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Oscillation and Asymptotic Properties of Differential Equations of Third-Order

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Abstract: The main purpose of this study is aimed at developing new criteria of the iterative nature to test the asymptotic and oscillation of nonlinear neutral delay differential equations of third order with noncanonical operator $(a(\iota)[(b(\iota)[x(\iota)+p(\iota)x(\iota-\tau)]')']^{\beta})' + \int_c^d q(\iota,\mu)x^{\beta}(\sigma(\iota,\mu))\,d\mu = 0$, where $\iota \geq \iota_0$ and $w(\iota) := x(\iota)+p(\iota)x(\iota-\tau)$. New oscillation results are established by using the generalized Riccati technique under the assumption of $\int_{\iota_0}^{\iota} a^{-1/\beta}(s)ds < \int_{\iota_0}^{\iota} \frac{1}{b(s)}ds = \infty$ as $\iota \to \infty$. Our new results complement the related contributions to the subject. An example is given to prove the significance of new theorem.

Keywords: oscillation; third-order; neutral differential equation; Riccati transformation; distributed deviating arguments

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1. Introduction

The objective of this paper is to provide oscillation theorems for the third order equation as follows:

$$\left(a(\iota)\left[\left(b(\iota)[x(\iota)+p(\iota)x(\iota-\tau)]'\right)'\right]^{\beta}\right)' + \int_c^d q(\iota,\mu)x^{\beta}(\sigma(\iota,\mu))\,d\mu = 0, \tag{1}$$

where $a(\iota), b(\iota), p(\iota), q(\iota) \in C([\iota_0,+\infty))$, $a(\iota), b(\iota) > 0$, $a'(\iota) \geq 0$, $q(\iota) \geq 0$, $\beta \geq 1$ and $0 \leq p(\iota) \leq p_0 \leq 1$. The main results are obtained under the following assumptions:

(A_1) $q(\iota,\mu) \in C([\iota_0,+\infty) \times [c,d], [0,+\infty))$ and $q(\iota,\mu)$ does not vanish identically for any half line $[\iota_*,+\infty) \times [c,d]$, $\iota_* \geq \iota$;

(A_2) $\sigma(\iota,\mu) \in C([\iota_0,+\infty) \times [c,d], [0,+\infty))$, $\sigma(\iota,\mu)+\tau \leq \iota$, $\sigma(\iota,\mu)$ is nondecreasing with respect to ι and μ respectively, $\liminf_{\iota\to+\infty} \sigma(\iota,\mu) = \infty$ and $\liminf_{\iota\to+\infty} \tau(\iota) = \infty$.

We set

$$w(\iota) := x(\iota) + p(\iota)x(\iota-\tau)$$

and

$$A(\iota,\iota_0) = \int_{\iota_0}^{\iota} a^{-1/\beta}(s)ds, \quad B(\iota,\iota_0) = \int_{\iota_0}^{\iota} \frac{1}{b(s)}ds.$$

We intend that for a solution of (1), we mean a function $x(\iota) \in C([T_x,\infty))$, $T_x \geq \iota_0$, which has the property $w' \in C^1([T_x,\infty))$, $b\,w' \in C^1([T_x,\infty))$, $a\left((b\,w')'\right)^{\beta} \in C^1([T_x,\infty))$ and satisfies (1) on $[T_x, \infty)$. We only consider those solutions x of (1) which satisfy $\sup\{|x(\iota)| : \iota \geq T\} > 0$ for all $T \geq T_x$. We start with the assumption that Equation (1) does possess a proper solution. A proper solution of Equation (1) is called oscillatory if it has a sequence of large zeros lending to ∞; otherwise, we call it non-oscillatory.

Neutral/delay differential equations of the third order are used in a variety of problems in economics, biology, and physics, including lossless transmission lines, vibrating masses attached to an elastic bar, and as the Euler equation in some variational problems; see Hale [1]. As a result, there is an ongoing interest in obtaining several sufficient conditions for the oscillation or non-oscillation of the solutions of different kinds of differential equations; see [2–24] as examples of instant results on this topic.

However, to the best of our knowledge, only a few papers have studied the oscillation of nonlinear neutral delay differential equations of third order with distributed deviating arguments; see, for example, [2–5]. Recently, Haifei Xiang [6] and Haixia Wang et. al [7] studied the oscillatory behavior of Equation (1) under the following assumption:

$$A(\iota, \iota_0) = \infty, \quad B(\iota, \iota_0) = \infty \text{ as } \iota \to \infty.$$

Motivated by this above observation, in this paper, we extend the results under the following assumption:

$$A(\iota, \iota_0) < B(\iota, \iota_0) = \infty \quad \text{as} \quad \iota \to \infty. \tag{2}$$

Motivated by these reasons mentioned above, in this paper, we extend the results using generalized Riccati transformation and the integral averaging technique. We establish criteria for Equation (1) to be oscillatory or converge to zero asymptotically with the assumption of (2). As is customary, all observed functional inequalities are assumed to support eventually; that is, they are satisfied for all ι that are large enough.

2. Main Results

For our further reference, let us denote the following:

$$E_0 w = w, \quad E_1 w = b(E_0 w)', \quad E_2 w = a((E_1 w)')^\beta, \quad E_3 w = (E_2 w)', \tag{3}$$

and

$$C(\iota, \iota_0) = \int_{\iota_2}^{\iota} \frac{A(s, \iota_1)}{b(s)} ds, \quad D(\iota) := \int_{\iota}^{\infty} \frac{1}{a^{1/\beta}(s)} ds, \quad \sigma_1(\iota) = \sigma(\iota, c),$$
$$q_1(\iota) = (1 - p_0)^\beta \int_c^d q(\iota, \mu)\, d\mu, \quad \varphi'_+(\iota) = \max\{0, \varphi'(\iota)\}.$$

Theorem 1. *Assume $(A_1) - (A_2)$ and (2) hold. If there exists a $\varphi \in C^1([\iota_0, \infty), \mathbb{R})$, such that $\iota_1 \geq \iota_0$ for some $\iota_m > \iota_1$, we have the following:*

$$\limsup_{\iota \to \infty} \int_{\iota_3}^{\iota} \left(\varphi(s) q_1(s) \frac{C^\beta(\sigma_1(s), \iota_2)}{A^\beta(\sigma_1(s), \iota_1)} - \frac{1}{(1+\beta)^{1+\beta}} \frac{a(s)(\varphi'_+(s))^{1+\beta}}{\varphi^\beta(s)} \right) ds = \infty, \tag{4}$$

$$\int_{\iota_4}^{\infty} b^{-1}(v) \int_v^{\infty} \left(a^{-1}(u) \int_u^{\infty} \int_c^d q(s, \mu) d\mu\, ds \right)^{1/\beta} du\, dv = \infty, \tag{5}$$

and

$$\limsup_{\iota \to \infty} \int_{\iota_6}^{\iota} \left(D^\beta(s) q_1(s) B^\beta(\sigma_1(s), \iota_5) - \left(\frac{\beta}{\beta+1} \right)^{\beta+1} \frac{1}{D^\beta(s) a^{1/\beta}(s)} \right) ds = \infty, \tag{6}$$

Then, every solution $x(\iota)$ of (1) is either oscillatory or tends to 0.

Proof. Suppose that (1) has a non-oscillatory solution x. Now, we may take $x(\iota) > 0$, $x(\iota - \tau) > 0$ and $x(\sigma(\iota, \mu)) > 0$ for $\iota \geq \iota_1$ some $\iota_1 \geq \iota_0$ and $\mu \in [c, d]$. By condition (2), there exist three possible cases:

(I) $w(\iota) > 0, w'(\iota) > 0, (b(\iota)w'(\iota))' > 0, \left(a(\iota)[(b(\iota)w'(\iota))']^\beta\right)' < 0,$

(II) $w(\iota) > 0, w'(\iota) < 0, (b(\iota)w'(\iota))' > 0, \left(a(\iota)[(b(\iota)w'(\iota))']^\beta\right)' < 0,$ or

(III) $w(\iota) > 0, w'(\iota) > 0, (b(\iota)w'(\iota))' < 0, \left(a(\iota)[(b(\iota)w'(\iota))']^\beta\right)' < 0,$ for $\iota \geq \iota_1$, ι_1 is large enough.

Assume first the case (I) holds for $\iota \geq \iota_2$. From the definition of $w(\iota)$, $w(\iota) \geq x(\iota)$ for $\iota \geq \iota_2$ and

$$w(\sigma(\iota, \mu)) \geq w(\sigma(\iota, \mu) - \tau) \geq x(\sigma(\iota, \mu) - \tau), \quad \iota \geq \iota_3 \geq \iota_2, \tag{7}$$

and

$$x(\iota) = w(\iota) - p(\iota)x(\iota - \tau) \geq w(\iota) - p(\iota)w(\iota - \tau) \geq (1 - p_0)w(\iota). \tag{8}$$

Thus from (1) and (8), we have the following:

$$\begin{aligned} E_3 w(\iota) &= -\int_c^d q(\iota, \mu) x^\beta(\sigma(\iota, \mu))\, d\mu \\ &\leq -(1 - p_0)^\beta \int_c^d q(\iota, \mu) w^\beta(\sigma(\iota, \mu))\, d\mu \\ &\leq -(1 - p_0)^\beta w^\beta(\sigma(\iota, c)) \int_c^d q(\iota, \mu)\, d\mu \\ &= -q_1(\iota) w^\beta(\sigma_1(\iota)). \end{aligned} \tag{9}$$

Using the fact that $w'(\iota) > 0$, we have the following:

$$E_1 w(\iota) \geq \int_{\iota_1}^{\iota} \frac{a^{1/\beta}(s)(E_1 w(s))'}{a^{1/\beta}(s)} ds \geq a^{1/\beta}(\iota)(E_1 w(\iota))' A(\iota, \iota_1).$$

Thus, we have the following:

$$w(\iota) = w(\iota_2) + \int_{\iota_2}^{\iota} \frac{E_1 w(s)}{A(s, \iota_1)} \frac{A(s, \iota_1)}{b(s)} ds \geq \frac{E_1 w(\iota)}{A(\iota, \iota_1)} \int_{\iota_2}^{\iota} \frac{A(s, \iota_1)}{b(s)} ds.$$

Then, we have the following:

$$\frac{w(\sigma_1(\iota))}{E_1 w(\sigma_1(\iota))} \geq \frac{C(\sigma_1(\iota), \iota_2)}{A(\sigma_1(\iota), \iota_1)}, \tag{10}$$

and

$$\frac{E_1 w(\sigma_1(\iota))}{E_1 w(\iota)} \geq \frac{A(\sigma_1(\iota), \iota_1)}{A(\iota, \iota_1)} \tag{11}$$

We define a function as follows:

$$\psi(\iota) := \varphi(\iota) \frac{E_2 w(\iota)}{E_1^\beta w(\iota)}, \tag{12}$$

and note that $\psi(\iota) > 0$ for $\iota \geq \iota_1$. Differentiating (12), we obtain the following:

$$\psi'(\iota) = \frac{\varphi'(\iota)}{\varphi(\iota)} \psi(\iota) + \varphi(\iota) \frac{E_3 w(\iota)}{E_1^\beta w(\iota)} - \beta \varphi(\iota) a(\iota) \left[\frac{(E_1 w(\iota))'}{E_1 w(\iota)} \right]^{\beta+1}. \tag{13}$$

It follows from (9), (12), and (13) that the following holds:

$$\begin{aligned}
\psi'(\iota) &\leq \frac{\varphi'(\iota)}{\varphi(\iota)}\psi(\iota) - \varphi(\iota)q_1(\iota)\left(\frac{w(\sigma(\iota))}{E_1 w(\iota)}\right)^{\beta} - \beta\frac{\psi^{\frac{(\beta+1)}{\beta}}(\iota)}{[\varphi(\iota)a(\iota)]^{1/\beta}} \\
&= \frac{\varphi'(\iota)}{\varphi(\iota)}\psi(\iota) - \varphi(\iota)q_1(\iota)\left(\frac{w(\sigma_1(\iota))}{E_1 w(\sigma_1(\iota)))}\frac{E_1 w(\sigma_1(\iota)))}{E_1 w(\iota))}\right)^{\beta} - \beta\frac{\psi^{\frac{(\beta+1)}{\beta}}(\iota)}{[\varphi(\iota)a(\iota)]^{1/\beta}}. \quad (14)
\end{aligned}$$

Now, (10) and (14) implies the following:

$$\psi'(\iota) \leq \frac{\varphi'_+(\iota)}{\varphi(\iota)}\psi(\iota) - \beta\frac{\psi^{\frac{(\beta+1)}{\beta}}(\iota)}{[\varphi(\iota)a(\iota)]^{1/\beta}} - \varphi(\iota)q_1(\iota)\frac{C^{\beta}(\sigma_1(\iota),\iota_2)}{A^{\beta}(\sigma_1(\iota),\iota_1)}. \quad (15)$$

Then, using (15) and inequality, we have the following:

$$Bu - Au^{(m+1)/m} \leq \frac{m^m}{(m+1)^{m+1}}\frac{B^{m+1}}{A^m}. \quad (16)$$

We find the following:

$$\psi'(\iota) \leq -\varphi(\iota)q_1(\iota)\frac{C^{\beta}(\sigma_1(\iota),\iota_2)}{A^{\beta}(\sigma_1(\iota),\iota_1)} + \frac{1}{(1+\beta)^{1+\beta}}\frac{a(\iota)(\varphi'_+(\iota))^{1+\beta}}{\varphi^{\beta}(\iota)}.$$

Integrating the last inequality from ι_3 $(> \iota_2)$ to ι gives

$$\limsup_{\iota\to\infty}\int_{\iota_3}^{\iota}\left(\varphi(s)q_1(s)\frac{C^{\beta}(\sigma_1(s),\iota_2)}{A^{\beta}(\sigma_1(s),\iota_1)} - \frac{1}{(1+\beta)^{1+\beta}}\frac{a(s)(\varphi'_+(s))^{1+\beta}}{\varphi^{\beta}(s)}\right)ds \leq \psi(\iota_3), \quad (17)$$

which contradicts (4).

Next, if (II) holds. Since $w(\iota) > 0$ and $w'(\iota) < 0$, we have $w(\iota) \to l \geq 0$. If $L > 0$, then for $\epsilon = \frac{L(1-p_0)}{2p_0} > 0$, there exists $\iota_4 \geq \iota_1$ such that $L < w(\iota) < L + \epsilon$ for $\iota \geq \iota_4$. Then, for $\iota \geq \iota_4$, we have the following:

$$x(\iota) = w(\iota) - p(\iota)x(\iota - \tau) > L - p_0 w(\iota) > L - p_0(L+\epsilon) = L_1$$

Using the above inequality, which we obtained from (9), we have the following:

$$E_3 w(\iota) > -L_1^{\beta}\int_c^d q(\iota,\mu)\,d\mu$$

Integrating from $\iota(\geq \iota_4)$ to ∞ and using the fact that $a(\iota)\left[\left(b(\iota)w'(\iota)\right)'\right]^{\beta}$ is positive and decreasing, we obtain the following:

$$(E_1 w(\iota))' \geq L_1\left(\frac{1}{a(\iota)}\int_{\iota}^{\infty}\int_c^d q(s,\mu)d\mu ds\right)^{1/\beta}.$$

Again integrating the following,

$$E_1 w(\iota) \geq -L_1\int_{\iota}^{\infty}\left(\frac{1}{a(u)}\int_{\iota}^{\infty}\int_c^d q(s,\mu)d\mu\,ds\right)^{1/\beta}du,$$

and again with the integration from ι_4 to ∞, we obtain the following:

$$w(\iota_4) \geq L_1\int_{\iota_4}^{\infty}\frac{1}{b(v)}\int_{\iota}^{\infty}\left(\frac{1}{a(u)}\int_{\iota}^{\infty}\int_c^d q(s,\mu)d\mu\,ds\right)^{1/\beta}du\,dv,$$

which contradicts (5) and shows that $L = 0$, i.e., $w(\iota) \to 0$. Since $0 < x(\iota) < w(\iota)$, we have $x(\iota) \to 0$ as $\iota \to \infty$.

Finally, assume that case (III) holds, $E_3 w(\iota) \leq 0$, and is non-increasing. Thus, we obtain the following:

$$E_2 w(s) \leq E_2 w(\iota), \quad s \geq \iota \geq \iota_5. \tag{18}$$

for some $\iota_5 \geq \iota_0$. Dividing (18) by $a(s)$ and integrating from ι to l, we obtain the following:

$$E_1 w(l) \leq E_1 w(\iota) + a^{1/\beta}(\iota) \, (E_1 w(\iota))' \, A(l, \iota).$$

Letting $l \to \infty$, we have the following:

$$-\frac{a^{1/\beta}(\iota)(E_1 w(\iota))'}{E_1 w(\iota)} D(\iota) \leq 1. \tag{19}$$

Define function ϕ by the following:

$$\phi(\iota) := \frac{E_2 w(\iota)}{E_1^\beta w(\iota)}, \quad \iota \geq \iota_5. \tag{20}$$

Then $\phi(\iota) < 0$ for $\iota \geq \iota_5$. Hence, from (19) and (20), we obtain the following:

$$-D^\beta(\iota)\phi(\iota) \leq 1. \tag{21}$$

Differentiating (20) gives the following:

$$\phi'(\iota) = \frac{E_3 w(\iota)}{E_1^\beta w(\iota)} - \beta a(\iota) \left[\frac{(E_1 w(\iota))'}{E_1 w(\iota)} \right]^{\beta+1}.$$

Now $w'(\iota) > 0$, so from (9) and (20), we have the following:

$$\phi'(\iota) \leq -q_1(\iota) \left[\frac{w(\sigma_1(\iota))}{E_1 w(\iota)} \right]^\beta - \beta \frac{\phi^{1+\frac{1}{\beta}}(\iota)}{a^{1/\beta}(\iota)}. \tag{22}$$

In case (III), we see that the following holds:

$$w(\iota) \geq b(\iota) w'(\iota) \int_{\iota_5}^{\iota} \frac{ds}{b(s)} = E_1 w(\iota) \, B(\iota, \iota_5). \tag{23}$$

Hence

$$\left[\frac{w(\iota)}{B(\iota, \iota_5)} \right]' \leq 0,$$

which implies the following:

$$\frac{w(\sigma_1(\iota))}{w(\iota)} \geq \frac{B(\sigma_1(\iota), \iota_5)}{B(\iota, \iota_5)}. \tag{24}$$

Using (23) and (24) in (22), we obtain the following:

$$\phi'(\iota) \geq -q_1(\iota) B^\beta(\sigma_1(\iota), \iota_5) - \beta \frac{\phi^{1+\frac{1}{\beta}}(\iota)}{a^{1/\beta}(\iota)}. \tag{25}$$

Hence from (25), we have the following:

$$\phi(\iota)D^{\beta}(\iota)-\phi(\iota_6)D^{\beta}(\iota_6)\le-\int_{\iota_6}^{\iota}q_1(s)D^{\beta}(s)B^{\beta}(\sigma_1(s),\iota_5)ds \\ -\int_{\iota_6}^{\iota}\beta\frac{D^{\beta-1}(s)\phi(s)}{a^{1/\beta}(s)}ds-\int_{\iota_6}^{\iota}\beta\frac{D^{\beta}(s)\phi^{1+\frac{1}{\beta}}(s)}{a^{1/\beta}(s)}ds,$$

or

$$\phi(\iota)D^{\beta}(\iota)-\phi(\iota_6)D^{\beta}(\iota_6)\le-\int_{\iota_6}^{\iota}q_1(s)D^{\beta}(s)B^{\beta}(\sigma_1(s),\iota_5)ds \\ -\int_{\iota_6}^{\iota}\beta\left[\frac{D^{\beta-1}(s)\phi(s)}{a^{1/\beta}(s)}+\frac{D^{\beta}(s)\phi^{1+\frac{1}{\beta}}(s)}{a^{1/\beta}(s)}\right]ds. \quad (26)$$

Set $\phi:=-u(s)$ and using inequality

$$Au^{(\beta+1)/\beta}-Bu\le-\frac{\beta^{\beta}}{(\beta+1)^{\beta+1}}\frac{B^{\beta+1}}{A^{\beta}},\quad A>0,$$

we obtain the following:

$$\int_{\iota_6}^{\iota}\left[q_1(s)D^{\beta}(s)B^{\beta}(\sigma_1(s),\iota_5)-\left(\frac{\beta}{\beta+1}\right)^{\beta+1}\frac{1}{D^{\beta}(s)a^{1/\beta}(s)}\right]ds\le-\phi(\iota)D^{\beta}(\iota)+\phi(\iota_6)D^{\beta}(\iota_6) \quad (27)$$

Using (21) in (26) and then taking $\iota\to\infty$, we obtain the following:

$$\int_{\iota_6}^{\infty}\left[q_1(s)D^{\beta}(s)B^{\beta}(\sigma_1(s),\iota_5)-\left(\frac{\beta}{\beta+1}\right)^{\beta+1}\frac{1}{D^{\beta}(s)a^{1/\beta}(s)}\right]ds\le 1+\phi(\iota_6)D^{\beta}(\iota_6)$$

which contradicts (6). This completes the proof. □

We will present an example to illustrate the main results.

Example 1. *Consider the following 3rd-order equation:*

$$\left(\iota^2[x(\iota)+p(\iota)x(\iota-\pi)]''\right)'+\int_{\pi}^{3\pi/2}x(\iota-\mu)d\mu=0. \quad (28)$$

where $a(\iota)=\iota^2$, $b(\iota)=1$, $\tau(\iota)=\iota-\pi$, $p(\iota)=1$, $\sigma(\iota,\mu)=\iota-\mu$, $\beta=1$, $a=\pi$, $b=3\pi/2$. *Moreover* $0<p(\iota)\le p_0$ *and* $\varphi(\iota)=1$. *Then, we obtain the following:* $q_1(\iota)=(1-p_0)\pi/2$, $D(\iota)=1/\iota$, $\int_{\iota_5}^{\iota-\mu}\frac{ds}{b(s)}=\iota-\mu-\iota_5$. *The condition* (4) *becomes the following:*

$$\int_{\iota_3}^{\infty}\Phi(s)ds=\frac{\pi(1-p_0)}{2}\int_{\iota_3}^{\infty}\frac{\frac{\iota_1}{\iota_2}(\iota-\mu)^2-(\iota-\mu)(\iota_1\log(\iota-\mu)+\iota_1\log\iota_2-\iota_2)}{(\iota-\iota_1-\mu)}=\infty,$$

and

$$\int_{\iota_6}^{\infty}\left(D^{\beta}(s)q_1(s)\int_{\iota_5}^{\sigma_1(\iota)}\frac{dv}{b(v)}-\frac{1}{D(s)a^{1/\beta}(s)}\right)ds \\ =\frac{\pi(1-p_0)}{2}\int_{\iota_6}^{\infty}\left(1-\frac{(\mu-\iota_5-1)}{s}\right)ds=\infty,$$

so condition (6) *also holds. Hence, by Theorem* (1), *it holds that every solution x of* (28) *is almost oscillatory.*

3. A Concluding Remark

We established new oscillation theorems for (1) in this paper. The main outcomes are proved via the means of the integral averaging condition, and the generalized Riccati technique under the assumptions of $\int_{\iota_0}^{\iota} a^{-1/\beta}(s)ds < \int_{\iota_0}^{\iota} \frac{1}{b(s)}ds = \infty$ as $\iota \to \infty$. Examples are given to prove the significance of the new results. The main results in this paper are presented in an essentially new form and of a high degree of generality. For future consideration, it will be of great importance to study the oscillation of (1) when $-\infty < p(\iota) \leq -1$ and $|p(\iota)| < \infty$.

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References

1. Hale, J.K. *Theory of Functional Differential Equations*; Springer: New York, NY, USA, 1977.
2. Sathish Kumar, M.; Ganesan,V. Asymptotic behavior of solutions of third-order neutral differential equations with discrete and distributed delay. *AIMS Math.* **2020**, *5*, 3851–3874. [CrossRef]
3. Sathish Kumar, M.; Bazighifan, O.; Almutairi, A.; Chalishajar, D.N. Philos-Type Oscillation Results for Third-Order Differential Equation with Mixed Neutral Terms. *Mathematics* **2021**, *9*, 1021. [CrossRef]
4. Tian, Y.; Cai, Y.; Fu, Y.; Li, T. Oscillation and asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. *Adv. Differ. Eq.* **2015**, *2015*, 267. [CrossRef]
5. Fu, Y.; Tian, Y.; Jiang, C.; Li, T. On the Asymptotic Properties of Nonlinear Third-Order Neutral Delay Differential Equations with Distributed Deviating Arguments. *J. Funct. Spaces* **2016**, *2016*, 3954354. [CrossRef]
6. Xiang, H.; Oscillation of third-order nonlinear neutral differential equations with distributed time delay. *Ital. J. Pure Appl. Math.* **2016**, *36*, 769–782.
7. Wang, H.; Chen, G.; Jiang, Y.; Jiang, C.; Li, T. Asymptotic behavior of third-order neutral differential equations with distributed deviating arguments. *J. Math. Comput. Sci.* **2017**, *17*, 194–199. [CrossRef]
8. Driver, R.D. A mixed neutral system. *Nonlinear Anal. Theory Methods Appl.* **1984**, *8*, 155–158. [CrossRef]
9. Angelov, V.G. On asymptotic behavior of solutions of third order neutral differential equations. In Proceedings of the International Conference VSU'2013, Sofia, Bulgaria, 6–7 June 2013; Volume I, pp. 112–116.
10. Angelov, V.G.; Bainov, D.D. Bounded solutions of functional differential equations of the neutral type with infinite delays, *Proc. Royal Soc. Edinburgh.* **1982**, *93A*, 33–39. [CrossRef]
11. Graef, J.R.; Savithri, R.; Thapani, E. Oscillatory properties of third order neutral delay differential equations. *Discret. Contin. Dyn. Syst.* **2003**, 342–350. [CrossRef]
12. Baculíková, B.; Džurina, J. Oscillation of third-order neutral differential equations. *Math. Comput. Model.* **2010**, *52*, 215–226. [CrossRef]
13. Elayaraja, R.; Kumar, M.S.; Ganesan, V. Nonexistence of Kneser solution for third order nonlinear neutral delay differential equations. *J. Phys. Conf. Ser.* **2021**, *1850*, 012054. [CrossRef]
14. Han, Z.; Li, T.; Sun, S.; Zhang, C. Oscillation behavior of third-order neutral Emden-Fowler delay dynamic equations on time scales. *Adv. Differ. Eq.* **2010**, 586312. [CrossRef]
15. Thandapani, E.; Vijaya, M.; Li, T. On the oscillation of third order half-linear neutral type difference equations. *Electron. J. Qual. Theory Differ. Eq.* **2011**, *76*, 1–13. [CrossRef]
16. Ganesan, V.; Kumar, M.S. On the oscillation of a third order nonlinear differential equations with neutral type. *Ural Math. J.* **2017**, *3*, 122–129. [CrossRef]
17. Sathish Kumar, M.; Janaki, S.; Ganesan,V. Some new oscillatory behavior of certain third-order nonlinear neutral differential equations of mixed type. *Int. J. Appl. Comput. Math.* **2018**, *78*, 1–14. [CrossRef]

18. Qin, G.; Huang, C.; Xie, Y.; Wen, F. Asymptotic behavior for third-order quasi-linear differential equations. *Adv. Differ. Eq.* **2013**, *305*. [CrossRef]
19. Thandapani, E.; Li, T. On the oscillation of third-order quasi-linear neutral functional differential equations. *Arch. Math.* **2011**, *47*, 181–199.
20. Althobati, S.; Bazighifan, O.; Yavuz, M. Some Important Criteria for Oscillation of Non-Linear Differential Equations with Middle Term. *Mathematics* **2021**, *9*, 346. [CrossRef]
21. Agarwal, R.P.; Bazighifan, O.; Ragusa, M.A. Nonlinear Neutral Delay Differential Equations of Fourth-Order: Oscillation of Solutions. *Entropy* **2021**, *23*, 129. [CrossRef] [PubMed]
22. Bazighifan, O.; Alotaibi, H.; Mousa, A.A.A. Neutral Delay Differential Equations: Oscillation Conditions for the Solutions. *Symmetry* **2021**, *13*, 101. [CrossRef]
23. Baskonus, H.M.; Bulut, H.; Sulaiman, T.A. New Complex Hyperbolic Structures to the Lonngren-Wave Equation by Using Sine-Gordon Expansion Method. *Appl. Math. Nonlinear Sci.* **2019**, *4*, 129–138. [CrossRef]
24. Yel, G.; Aktürk, T. A New Approach to $(3+1)$ Dimensional Boiti-Leon-Manna-Pempinelli Equation. *Appl. Math. Nonlinear Sci.* **2020**, *5*, 309–316. [CrossRef]