{a};\mathbf{b};z)}{{}_3F_2(\mathbf{a};b_1,b_2+1;z)}. $$

In this paper, we construct and study the branched continued fraction expansions for four ratios of generalized hypergeometric function (1), among them two ratios of type (3), and two more of a new type. Explicit formulas for calculating the coefficients of constructed expansion by the coefficients of the function (1) will also be given. In Section 2.3, we derive some convergence criteria for the above-mentioned branched continued fractions and prove their convergence to functions, which are an analytic continuation of the ratios of generalized hypergeometric function ${}_3F_2$ in a certain region (here, region is an domain (open connected set) together with all, part or none of its boundary). Finally, we show an effective approximation of the analytic function, which under certain conditions is the solution of the differential Equation (2), using the constructed expansion.

2. Main Results

One of the problems in approaches to constructing branched continued fraction expansions for special functions of one or several variables (such as generalized hypergeometric functions, hypergeometric functions of Appell, Lauricella and Horn, etc.) is to obtain the simplest structure of branched continued fractions (elements of which are simple polynomials), as well as problems that arise in the development and implementation of effective methods for investigating the convergence of branched continued fractions.

Let us start with recurrence relations, which are the starting point in constructing the expansions of ratios of hypergeometric functions.

2.1. Recurrence Relations

It is know (see, for example, ([26], Lemma 14.1)) that for function (1) the following three-term recurrence relations hold

$$ {}_3F_2(\mathbf{a};\mathbf{b};z) = {}_3F_2(a_1+1,a_2,a_3;b_1+1,b_2;z) - \frac{(b_1-a_1)a_2a_3}{(b_1+1)b_1b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+1;b_1+2,b_2+1;z), \tag{4} $$

$$ {}_3F_2(\mathbf{a};\mathbf{b};z) = {}_3F_2(a_1,a_2+1,a_3;b_1,b_2+1;z) - \frac{(b_2-a_2)a_1a_3}{(b_2+1)b_1b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+1;b_1+1,b_2+2;z), \tag{5} $$

$$ {}_3F_2(\mathbf{a};\mathbf{b};z) = {}_3F_2(a_1,a_2,a_3+1;b_1+1,b_2;z) - \frac{(b_1-a_3)a_1a_2}{(b_1+1)b_1b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+1;b_1+2,b_2+1;z), \tag{6} $$

$$ {}_3F_2(\mathbf{a};\mathbf{b};z) = {}_3F_2(a_1,a_2,a_3+1;b_1,b_2+1;z) - \frac{(b_2-a_3)a_1a_2}{(b_2+1)b_1b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+1;b_1+1,b_2+2;z). \tag{7} $$

We note that these relations can be checked by direct verification.

Lemma 1. *The following four-term recurrence relations hold*

$$ \begin{aligned} {}_3F_2(\mathbf{a};\mathbf{b};z) = {}&{}_3F_2(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z) - \frac{(b_1-a_3)a_1a_2}{(b_1+1)b_1b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+1;b_1+2,b_2+1;z) \\ &- \frac{(b_2-a_2)(a_3+1)a_1}{(b_1+1)(b_2+1)b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+2;b_1+2,b_2+2;z), \end{aligned} \tag{8} $$

$$ \begin{aligned} {}_3F_2(\mathbf{a};\mathbf{b};z) = {}&{}_3F_2(a_1+1,a_2,a_3+1;b_1+1,b_2+1;z) - \frac{(b_2-a_3)a_1a_2}{(b_2+1)b_1b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+1;b_1+1,b_2+2;z) \\ &- \frac{(b_1-a_1)(a_3+1)a_2}{(b_1+1)(b_2+1)b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+2;b_1+2,b_2+2;z). \end{aligned} \tag{9} $$

Proof. From the formula (5), replacing a_3 by a_3+1 and b_1 by b_1+1, we get

$$ \begin{aligned} {}_3F_2(a_1,a_2,a_3+1;b_1+1,b_2;z) = {}&{}_3F_2(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z) \\ &- \frac{(b_2-a_2)a_1(a_3+1)}{(b_2+1)(b_1+1)b_2} z\, {}_3F_2(a_1+1,a_2+1,a_3+2;b_1+2,b_2+2;z). \end{aligned} $$

Applying this relation to the formula (6), we obtain the relation (8).

By analogy, combining the formulas (4) and (7), we have the relation (9). □

2.2. Expansions

In this subsection, we construct four closely related formal branched continued fraction expansions for the ratios of function (1).

Let $(ij)_0 = (i_0, j_0)$ and

$$ \mathcal{I} = \{(1,1);(1,2);(2,1);(2,2)\}. $$

Then, for each pair $(i_0, j_0) \in \mathcal{I}$ we set

$$ R_{(ij)_0}(\mathbf{a};\mathbf{b};z) = \frac{{}_3F_2(\mathbf{a};\mathbf{b};z)}{{}_3F_2(a_1+\delta^1_{i_0}\delta^1_{j_0}+\delta^2_{i_0}\delta^2_{j_0}, a_2+\delta^1_{i_0}\delta^2_{j_0}+\delta^2_{i_0}\delta^1_{j_0}, a_3+\delta^2_{i_0}; b_1+\delta^1_{i_0}\delta^1_{j_0}+\delta^2_{i_0}, b_2+\delta^1_{i_0}\delta^2_{j_0}+\delta^2_{i_0};z)}, \tag{10} $$

where δ^p_k is the Kronecker symbol.

Applying the formula (4) to the relation (10) with $i_0 = j_0 = 1$ one obtains

$$ \begin{aligned} R_{1,1}(\mathbf{a};\mathbf{b};z) &= 1 - \frac{(b_1-a_1)a_2a_3}{(b_1+1)b_1b_2} z \frac{{}_3F_2(a_1+1,a_2+1,a_3+1;b_1+2,b_2+1;z)}{{}_3F_2(a_1+1,a_2,a_3;b_1+1,b_2;z)} \\ &= 1 - \frac{(b_1-a_1)a_2a_3}{(b_1+1)b_1b_2} z \frac{1}{R_{2,1}(a_1+1,a_2,a_3;b_1+1,b_2;z)}. \end{aligned} \tag{11} $$

By analogy, with the use of the formula (5) to (10) the following relation gives

$$R_{1,2}(\mathbf{a};\mathbf{b};z)=1-\frac{(b_2-a_2)a_1a_3}{(b_2+1)b_1b_2}z\frac{1}{R_{2,2}(a_1,a_2+1,a_3;b_1,b_2+1;z)}. \tag{12}$$

Dividing formula (8) by ${}_3F_2(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z)$ we obtain

$$\begin{aligned}R_{2,1}(\mathbf{a};\mathbf{b};z)&=1-\frac{(b_1-a_3)a_1a_2}{(b_1+1)b_1b_2}z\frac{{}_3F_2(a_1+1,a_2+1,a_3+1;b_1+2,b_2+1;z)}{{}_3F_2(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z)}\\&-\frac{(b_2-a_2)(a_3+1)a_1}{(b_1+1)(b_2+1)b_2}z\frac{{}_3F_2(a_1+1,a_2+1,a_3+2;b_1+2,b_2+2;z)}{{}_3F_2(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z)}\\&=1-\frac{(b_1-a_3)a_1a_2}{(b_1+1)b_1b_2}z\frac{1}{R_{1,1}(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z)}\\&-\frac{(b_2-a_2)(a_3+1)a_1}{(b_1+1)(b_2+1)b_2}z\frac{1}{R_{2,2}(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z)}.\end{aligned} \tag{13}$$

By analogy, dividing formula (9) by ${}_3F_2(a_1,a_2+1,a_3+1;b_1+1,b_2+1;z)$, the following relation gives

$$\begin{aligned}R_{2,2}(\mathbf{a};\mathbf{b};z)&=1-\frac{(b_2-a_3)a_1a_2}{(b_2+1)b_1b_2}z\frac{1}{R_{1,2}(a_1+1,a_2,a_3+1;b_1+1,b_2+1;z)}\\&-\frac{(b_1-a_1)(a_3+1)a_2}{(b_1+1)(b_2+1)b_1}z\frac{1}{R_{2,1}(a_1+1,a_2,a_3+1;b_1+1,b_2+1;z)}.\end{aligned} \tag{14}$$

Now we combine formulas (11)–(14) into one. To this end, for each pair $(i_0,j_0)\in\mathcal{I}$ let us introduce the following set of multiindices

$$\mathcal{I}_{(ij)_0}=\{(ij)_k:\ (ij)_k=(i_1,j_1,i_2,j_2,\dots,i_k,j_k),\ 1+\delta^1_{i_{k-1}}\le i_k\le 2,\ j_k\in\{1,2\},\ |i_k-j_k|\ne|i_{k-1}-j_{k-1}|,\ k\ge 1\}.$$

Then, for all $k\ge 1$ and for all $(ij)_k\in\mathcal{I}_{(ij)_0}$, $(ij)_0\in\mathcal{I}$ we set

$$\begin{aligned}\mathbf{a}^{(ij)_0}_{(ij)_k}&=\left(a_1+\sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}\delta^2_{j_p}),a_2+\sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}\delta^1_{j_p}),a_3+\sum_{p=0}^{k-1}\delta^2_{i_p}\right),\\ \mathbf{b}^{(ij)_0}_{(ij)_k}&=\left(b_1+\sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}),b_2+\sum_{p=0}^{k-1}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p})\right),\end{aligned}$$

and also we set

$$c^{(ij)_0}_{(ij)_k}=\frac{(b_1-a_3+\sum_{p=0}^{k-2}\delta^1_{i_p}\delta^1_{j_p})(a_1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}\delta^2_{j_p}))(a_2+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}\delta^1_{j_p}))}{(b_1+1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}))(b_1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}))(b_2+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}))}, \tag{15}$$

if $i_{k-1}=2$, $j_{k-1}=i_k=j_k=1$,

$$c^{(ij)_0}_{(ij)_k}=\frac{(b_2-a_3+\sum_{p=0}^{k-2}\delta^1_{i_p}\delta^2_{j_p})(a_1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}\delta^2_{j_p}))(a_2+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}\delta^1_{j_p}))}{(b_2+1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}))(b_1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}))(b_2+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}))}, \tag{16}$$

if $i_{k-1}=j_{k-1}=j_k=2$, $i_k=1$,

$$c^{(ij)_0}_{(ij)_k}=\frac{(b_1-a_1+\sum_{p=0}^{k-2}\delta^2_{i_p}\delta^1_{j_p})(a_2+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}\delta^1_{j_p}))(a_3+\sum_{p=0}^{k-2}\delta^2_{i_p})}{(b_1+1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}))(b_1+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^1_{j_p}+\delta^2_{i_p}))(b_2+\sum_{p=0}^{k-2}(\delta^1_{i_p}\delta^2_{j_p}+\delta^2_{i_p}))}, \tag{17}$$

if $i_{k-1}=j_{k-1}=j_k=1$, $i_k=2$,

$$c_{(ij)_k}^{(ij)_0} = \frac{(b_1 - a_1 + \sum_{p=0}^{k-2} \delta_{i_p}^2 \delta_{j_p}^1)(a_3 + 1 + \sum_{p=0}^{k-2} \delta_{i_p}^2)(a_2 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^2 + \delta_{i_p}^2 \delta_{j_p}^1))}{(b_1 + 1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^1 + \delta_{i_p}^2))(b_2 + 1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^2 + \delta_{i_p}^2))(b_1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^1 + \delta_{i_p}^2))}, \tag{18}$$

if $i_{k-1} = j_{k-1} = i_k = 2$, $j_k = 1$,

$$c_{(ij)_k}^{(ij)_0} = \frac{(b_2 - a_2 + \sum_{p=0}^{k-2} \delta_{i_p}^2 \delta_{j_p}^2)(a_1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^1 + \delta_{i_p}^2 \delta_{j_p}^2))(a_3 + \sum_{p=0}^{k-2} \delta_{i_p}^2)}{(b_2 + 1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^2 + \delta_{i_p}^2))(b_1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^1 + \delta_{i_p}^2))(b_2 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^2 + \delta_{i_p}^2))}, \tag{19}$$

if $i_{k-1} = 1$, $j_{k-1} = i_k = j_k = 2$,

$$c_{(ij)_k}^{(ij)_0} = \frac{(b_2 - a_2 + \sum_{p=0}^{k-2} \delta_{i_p}^2 \delta_{j_p}^2)(a_3 + 1 + \sum_{p=0}^{k-2} \delta_{i_p}^2)(a_1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^1 + \delta_{i_p}^2 \delta_{j_p}^2))}{(b_1 + 1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^1 + \delta_{i_p}^2))(b_2 + 1 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^2 + \delta_{i_p}^2))(b_2 + \sum_{p=0}^{k-2}(\delta_{i_p}^1 \delta_{j_p}^2 + \delta_{i_p}^2))}, \tag{20}$$

if $j_{k-1} = 1$, $i_{k-1} = i_k = j_k = 2$.

This allows us to write the relations (11)–(14) in the form

$$R_{(ij)_0}(\mathbf{a};\mathbf{b};z) = 1 - \sum_{\substack{i_1 = 1+\delta_{i_0}^1 \\ |i_1 - j_1| \neq |i_0 - j_0|,\ j_1 \in \{1,2\}}}^{2} \frac{c_{(ij)_1}^{(ij)_0} z}{R_{i_1,j_1}(\mathbf{a}_{(ij)_1}^{(ij)_0};\mathbf{b}_{(ij)_1}^{(ij)_0};z)} \quad \text{for all} \quad (ij)_0 \in \mathcal{I}, \tag{21}$$

where $c_{(ij)_1}^{(ij)_0}$, $(ij)_1 \in \mathcal{I}_{(ij)_0}$, $(ij)_0 \in \mathcal{I}$, are defined by formulas (15)–(20) for $k = 1$.

By analogy, it is clear that for any $k \geq 2$ and for any $(ij)_{k-1} \in \mathcal{I}_{(ij)_0}$, $(ij)_0 \in \mathcal{I}$ the following recurrence relation holds

$$R_{i_{k-1},j_{k-1}}(\mathbf{a}_{(ij)_{k-1}}^{(ij)_0};\mathbf{b}_{(ij)_{k-1}}^{(ij)_0};z) = 1 - \sum_{\substack{i_k = 1+\delta_{i_{k-1}}^1 \\ |i_k - j_k| \neq |i_{k-1} - j_{k-1}|,\ j_k \in \{1,2\}}}^{2} \frac{c_{(ij)_k}^{(ij)_0} z}{R_{i_k,j_k}(\mathbf{a}_{(ij)_k}^{(ij)_0};\mathbf{b}_{(ij)_k}^{(ij)_0};z)}, \tag{22}$$

where $c_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, $(ij)_0 \in \mathcal{I}$, are defined by formulas (15)–(20).

Next, we will construct branched continued fraction expansions for $R_{(ij)_0}(\mathbf{a};\mathbf{b};z)$ for all $(ij)_0 \in \mathcal{I}$. Let $(ij)_0$ be arbitrary pair from the set $\mathcal{I}$. Then, substituting relation (22) at $k = 2$ in formula (21) on the first step we obtain

$$R_{(ij)_0}(\mathbf{a};\mathbf{b};z) = 1 - \sum_{\substack{i_1 = 1+\delta_{i_0}^1 \\ |i_1 - j_1| \neq |i_0 - j_0|,\ j_1 \in \{1,2\}}}^{2} \frac{c_{(ij)_1}^{(ij)_0} z}{1 - \sum\limits_{\substack{i_2 = 1+\delta_{i_1}^1 \\ |i_2 - j_2| \neq |i_1 - j_1|,\ j_2 \in \{1,2\}}}^{2} \dfrac{c_{(ij)_2}^{(ij)_0} z}{R_{i_2,j_2}(\mathbf{a}_{(ij)_2}^{(ij)_0};\mathbf{b}_{(ij)_2}^{(ij)_0};z)}},$$

where $c_{(ij)_r}^{(ij)_0}$, $(ij)_r \in \mathcal{I}_{(ij)_0}$, $r = 1, 2$, are defined by formulas (15)–(20).

Next, applying recurrence relation (22) after $(n-1)$ steps, we get

$$R_{(ij)_0}(\mathbf{a};\mathbf{b};z) = 1 - \sum_{\substack{i_1 = 1+\delta_{i_0}^1 \\ |i_1 - j_1| \neq |i_0 - j_0|,\ j_1 \in \{1,2\}}}^{2} \frac{c_{(ij)_1}^{(ij)_0} z}{1} - \cdots - \sum_{\substack{i_n = 1+\delta_{i_{n-1}}^1 \\ |i_n - j_n| \neq |i_{n-1} - j_{n-1}|,\ j_n \in \{1,2\}}}^{2} \frac{c_{(ij)_n}^{(ij)_0} z}{R_{i_n,j_n}(\mathbf{a}_{(ij)_n}^{(ij)_0};\mathbf{b}_{(ij)_n}^{(ij)_0};z)}, \tag{23}$$

where $c_{(ij)_r}^{(ij)_0}$, $(ij)_r \in \mathcal{I}_{(ij)_0}$, $1 \leq r \leq n$, are defined by formulas (15)–(20). Note that here we used the more convenient notation of branched continued fraction, proposed by J.F.W. Herschel ([38], p. 148) following the example of H.H. Bürmann.

Finally, by the relation (22), one obtains

$$R_{(ij)_0}(\mathbf{a};\mathbf{b};z) \sim 1 - \sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c_{(ij)_1}^{(ij)_0} z}{1} - \sum_{\substack{i_2=1+\delta_{i_1}^1 \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \frac{c_{(ij)_2}^{(ij)_0} z}{1} - \cdots - \sum_{\substack{i_k=1+\delta_{i_{k-1}}^1 \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{c_{(ij)_k}^{(ij)_0} z}{1} - \cdots, \tag{24}$$

where the symbol $\sim$ denotes a formal branched continued fraction expansion, the coefficients $c_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, are defined by formulas (15)–(20).

Thus, in (24) for each pair $(i_0, j_0) \in \mathcal{I}$ we have a formal expansion, which is used to construct branched continued fraction. For example, for $R_{1,1}(\mathbf{a};\mathbf{b};z)$ we have the following formal expansion

$$1 - \cfrac{c_{2,1}^{1,1} z}{1 - \cfrac{c_{2,1,1,1}^{1,1} z}{1 - \cfrac{c_{2,1,1,1,2,1}^{1,1} z}{1 - \cfrac{c_{2,1,1,1,2,1,1,1}^{1,1} z}{1 - \ddots} - \cfrac{c_{2,1,1,1,2,1,2,2}^{1,1} z}{1 - \ddots}}} - \cfrac{c_{2,1,2,2}^{1,1} z}{1 - \cfrac{c_{2,1,2,2,1,2}^{1,1} z}{1 - \cfrac{c_{2,1,2,2,1,2,2,2}^{1,1} z}{1 - \ddots}} - \cfrac{c_{2,1,2,2,2,1}^{1,1} z}{1 - \cfrac{c_{2,1,2,2,2,1,1,1}^{1,1} z}{1 - \ddots} - \cfrac{c_{2,1,2,2,2,1,2,2}^{1,1} z}{1 - \ddots}}}}. \tag{25}$$

We note one of the interesting properties of this branched continued fraction. To formulate it, we will give a few definitions ([18], p. 17).

The ratios $c_{(ij)_k}^{(ij)_0} z/1$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, are called the kth partial quotients, and the set of all kth quotients forms the kth floor on the branched continued fraction (25).

Proposition 1. *Let Q_k be the numbers of all kth partial quotients of the kth floor of the branched continued fraction (25). Then the sequence $\{Q_k\}$ is a sequence of Fibonacci numbers starting from the third number.*

Proof. In view on (11)–(14), the partial quotient in the corresponding ratio with first index equal to 1, generates one partial quotient on the next floor, and with an index equal to 2, generates two partial quotients.

Therefore, according to (11) on the first floor is only one partial quotient, which, in turn, according to (13) on the second floor generates two partial quotients.

Let p_k and q_k be the numbers of partial quotients of the kth floor of finite branched continued fraction (25), the denominators of which have ratios with the first index equal to 1 and 2, respectively. It is obvious that $p_1 = 0$ and $q_1 = 1$. Then on kth floor we have $Q_k = p_k + q_k$ partial quotients. Each of the p_k partial quotients generates one partial quotient with the first index equal to 2, and each of the q_k partial quotients generates two partial quotients with the first index equal to 1 and 2 in the corresponding ratios in the denominators.

Thus, if $k \geq 2$, then

$$p_{k+1} = q_k \text{ and } q_{k+1} = p_k + q_k.$$

It follows that for $k \geq 2$

$$Q_{k+1} = p_{k+1} + q_{k+1} = p_k + q_k + q_k = Q_k + p_{k-1} + q_{k-1} = Q_k + Q_{k-1},$$

that proves this proposition. □

It is clear that the other three expansions in the right-hand side (24) also have similar properties.

2.3. Convergence

One of the fundamental problems of the study of branched continued fractions is the proving of their convergence. New methods were developed and applied in [39–45] to establish convergence criteria and in [46–50] to find estimates of convergence rate.

We will remind some concepts on the theory of branched continued fractions (see, for example, [18]).

Let here and further $(ij)_0$ be an arbitrary pair from the set $\mathcal{I}$. Let ${}^{n}Q^{(ij)_0}_{(ij)_k}(z)$ denote the 'tails' of branched continued fraction

$$1 - \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_1} z}{1} - \sum_{\substack{i_2=1+\delta^1_{i_1} \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_2} z}{1} - \ldots, \tag{26}$$

that is

$${}^{n}Q^{(ij)_0}_{(ij)_n}(z) = 1,\ (ij)_n \in \mathcal{I}_{(ij)_0},\ n \geq 1, \tag{27}$$

and

$${}^{n}Q^{(ij)_0}_{(ij)_k}(z) = 1 - \sum_{\substack{i_{k+1}=1+\delta^1_{i_k} \\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{k+1}} z}{1} - \ldots - \sum_{\substack{i_n=1+\delta^1_{i_{n-1}} \\ |i_n-j_n|\neq|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_n} z}{1}, \tag{28}$$

where $(ij)_k \in \mathcal{I}$, $1 \leq k \leq n-1$, $n \geq 2$. Then it is clear that the following recurrence relation holds

$${}^{n}Q^{(ij)_0}_{(ij)_k}(z) = 1 - \sum_{\substack{i_{k+1}=1+\delta^1_{i_k} \\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{k+1}} z}{{}^{n}Q^{(ij)_0}_{(ij)_{k+1}}(z)},\ (ij)_k \in \mathcal{I},\ 1 \leq k \leq n-1,\ n \geq 2. \tag{29}$$

If f_n denotes the nth approximant of (26), then

$$\begin{aligned} f^{(ij)_0}_n(z) &= 1 - \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_1} z}{1} - \sum_{\substack{i_2=1+\delta^1_{i_1} \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_2} z}{1} - \ldots - \sum_{\substack{i_n=1+\delta^1_{i_{n-1}} \\ |i_n-j_n|\neq|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_n} z}{1} \\ &= 1 - \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_1} z}{{}^{n}Q^{(ij)_0}_{(ij)_1}(z)},\ n \geq 1. \end{aligned}$$

The branched continued fraction (26) is said to converges at $z = z_0$ if its sequence of approximants $\{f^{(ij)_0}_n(z)\}$ converges, and

$$\lim_{n\to\infty} f^{(ij)_0}_n(z)$$

is called its value.

The branched continued fraction (26), whose elements are functions in the certain domain D, $D \subset \mathbb{C}$, is called uniformly convergent on set E, $E \subset D$, if its sequence of approximants $\{f^{(ij)_0}_n(z)\}$ converges uniformly on E. When this occurs for an arbitrary set E such that $\overline{E} \subset D$ (here $\overline{E}$ is the closure of the set E) we say that the branched continued fraction converges uniformly on every compact subset of D.

We adopt the convention that a branched continued fraction (26) and all of its approximants have value 1 at $z = 0$.

If ${}^{n}Q^{(ij)_0}_{(ij)_k}(z) \neq 0$ for all $(ij)_k \in \mathcal{I}$, $1 \le k \le n$, $n \ge 1$, and for all z in the certain set D, $D \subset \mathbb{C}$, then for each $m > n \ge 1$ the following formula is valid (see ([18], p. 28))

$$f_m^{(ij)_0}(z) - f_n^{(ij)_0}(z) = (-1)^n \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_1}\cdot(-z)}{{}^{m}Q^{(ij)_0}_{(ij)_1}(z)\cdot {}^{n}Q^{(ij)_0}_{(ij)_1}(z)} \cdots \sum_{\substack{i_n=1+\delta^1_{i_{n-1}} \\ |i_n-j_n|\neq|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_n}\cdot(-z)}{{}^{m}Q^{(ij)_0}_{(ij)_n}(z)\cdot {}^{n}Q^{(ij)_0}_{(ij)_n}(z)}$$
$$\times \sum_{\substack{i_{n+1}=1+\delta^1_{i_n} \\ |i_{n+1}-j_{n+1}|\neq|i_n-j_n|,\, j_{n+1}\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{n+1}}\cdot(-z)}{{}^{m}Q^{(ij)_0}_{(ij)_{n+1}}(z)}. \tag{30}$$

The following result is valid.

Theorem 1. *Let (26) be a branched continued fraction with $c^{(ij)_0}_{(ij)_k}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, defined by (15)–(20) and such that*

$$b_r \ge a_r \ge 0,\ b_r \ge a_3 \ge 0,\ b_r \neq 0,\ r = 1,2. \tag{31}$$

Then:

(A) the branched continued fraction (26) converges to a finite value $f^{(ij)_0}(z)$ for each $z \in D$, where

$$D = \{z \in \mathbb{R} :\ z \le 0\}, \tag{32}$$

and it converges uniformly on every compact subset of Int D*;*

(B) if $f_n^{(ij)_0}(z)$ denotes the nth approximant of the branched continued fraction (26), then for each $z \in D$

$$|f^{(ij)_0}(z) - f_n^{(ij)_0}(z)| \le \frac{(2|z|)^{n+1}}{(2|z|+1)^n},\ n \ge 1.$$

Proof. We will find the upper bound of $|f_m^{(ij)_0}(z) - f_n^{(ij)_0}(z)|$ for $m > n \ge 1$ and $z \in D$.

From (15)–(20) it is clear that for each $(ij)_k \in \mathcal{I}_{(ij)_0}$ the coefficients $c^{(ij)_0}_{(ij)_k}$ of branched continued fraction (26) take non-negative values in the assumption of this theorem. And, consequently, in view of the formulas (27) and (28) it follows that

$${}^{n}Q^{(ij)_0}_{(ij)_k}(z) \ge 1 \quad \text{for all} \quad z \in D \quad \text{and} \quad (ij)_k \in \mathcal{I},\ 1 \le k \le n,\ n \ge 1. \tag{33}$$

In addition, the following inequality holds

$$c^{(ij)_0}_{(ij)_k} \le 1 \quad \text{for all} \quad (ij)_k \in \mathcal{I}_{(ij)_0}. \tag{34}$$

Indeed, for any $(ij)_{k-1} \in \mathcal{I}_{(ij)_0}$ if $k > 1$ and $i_{k-1} = 2$, $j_{k-1} = i_k = j_k = 1$ we have

$$\begin{aligned} b_1 - a_3 + \sum_{p=0}^{k-2} \delta^1_{i_p} \delta^1_{j_p} &\leq b_1 - a_3 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^1_{j_p} + \delta^2_{i_p}) \\ &\leq b_1 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^1_{j_p} + \delta^2_{i_p}), \\ a_1 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^1_{j_p} + \delta^2_{i_p} \delta^2_{j_p}) &\leq a_1 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^1_{j_p} + \delta^2_{i_p}) \\ &\leq b_1 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^1_{j_p} + \delta^2_{i_p}), \\ a_2 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^2_{j_p} + \delta^2_{i_p} \delta^1_{j_p}) &\leq a_2 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^2_{j_p} + \delta^2_{i_p}) \\ &\leq b_2 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^2_{j_p} + \delta^2_{i_p}). \end{aligned}$$

In view of the formula (15), in this case it follows (34). Now, since

$$\begin{aligned} b_2 - a_3 + \sum_{p=0}^{k-2} \delta^1_{i_p} \delta^2_{j_p} &\leq b_2 - a_3 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^2_{j_p} + \delta^2_{i_p}) \\ &\leq b_2 + \sum_{p=0}^{k-2} (\delta^1_{i_p} \delta^2_{j_p} + \delta^2_{i_p}), \end{aligned}$$

from (16) we get (34), when $i_{k-1} = j_{k-1} = j_k = 2$ and $i_k = 1$. Finally, by analogy, we are convinced of the validity of inequality (34) in other cases (see formulas (17)–(20)).

From the inequality (33) it follows that ${}^n Q^{(ij)_0}_{(ij)_k}(z) \neq 0$ for all $(ij)_k \in \mathcal{I}$, $1 \leq k \leq n$, $n \geq 1$, and $z \in D$. Therefore, from (30) for each $m > n \geq 1$ and $\mathbf{z} \in D$ we get

$$\begin{aligned} f^{(ij)_0}_m(z) - f^{(ij)_0}_n(z) = (-1)^n & \sum_{\substack{i_1 = 1 + \delta^1_{i_0} \\ |i_1 - j_1| \neq |i_0 - j_0|,\, j_1 \in \{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_1} \cdot (-z)}{{}^r Q^{(ij)_0}_{(ij)_1}(z)} \cdots \sum_{\substack{i_n = 1 + \delta^1_{i_{n-1}} \\ |i_n - j_n| \neq |i_{n-1} - j_{n-1}|,\, j_n \in \{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_n} \cdot (-z)}{{}^n Q^{(ij)_0}_{(ij)_{n-1}}(z) \cdot {}^n Q^{(ij)_0}_{(ij)_n}(z)} \\ \times & \sum_{\substack{i_{n+1} = 1 + \delta^1_{i_n} \\ |i_{n+1} - j_{n+1}| \neq |i_n - j_n|,\, j_{n+1} \in \{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{n+1}} \cdot (-z)}{{}^m Q^{(ij)_0}_{(ij)_n}(z) \cdot {}^m Q^{(ij)_0}_{(ij)_{n+1}}(z)}, \end{aligned} \tag{35}$$

where $r = m$, if n is even, and $r = n$, if n is odd.

Next, using the inequality (34), for any $z \in D$ and $k \geq 1$ we have

$$\max_{(ij)_k \in \mathcal{I}_{(ij)_0}} \sum_{\substack{i_{k+1} = 1 + \delta^1_{i_k} \\ |i_{k-1} - j_{k-1}| \neq |i_k - j_k|,\, j_{k+1} \in \{1,2\}}}^{2} c^{(ij)_0}_{(ij)_{k+1}} \cdot (-z) \leq 2|z|,$$

which by the relations (28) and (29) and the inequality (33), for any $r \geq k + 1$ and $(ij)_k \in \mathcal{I}_{(ij)_0}$ gives us

$$
\sum_{\substack{i_{k+1}=1+\delta_{i_k}^1\\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{c_{(ij)_{k+1}}^{(ij)_0}\cdot(-z)}{{}^rQ_{(ij)_{k+1}}^{(ij)_0}(z)\cdot {}^rQ_{(ij)_k}^{(ij)_0}(z)} \leq \frac{\sum\limits_{\substack{i_{k+1}=1+\delta_{i_k}^1\\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} c_{(ij)_{k+1}}^{(ij)_0}\cdot(-z)}{1+\sum\limits_{\substack{i_{k+1}=1+\delta_{i_k}^1\\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} c_{(ij)_{k+1}}^{(ij)_0}\cdot(-z)}
\leq \frac{2|z|}{1+2|z|}. \tag{36}
$$

Now, by a successive application of inequality (36) and relations (33) and (34) to the formula (35), for any $m > n \geq 1$ and $z \in D$ we arrive at

$$
|f_m^{(ij)_0}(z) - f_n^{(ij)_0}(z)| \leq \left(\frac{2|z|}{1+2|z|}\right)^n \sum_{\substack{i_1=1+\delta_{i_0}^1\\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c_{(ij)_1}^{(ij)_0}\cdot|z|}{{}^rQ_{(ij)_1}^{(ij)_0}(z)}
\leq \frac{(2|z|)^{n+1}}{(1+2|z|)^n},
$$

where $r = m$, if n is even, and $r = n$, if n is odd. Hence, due to the arbitrariness of m and taking into account that for any fixed $z \in D$

$$
\frac{(2|z|)^{n+1}}{(1+2|z|)^n} \to 0 \text{ as } n \to \infty,
$$

it follows that the branched continued fraction (26) converges to a finite value $f^{(ij)_0}(z)$ for each $z \in D$.

Let K be an arbitrary compact subset of $\operatorname{Int} D$. Then there exists $M > 0$ such that for any $m > n \geq 1$

$$
|f_m^{(ij)_0}(z) - f_n^{(ij)_0}(z)| \leq \frac{(2M)^{n+1}}{(1+2M)^n} \text{ for all } z \in K.
$$

In addition, if m and r are arbitrary natural numbers such that $m > r \geq n$, then

$$
|f_m^{(ij)_0}(z) - f_r^{(ij)_0}(z)| \leq |f_m^{(ij)_0}(z) - f_n^{(ij)_0}(z)| + |f_r^{(ij)_0}(z) - f_n^{(ij)_0}(z)| \text{ for all } z \in K.
$$

Hence, taking into account that

$$
\frac{(2M)^{n+1}}{(1+2M)^n} \to 0 \text{ as } n \to \infty,
$$

it follows that the branched continued fraction (26) converges uniformly on every compact subset of $\operatorname{Int} D$.

Finally, passing to the limit as $m \to \infty$, we obtain (B). □

Note that it follows from the proof of Theorem 1 that (26) is a branched continued fraction with positive elements for each nonzero $z = z_0$ from the set (32). This means that (see, ([18], p. 29))

$$
f_{2n-2}^{(ij)_0}(z_0) < f_{2n}^{(ij)_0}(z_0) < f_{2n+1}^{(ij)_0}(z_0) < f_{2n-1}^{(ij)_0}(z_0),\ n \geq 1,
$$

(here $f_0^{(ij)_0}(z_0) = 1$), so that the even and odd parts of (26) both converge to finite value $f^{(ij)_0}(z_0)$. This system of inequalities expresses a so-called 'fork property' for branched continued fractions.

Theorem 2. *Let (1) be a generalized hypergeometric function ${}_3F_2(\mathbf{a};\mathbf{b};z)$ with parameters satisfying the inequalities (31).*

Then:

(A) for each $z \in H_\varepsilon$, where

$$H_\varepsilon = \bigcup_{-\pi/(2+2\varepsilon)<\varphi<\pi/(2+2\varepsilon)} H_{\varphi,\varepsilon}, \tag{37}$$

$$H_{\varphi,\varepsilon} = \left\{ z \in \mathbb{C} : \ |z| + \operatorname{Re}(ze^{-2i\varphi}) \leq \frac{1-\varepsilon}{4}\cos^2\varphi \right\}, \ 0 < \varepsilon < 1, \tag{38}$$

the branched continued fraction (26), where $c_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, defined by (15)–(20), converges to a finite value $f^{(ij)_0}(z)$;

(B) the convergence is uniform on every compact subset of $\operatorname{Int} H_\varepsilon$, and $f^{(ij)_0}(z)$ is holomorphic on $\operatorname{Int} H_\varepsilon$;

(C) the function $f^{(ij)_0}(z)$ is an analytic continuation of (10) in H_ε.

In our proof we will use the auxiliary lemma, which follows from ([51], Theorem 2).

Lemma 2. *Let the elements $d_{(ij)_k}^{(ij)_0}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, of branched continued fraction*

$$\sum_{\substack{i_1=1+\delta_{i_0}^1 \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{d_{(ij)_1}^{(ij)_0}}{1} - \sum_{\substack{i_2=1+\delta_{i_1}^1 \\ |i_2-j_2|\neq|i_1-j_1|,\, j_2\in\{1,2\}}}^{2} \frac{d_{(ij)_2}^{(ij)_0}}{1} - \cdots \tag{39}$$

satisfy the following conditions

$$\sum_{\substack{i_k=1+\delta_{i_{k-1}}^1 \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{|d_{(ij)_k}^{(ij)_0}| - \operatorname{Re}(d_{(ij)_k}^{(ij)_0} e^{-i(\varphi_{(ij)_{k-1}}+\varphi_{(ij)_k})})}{p_{(ij)_{k-1}}(\cos\varphi_{(ij)_k} - p_{(ij)_k})} \leq 2(1-\varepsilon), \ (ij)_k \in \mathcal{I}_{(ij)_0},$$

where $\varphi_{(ij)_0}$, $p_{(ij)_0}$, and $\varphi_{(ij)_k}$, $p_{(ij)_k}$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, are real numbers such that

$$|\varphi_{(ij)_0}| \leq \varphi^*, \ p_{(ij)_0} \geq 0, \ |\varphi_{(ij)_k}| \leq \varphi^*, \ 0 \leq p_{(ij)_k} \leq (1-\varepsilon)\cos\varphi_{(ij)_k}, \ (ij)_k \in \mathcal{I}_{(ij)_0},$$

where ε and φ^* are constants such that $0 < \varepsilon < 1$ and $0 < \varphi^* < \pi/(2+2\varepsilon)$.

Then:

(A) the approximants of branched continued fraction (39) are all finite and lie in the half-plane

$$V_{\varphi_{(ij)_0}, p_{(ij)_0}} = \{\omega : \ \operatorname{Re}(\omega e^{-i\varphi_{(ij)_0}}) \geq -p_{(ij)_0}\};$$

(B) a branched continued fraction (39) converges if the series

$$\sum_{k=1}^{\infty} \left(\max_{(ij)_k \in \mathcal{I}_{(ij)_0}} |d_{(ij)_k}^{(ij)_0}| \right)^{-1}$$

diverges.

We will now add the necessary notations and definitions. Let

$$L(z) = \sum_{k=0}^{\infty} a_k z^k,$$

where $a_k \in \mathbb{C}$, $k \geq 0$, $z \in \mathbb{C}$, be a formal power series at $z = 0$. Let $F(z)$ be a function holomorphic in a neighbourhood of the origin ($z = 0$). Let the mapping $\Lambda : F(z) \to \Lambda(F)$ associate with $F(z)$ its Taylor expansion in a neighbourhood of the origin.

A sequence $\{F_n(z)\}$ of functions holomorphic at the origin is said to correspond at $z = 0$ to a formal power series $L(z)$ if

$$\lim_{n\to\infty} \lambda\big(L - \Lambda(F_n)\big) = \infty,$$

where λ is the function defined as follows: $\lambda : \mathbb{L} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$; if $L(z) \equiv 0$ then $\lambda(L) = \infty$; if $L(z) \not\equiv 0$ then $\lambda(L) = m$, where m is the smallest degree of terms for which $a_k \neq 0$, that is $m = k + l$.

If $\{F_n(z)\}$ corresponds at $z = 0$ to a formal power series $L(z)$, then the order of correspondence of $F_n(z)$ is defined to be

$$\nu_n = \lambda\big(L - \Lambda(F_n)\big).$$

By the definition of λ, the series $L(z)$ and $\Lambda(F_n)$ agree for all terms up to and including degree $(\nu_n - 1)$.

A branched continued fraction is said to correspond at $z = 0$ to a formal double power series $L(z)$ if its sequence of approximants corresponds to $L(z)$.

For more details on the concept of correspondence, we refer to ([12], pp. 148–160) (see, also ([8], pp. 30–35)).

Proof of Theorem 2. Let φ be an arbitrary real in $(-\pi/(2+2\varepsilon), \pi/(2+\varepsilon))$ and z be an arbitrary point in (38). We choose

$$p_{(ij)_0} = \frac{1}{2}\cos\varphi,\ p_{(ij)_k} = \frac{1}{2}\cos\varphi,\ (ij)_k \in \mathcal{I}_{(ij)_0}. \tag{40}$$

Then for any $(ij)_k \in \mathcal{I}_{(ij)_0}$ we have

$$\begin{aligned}
\sum_{\substack{i_k=1+\delta^1_{i_{k-1}} \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{|c^{(ij)_0}_{(ij)_k} z| - \mathrm{Re}(c^{(ij)_0}_{(ij)_k}(-z)e^{-2i\varphi})}{\cos\varphi - p_{(ij)_k}} &= \sum_{\substack{i_k=1+\delta^1_{i_{k-1}} \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} \frac{2c^{(ij)_0}_{(ij)_k}(|z| + \mathrm{Re}(ze^{-2i\varphi}))}{\cos\varphi} \\
&\leq \frac{1}{2}(1-\varepsilon)\cos\varphi \sum_{\substack{i_k=1+\delta^1_{i_{k-1}} \\ |i_k-j_k|\neq|i_{k-1}-j_{k-1}|,\, j_k\in\{1,2\}}}^{2} c^{(ij)_0}_{(ij)_k} \\
&\leq (1-\varepsilon)\cos\varphi \\
&\leq 2(1-\varepsilon)p_{(ij)_{k-1}}.
\end{aligned}$$

Thus, the elements of (26) satisfy the conditions of Lemma 2, with $\varphi_{(ij)_0} = \varphi$ and $\varphi_{(ij)_k} = \varphi$, $(ij)_k \in \mathcal{I}_{(ij)_0}$, iff $z \in H_{\varphi,\varepsilon}$.

By the inequality (34) we obtain

$$\max_{(ij)_k \in \mathcal{I}_{(ij)_0}} |c^{(ij)_0}_{(ij)_k} \cdot (-z)| \leq |z|,$$

that gives us

$$\sum_{k=1}^{\infty}\left(\max_{(ij)_k\in\mathcal{I}_{(ij)_0}}|c_{(ij)_k}^{(ij)_0}\cdot(-z)|\right)^{-1}\geq\frac{1}{|z|}\sum_{k=1}^{\infty}1,$$

this means that this series is divergent for each nonzero $z \in H_{\varphi,\varepsilon}$.

Recall that we adopted the convention according to which a branched continued fraction (26) and all of its approximants have value 1 at $z = 0$.

Thus, it follows from (B) of Lemma 2 that the branched continued fraction (26) converges to finite value for all $z \in H_{\varphi,\varepsilon}$ and, consequently, for all $z \in H_\varepsilon$ by virtue of arbitrariness φ. This proves (A).

Now, we prove (B). From (A) of Lemma 2 it follows that for every index $(ij)_1 \in \mathcal{I}_{(ij)_0}$ and point $z \in H_{\varphi,\varepsilon}$ the values of all 'tails' ${}^{n}Q_{(ij)_1}^{(ij)_0}(z)$, $n \geq 1$, of (26) are finite and lie in the half-plane

$$V_{(ij)_1}(\varphi,p_{(ij)_1})=\left\{\omega:\ \operatorname{Re}(\omega e^{-i\varphi})\geq\cos\varphi-p_{(ij)_1}\right\}. \tag{41}$$

It follows from (40) that ${}^{n}Q_{(ij)_1}^{(ij)_0}(z) \neq 0$ for all indices and points $z \in H_{\varphi,\varepsilon}$. Thus, the approximants $f_n^{(ij)_0}(z)$, $n \geq 1$, of (26) form a sequence of holomorphic functions in $\operatorname{Int} H_{\varphi,\varepsilon}$ and, consequently, in $\operatorname{Int} H_\varepsilon$ by virtue of arbitrariness φ.

Let K be an arbitrary compact subset of $\operatorname{Int} H_\varepsilon$. Then there exists an open disk

$$Q_r=\{z\in\mathbb{C}:\ |z|<r\},$$

containing K. Let us cover K with domains of the form $P_{\varphi,\varepsilon,r} = \operatorname{Int} H_{\varphi,\varepsilon} \bigcap Q_r$. From this cover we choose the finite subcover

$$P_{\varphi_1,\varepsilon,r},\ P_{\varphi_2,\varepsilon,r},\ \ldots,\ P_{\varphi_k,\varepsilon,r}.$$

Using (34), (40) and (41), for the arbitrary $s \in \{1,\ 2,\ \ldots,\ k\}$ we obtain for any $z \in P_{\varphi_s,\varepsilon,r}$ and $n \geq 1$

$$\begin{aligned}|f_n^{(ij)_0}(z)|&\leq 1+\sum_{\substack{i_1=1+\delta_{i_0}^1\\|i_1-j_1|\neq|i_0-j_0|,\ j_1\in\{1,2\}}}^{2}\frac{c_{(ij)_1}^{(ij)_0}|z|}{\operatorname{Re}({}^{n}Q_{(ij)_1}^{(ij)_0}(z)e^{-i\varphi_s})}\\&<1+\sum_{\substack{i_1=1+\delta_{i_0}^1\\|i_1-j_1|\neq|i_0-j_0|,\ j_1\in\{1,2\}}}^{2}\frac{|z|}{\cos\varphi_s-p_{(ij)_1}}\\&<1+\frac{4r}{\cos\varphi_s}\\&=M(P_{\varphi_s,\varepsilon,r}).\end{aligned}$$

Set

$$M(K)=\max_{1\leq s\leq k}M(P_{\varphi_s,\varepsilon,r}).$$

Then for arbitrary $z \in K$ we obtain $|f_n^{(ij)_0}(z)| \leq M(K)$, for $n \geq 1$, i.e., the sequence $\{f_n^{(ij)_0}(z)\}$ is uniformly bounded on every compact subset of $\operatorname{Int} H_\varepsilon$. An application of Theorem 24.2 [16] yields the uniform convergence of (26) to holomorphic functions on all compact subsets of $\operatorname{Int} H_\varepsilon$. This proves (B).

Finally, we prove (C). We set

$${}^{n}F_{(ij)_n}^{(ij)_0}(z)=R_{i_n,j_n}(\mathbf{a}_{(ij)_n}^{(ij)_0};\mathbf{b}_{(ij)_n}^{(ij)_0};z),\ (ij)_n\in\mathcal{I},\ n\geq 1, \tag{42}$$

where the expression in the right-hand side is defined by (23), and

$$ {}^{n}F^{(ij)_0}_{(ij)_k}(z) = 1 - \sum_{\substack{i_{k+1}=1+\delta^1_{i_k} \\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{k+1}} z}{1} - \cdots - \sum_{\substack{i_n=1+\delta^1_{i_{n-1}} \\ |i_n-j_n|\neq|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_n} z}{{}^{n}F^{(ij)_0}_{(ij)_n}(z)}, $$

where $(ij)_k \in \mathcal{I}$, $1 \le k \le n-1$, $n \ge 2$. Then for all $i(k) \in \mathcal{I}$, $1 \le k \le n-1$, and $n \ge 2$

$$ {}^{n}F^{(ij)_0}_{(ij)_k}(z) = 1 - \sum_{\substack{i_{k+1}=1+\delta^1_{i_k} \\ |i_{k+1}-j_{k+1}|\neq|i_k-j_k|,\, j_{k+1}\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{k+1}} z}{{}^{n}F^{(ij)_0}_{(ij)_{k+1}}(z)}. \tag{43} $$

From (10) and (23) it follows that for each $n \ge 1$

$$ R_{(ij)_0}(\mathbf{a};\mathbf{b};z) = 1 - \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_1} z}{{}^{n+1}F^{(ij)_0}_{(ij)_1}(z)}. $$

Since ${}^{n}F^{(ij)_0}_{(ij)_k}(0) = 1$ and ${}^{n}Q^{(ij)_0}_{(ij)_k}(0) = 1$ for any $(ij)_k \in \mathcal{I}$, $1 \le k \le n$, $n \ge 1$, there exist $\Lambda(1/{}^{n}F^{(ij)_0}_{(ij)_k})$ and $\Lambda(1/{}^{n}Q^{(ij)_0}_{(ij)_k})$, i.e., the $1/{}^{n}F^{(ij)_0}_{(ij)_k}(z)$ and $1/{}^{n}Q^{(ij)_0}_{(ij)_k}(z)$ have Taylor expansions in a neighbourhood of the origin. In addition, since ${}^{n}F^{(ij)_0}_{(ij)_k}(0) \neq 0$ and ${}^{n}Q^{(ij)_0}_{(ij)_k}(0) \neq 0$ for all indices, taking into account (27), (29), (42), and (43), from (30) for each $n \ge 1$ one obtains

$$ R_{(ij)_0}(\mathbf{a};\mathbf{b};z) - f^{(ij)_0}_n(z) = \sum_{\substack{i_1=1+\delta^1_{i_0} \\ |i_1-j_1|\neq|i_0-j_0|,\, j_1\in\{1,2\}}}^{2} \frac{(-1)^n c^{(ij)_0}_{(ij)_1} \cdot (-z)}{{}^{n+1}F^{(ij)_0}_{(ij)_1}(z) \cdot {}^{n}Q^{(ij)_0}_{(ij)_1}(z)} \cdots \sum_{\substack{i_n=1+\delta^1_{i_{n-1}} \\ |i_n-j_n|\neq|i_{n-1}-j_{n-1}|,\, j_n\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_n} \cdot (-z)}{{}^{n+1}F^{(ij)_0}_{(ij)_n}(z) \cdot {}^{n}Q^{(ij)_0}_{(ij)_n}(z)} $$
$$ \times \sum_{\substack{i_{n+1}=1+\delta^1_{i_n} \\ |i_{n+1}-j_{n+1}|\neq|i_n-j_n|,\, j_{n+1}\in\{1,2\}}}^{2} \frac{c^{(ij)_0}_{(ij)_{n+1}} \cdot (-z)}{{}^{n+1}F^{(ij)_0}_{(ij)_{n+1}}(z)}. $$

From this formula for any $n \ge 1$ at $z = 0$ we have

$$ \Lambda(R_{(ij)_0}) - \Lambda(f^{(ij)_0}_n) = \sum_{k=n+1}^{+\infty} \alpha^{(n)}_k z^k, $$

where $a^{(n)}_k$, $k \ge n$, are some coefficients. It follows that

$$ \nu_n = \lambda\Big(\Lambda(R_{(ij)_0}) - \Lambda(f^{(ij)_0}_n)\Big) = n+1 $$

tends monotonically to ∞ as $n \to \infty$.

Thus, the branched continued fraction (26) corresponds at $z = 0$ to a formal power series $\Lambda(R_{(ij)_0})$. Therefore (C) is an immediate consequence of Theorem 5.13 [12]. $\square$

Setting $i_0 = j_0 = 1$, $a_1 = 0$ and replacing b_1 by $b_1 - 1$ in Theorem 2, we obtain a corollary.

Corollary 1. *Let (1) be a generalized hypergeometric function ${}_3F_2(1, a_2, a_3; \mathbf{b}; z)$ with parameters satisfying inequalities*

$$ b_1 > 1,\ b_1 - 1 \ge a_3 \ge 0,\ b_2 \neq 0,\ b_2 \ge a_k \ge 0,\ k = 2,3. \tag{44} $$

Then:

(A) *for each* $z \in H_\varepsilon$, *where* H_ε *defined by* (37), *the branched continued fraction*

$$\cfrac{1}{1-\cfrac{c^{1,1}_{2,1}z}{1-\cfrac{c^{1,1}_{2,1,1,1}z}{1-\cfrac{c^{1,1}_{2,1,1,1,2,1}z}{1-\ddots}}-\cfrac{c^{1,1}_{2,1,2,2}z}{1-\cfrac{c^{1,1}_{2,1,2,2,1,2}z}{1-\ddots}-\cfrac{c^{1,1}_{2,1,2,2,2,1}z}{1-\ddots}}}}, \tag{45}$$

where $c^{1,1}_{(ij)_k}$, $(ij)_k \in \mathcal{I}_{1,1}$, *defined by formulas* (15)–(20), *where* $a_1 = 0$ *and* b_1 *replaced by* $b_1 - 1$, *converges to a finite value* $f^{1,1}(z)$*;*

(B) *the convergence is uniform on every compact subset of* $\operatorname{Int} H_\varepsilon$, *and* $f^{1,1}(z)$ *is holomorphic on* $\operatorname{Int} H_\varepsilon$*;*

(C) *the function* $f^{1,1}(z)$ *is an analytic continuation of* ${}_3F_2(1, a_2, a_3; \mathbf{b}; z)$ *in* H_ε.

It is clear that we will get similar corollaries if:

(a) $i_0 = 1$, $j_0 = 2$, $a_2 = 0$ and b_2 replaced by $b_2 - 1$;

(b) $i_0 = 2$, $j_0 = 1$, $a_2 = 0$ (or $a_3 = 0$) and a_3 (or a_2), b_1, b_2 replaced by $a_3 - 1$ (or $a_2 - 1$), $b_1 - 1$, $b_2 - 1$, respectively;

(c) $i_0 = 2$, $j_0 = 2$, $a_1 = 0$ (or $a_3 = 0$) and a_1 (or a_3), b_1, b_2 replaced by $a_1 - 1$ (or $a_3 - 1$), $b_1 - 1$, $b_2 - 1$, respectively.

3. Numerical Experiments

In this section, we illustrate the use of branched continued fractions to approximate the solutions of differential equations. Approximation of some analytic functions by branched continued fractions can be found in [42,52–54].

It should be noted that if the conditions of Corollary 1 are satisfied, then the branched continued fraction (45) satisfies the differential Equation (2) in which $a_1 = 1$. This means that the approximations of (45) can be used to approximate the solution of this differential equation in the region (37).

For example, we set $a_2 = 1/2$, $a_3 = 1$, $b_1 = 6$, $b_2 = 2$. Then, it is obvious that the parameters satisfy the conditions (44) and, therefore, from (15)–(20), where $a_1 = 0$ and b_1 replaced by $b_1 - 1$, we have such approximations

$$f^{1,1}_1(z) = 1,\ f^{1,1}_2(z) = \frac{28}{28 - z},\ f^{1,1}_3(z) = \frac{168 - 11z}{168 - 18z},\ \text{etc.},$$

for the solution $u(z)$ of differential equation

$$z^3(1-z)\frac{d^3u}{dz^3} + z\left(\frac{z}{2} - 2\right)\frac{d^2u}{dz^2} + (z-6)\frac{du}{dz} + u = 0. \tag{46}$$

The values of these approximations are given in Table 1 together with the values of the partial sums $S_n(z)$ of ${}_3F_2(1, 1/2, 1; 6, 2; z)$ for $1 \le n \le 13$ and for the various value of z. Those numbers illustrate the rate of convergence of $f^{1,1}_n(z)$ and $S_n(z)$ to $u(z)$ as n increases. Comparing them, we see that the branched continued fraction (45) gives better approximations of the solution of differential Equation (46) than the generalized hypergeometric series ${}_3F_2(1, 1/2, 1; 6, 2; z)$.

Table 1. Approximation of the solution of differential Equation (46) by the branched continued fraction (45) and the generalized hypergeometric series ${}_3F_2(1,1/2,1;6,2;z)$.

n	$f_n^{1,1}(0.1)$	$S_n(0.1)$	$f_n^{1,1}(-1.0)$	$S_n(-1.0)$	$f_n^{1,1}(-4.0)$	$S_n(-4.0)$
1	1.000000000000	1.000000000000	1.000000000000	1.000000000000	1.000000000000	1.000000000000
2	1.004184100418	1.004166666667	0.960000000000	0.958333333333	0.857142857143	0.833333333333
3	1.004227053140	1.004226190476	0.963541666667	0.964285714290	0.893939393940	0.928571428570
4	1.004227621300	1.004227585565	0.963169003257	0.962890625000	0.883767535070	0.839285714286
5	1.004227630503	1.004227628968	0.963214851933	0.963324652778	0.886929122814	0.950396825397
6	1.004227630666	1.004227630596	0.963208892004	0.963161892361	0.885923553202	0.783730158730
7	1.004227630669	1.004227630666	0.963209728694	0.963231646825	0.886262535450	1.069444444444
8	1.004227630669	1.004227630669	0.963209609365	0.963198586116	0.886148086455	0.527777777778
9	1.004227630669	1.004227630669	0.963209627248	0.963215540326	0.886188283668	1.638888888889
10	1.004227630669	1.004227630669	0.963209624564	0.963206276061	0.886174256708	−0.789682539683
11	1.004227630669	1.004227630669	0.963209624981	0.963211610032	0.886179300167	4.803391053391
12	1.004227630669	1.004227630669	0.963209624917	0.963208401315	0.886177503402	−8.654942279943
13	1.004227630669	1.004227630669	0.963209624927	0.963210404948	0.886178158992	24.960442335442

It should be noted that analogous results can be observed in cases (a)–(c) given at the end of Section 2.3.

Finally, we consider the approximation of functions by constructed expansions.

The dilogarithm is the function defined by the power series (see, for example, [55])

$$\mathrm{Li}_2(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^2} \text{ for } |z| < 1$$

with an analytic continuation given by

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-u)}{u} du \text{ for } z \in \mathbb{C} \setminus [1, +\infty). \tag{47}$$

In [56], it is shown that

$$\mathrm{Li}_2(z) = \frac{z}{1+} \frac{a_2 z}{1} \frac{a_3 z}{+ \ 1} + \dots \text{ for } z \in \mathbb{C} \setminus [1, +\infty), \tag{48}$$

where

$$a_{2n} = \frac{A_n^{(1)} A_{n-1}^{(0)}}{A_n^{(0)} A_{n-1}^{(1)}},\ a_{2n+1} = \frac{A_{n-1}^{(1)} A_{n+1}^{(0)}}{A_n^{(0)} A_n^{(1)}},\ A_n^{(r)} = \det \left\| \frac{(-1)^{i+j+r}}{(r+i+j-1)^2} \right\|_{1 \le i,j \le n}.$$

In addition, $\mathrm{Li}_2(z)$ can be expressed as (see, ([57], Section 2.6))

$$\mathrm{Li}_2(z) = z\,{}_3F_2(1,1,1;2,2;z). \tag{49}$$

It follows from Corollary 1 that in the region (37) the function $\mathrm{Li}_2(z)$ is represented by a branched continued fraction of the form

$$\cfrac{z}{1 - \cfrac{c_{2,1}^{1,1} z}{1 - \cfrac{c_{2,1,1,1}^{1,1} z}{1 - \cfrac{c_{2,1,1,1,2,1}^{1,1} z}{1 - \ddots}} - \cfrac{c_{2,1,2,2}^{1,1} z}{1 - \cfrac{c_{2,1,2,2,1,2}^{1,1} z}{1 - \ddots} - \cfrac{c_{2,1,2,2,2,1}^{1,1} z}{1 - \ddots}}}}, \tag{50}$$

where $c^{1,1}_{(ij)_k}$, $(ij)_k \in \mathcal{I}_{1,1}$, defined by formulas (15)–(20), where $a_1 = 0$ and b_1 replaced by $b_1 - 1$.

Plots of the values of the nth approximants of the branched continued fraction (50) are shown in Figure 1. Here we can see the so-called 'fork property' for a branched continued fraction with positive elements (see [18] (p. 29)). That is, the plots of the values of even (odd) approximations of (50) approaches from above (below) to the plot of the function $\mathrm{Li}_2(z)$.

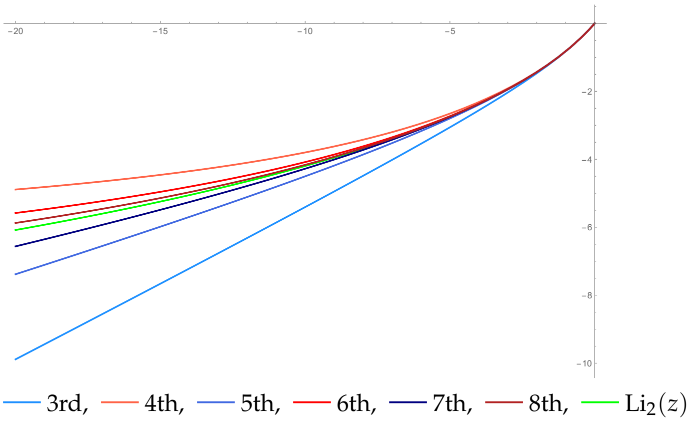


Figure 1. The plots of values of the nth approximants of (50) for $\mathrm{Li}_2(z)$.

Figure 2 shows the plots where the approximant $f_8^{1,1}(z)$ of the branched continued fraction (50) guarantees certain truncation error bounds for function $\mathrm{Li}_2(z)$.

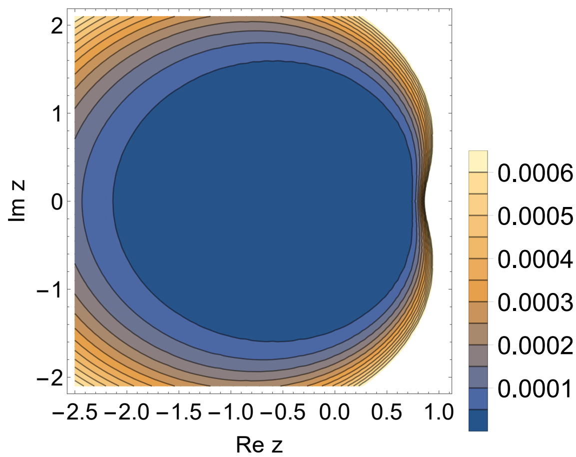


Figure 2. The plots where the approximant $f_8^{1,1}(z)$ of (50) guarantees certain truncation error bounds for $\mathrm{Li}_2(z)$.

The numerical illustration of (48)–(50) is given in the Table 2. Here we compare the relative errors of the approximation of function (47) by the partial sums of the power series and the approximants of the continued fraction and the branched continued fraction. As a results, the nth approximant of (50) is eventually a better approximation to (47) than the nth partial sum of (49) and the nth approximant of (48) is.

Table 2. Relative error of 5th partial sum and 5th approximants for $\mathrm{Li}_2(z)$.

z	**(47)**	**(49)**	**(48)**	**(50)**
0.2	0.2110037754	9.88869×10^{-6}	3.84790×10^{-7}	3.79061×10^{-7}
0.1	0.1026177911	2.92233×10^{-7}	9.50960×10^{-9}	9.36538×10^{-9}
-0.2	-0.1908001378	8.12951×10^{-6}	1.64637×10^{-7}	1.62051×10^{-7}
-0.4	-0.3658325775	2.40912×10^{-4}	3.70070×10^{-6}	3.64181×10^{-6}
-0.5	-0.4484142069	7.10129×10^{-4}	9.59022×10^{-6}	9.43694×10^{-6}
-0.7	-0.6051584023	3.58461×10^{-3}	3.80158×10^{-5}	3.74047×10^{-5}
-0.9	-0.7521631792	1.18992×10^{-2}	1.00919×10^{-4}	9.92914×10^{-5}
-5	-2.749279126	$3.48554 \times 10^{+1}$	1.83416×10^{-2}	1.80702×10^{-2}
-25	-6.785907900	$5.42019 \times 10^{+4}$	2.91475×10^{-1}	2.88304×10^{-1}

In [58], it is given that function

$$\operatorname{arcsinh}^2 \sqrt{z} = \ln^2(\sqrt{z} + \sqrt{z+1}) \tag{51}$$

(here the principal branch of the square root is assumed) has a generalized hypergeometric series in the form

$$z\,{}_3F_2(1,1,1;3/2,2;-z) = \sum_{n=0}^{\infty} \frac{(-1)^n((1)_n)^3}{(3/2)_n(2)_n}\frac{z^{n+1}}{n!}. \tag{52}$$

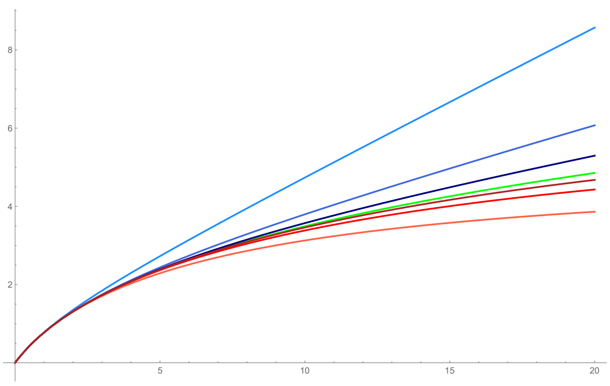
So, it follows from Corollary 1 that the branched continued fraction

$$\cfrac{z}{1+\cfrac{z/3}{1+\cfrac{z/15}{1+\cfrac{2z/15}{1+\ddots}}+\cfrac{8z/21}{1+\cfrac{4z/75}{1+\ddots}+\cfrac{3z/7}{1+\ddots}}}} \tag{53}$$

is an analytic continuation of function (51) in the region

$$H_\varepsilon = \{z \in \mathbb{C} : \ |\arg(z)| \leq \pi/(1+\varepsilon)\},\ 0 < \varepsilon < 1.$$

Plots of the values of the nth approximants of the branched continued fraction (53) for $\operatorname{arcsinh}^2 \sqrt{z}$ are shown in Figure 3.



3rd, 4th, 5th, 6th, 7th, 8th, $\operatorname{arcsinh}^2 \sqrt{z}$

Figure 3. The plots of values of the nth approximants of (53) for $\operatorname{arcsinh}^2 \sqrt{z}$.

Figure 4 shows the plots where the approximant $f_8^{1,1}(z)$ of the branched continued fraction (53) guarantees certain truncation error bounds for function $\operatorname{arcsinh}^2 \sqrt{z}$.

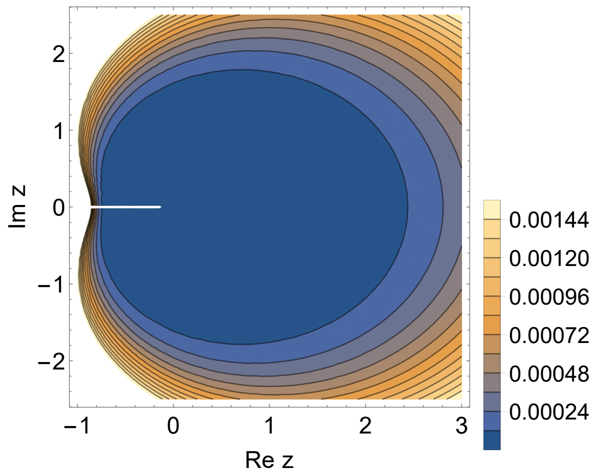


Figure 4. The plots where the approximant $f_8^{1,1}(z)$ of (53) guarantees certain truncation error bounds for $\operatorname{arcsinh}^2 \sqrt{z}$.

Finally, a numerical illustration of (52) and (53) is given in the Table 3.

Table 3. Relative error of 5th partial sum and 5th approximant for $\operatorname{arcsinh}^2 \sqrt{z}$.

z	**(51)**	**(52)**	**(53)**
0.1	0.09683377	5.89297×10^{-7}	9.38113×10^{-9}
0.2	0.18792863	1.81135×10^{-5}	2.44604×10^{-7}
0.4	0.35575900	5.39204×10^{-4}	5.37787×10^{-6}
0.7	0.57918140	8.01911×10^{-3}	5.37870×10^{-5}
0.9	0.71337919	2.62446×10^{-2}	1.40695×10^{-4}
1	0.77681940	4.26521×10^{-2}	2.07391×10^{-4}
3	1.73437810	8.86578×10^{-1}	6.54173×10^{-3}
10	3.49148329	9.99511×10^{-1}	8.03203×10^{-2}
50	7.04436144	9.99999×10^{-1}	4.29712×10^{-1}

Here we have results like to the results in the previous example.

4. Discussion

In [26], the authors constructed branched continued fraction expansions for some ratios of the generalized hypergeometric function ${}_3F_2$. In this work, we have constructed new expansions and investigated their convergence in some region together with the already known ones. This allows us to approximate the solutions of certain differential equations and also analytic functions, represented by generalized hypergeometric function ${}_3F_2$, using branched continued fractions. The result is a generalization of the classical continued fraction expansions of Gauss's hypergeometric function ratios.

Compared with power series or multiple power series under certain conditions, branched continued fractions have wider convergence regions and are endowed with the property of numerical stability. This makes them an effective tool for rational approximation in the theory approximation. Studying the branched continued fractions is to develop new and effective methods for establishing convergence criteria and finding estimates of the rate of convergence.

Despite the fact that the established convergence region for the constructed expansions is wider than the convergence region of the corresponding generalized hypergeometric function ${}_3F_2$, the problem of studying wider convergence regions and establishing estimates of the of convergence rate of the expansions mentioned above still remains open.

The proposed methods for constructing and studying the branched continued fraction expansions of the ratios of generalized hypergeometric function ${}_3F_2$ can also be applied to construct the expansions of other relations of generalizations of the Gauss hypergeometric function. This, in turn, will allow the use of branched continued fractions to approximate the solutions of some differential equations and their system, which can be used in applied problems in physics, astronomy, economics, etc. Here it is appropriate to mention one of the interesting applications of continued fractions in modelling the birth-death processes in the works [59–61]. Finally, we point to the works [62–64], where is no less interesting application of continued and branched continued fractions in chemical graph theory.

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