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Article

Properties of Certain Multivalent Analytic Functions Associated with the Lemniscate of Bernoulli

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Abstract: Using differential subordination, we consider conditions of β so that some multivalent analytic functions are subordinate to $(1+z)^\gamma$ $(0<\gamma\leq 1)$. Notably, these results are applied to derive sufficient conditions for $f\in A$ to satisfy the condition $\left|\left(\frac{zf'(z)}{f(z)}\right)^2-1\right|<1$. Several previous results are extended.

Keywords: analytic function; differential subordination; starlike function; lemniscate of Bernoulli

MSC: 30C45

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1. Introduction

Let $A(p)$ denote the class of multivalent functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k,\quad (p\in N=\{1,2,3,\cdots\}) \tag{1}$$

which are analytic in the open unit disk $D=\{z\in C:|z|<1\}$. Additionally, let $A:=A(1)$.

For the two functions f and g analytic in D, the function f is said to be subordinate to g, written as $f(z)\prec g(z)$ $(z\in D)$, if there exists a function w analytic in D with $w(0)=0$ and $|w(z)|<1$, such that $f(z)=g(w(z))$. Notably, if g is univalent in D, then $f(z)\prec g(z)$ is equivalent to $f(0)=g(0)$ and $f(D)\subset g(D)$.

In [1] Sokól and Stankiewicz defined and studied the class

$$SL:=\left\{f\in A:\left|\left(\frac{zf'(z)}{f(z)}\right)^2-1\right|<1,\quad z\in D\right\}. \tag{2}$$

From (2), one can see that a function $f\in SL$ if $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli, given by $|w^2-1|<1$. All functions in SL are univalent starlike functions. Several authors ([2–5]) considered differential subordination for functions belonging to the class SL.

Recently, many scholars introduced and investigated various subclasses of multivalent analytic functions (see, e.g., [3–15] and the references cited therein). Some properties, such as distortion bounds, inclusion relations and coefficient estimates, were considered. In [16], Seoudy and Shammaky introduced a class of multivalently Bazilevič functions involving the Lemniscate of Bernoulli and obtained subordination properties, inclusion relationship, convolution result, coefficients estimate, and Fekete–Szegŏ problems for this class. In [14], Xu and Liu investigated some geometric properties of multivalent analytic functions associated with the lemniscate of Bernoulli and obtained a radius of starlikeness of the order ρ. In [2], Ali, Cho, Ravichandran and Kumar considered conditions on β so that $1+\beta zp'(z)$ subordinate to $\sqrt{1+z}$. Furthermore, Srivastava [8] carried out a systematic investigation of various analytic function classes associated with operators of q-calculus and fractional q-calculus. In this paper, we will consider conditions of β so that some multivalent analytic functions are subordinate to $(1+z)^\gamma$ $(0<\gamma\le 1)$, and derive several sufficient conditions of multivalent analytic functions associated with the lemniscate of Bernoulli. Some previous results are extended.

In order to prove our results, the following lemmas will be recalled.

Lemma 1 ([17])**.** *Let q be univalent in D, and let φ be analytic in a domain containing $q(D)$. Also let $\frac{zq'(z)}{\varphi(q(z))}$ be starlike. If ϕ is analytic in D, $\phi(0)=q(0)$ and satisfies*

$$z\phi'(z)\varphi(\phi(z)) \prec zq'(z)\varphi(q(z)),$$

then $\phi(z)\prec q(z)$, and q is the most dominant.

Lemma 2 ([17])**.** *Let q be univalent in the unit disk D, and let θ and φ be analytic in a domain containing $q(D)$ with $\varphi(w)\ne 0$ when $w\in q(D)$. Set $Q(z)=zq'(z)\varphi(q(z))$, $h(z)=\theta(q(z))+Q(z)$. Suppose that*

(1) either h is convex, or Q is starlike univalent in D, and

(2) $\mathrm{Re}\frac{zh'(z)}{Q(z)}>0$ for $z\in D$.

If ϕ is analytic in D, $\phi(0)=q(0)$ and satisfies

$$\theta(\phi(z)) + z\phi'(z)\varphi(\phi(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)),$$

then $\phi(z)\prec q(z)$, and q is the best dominant.

2. Main Results

Theorem 1. *Let $0<\gamma\le 1$, $\beta_0=\frac{2-2^{1-\gamma}}{\gamma}$ and $f\in A(p)$ with $f(z)\ne 0$ when $z\ne 0$. If f satisfies the subordination*

$$1+\beta\left[\frac{zf'(z)}{pf(z)}+\frac{z^2f''(z)}{pf(z)}-\frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^2\right]\prec(1+z)^\gamma,\quad \beta\ge\beta_0, \tag{3}$$

then $\frac{zf'(z)}{pf(z)}\prec(1+z)^\gamma$. The lower bound β_0 is sharp.

Proof. We first prove the following conclusion. If ϕ is analytic in D and $\phi(0)=1$, then

$$1+\beta z\phi'(z)\prec(1+z)^\gamma \Rightarrow \phi(z)\prec(1+z)^\gamma, \tag{4}$$

where $\beta\ge\beta_0$ and the lower bound β_0 is the best possible.

Define the function $q(z)=(1+z)^\gamma$ with $q(0)=1$. Then $q(z)$ is univalent in D. It can been seen that $zq'(z)$ is starlike. By Lemma 1, we observe that if $1+\beta z\phi'(z)\prec 1+\beta zq'(z)$, then $\phi(z)\prec q(z)$.

Next, we need only to prove $q(z)\prec 1+\beta zq'(z)$. Consider the function h by

$$h(z) := 1+\beta zq'(z) = 1+\frac{\beta\gamma z}{(1+z)^{1-\gamma}}\quad (z\in D).$$

Since $q^{-1}(w)=w^{\frac{1}{\gamma}}-1$, we obtain

$$q^{-1}(h(z))=\left(1+\frac{\beta\gamma z}{(1+z)^{1-\gamma}}\right)^{\frac{1}{\gamma}}-1.$$

For $z=e^{it}$, $t\in[-\pi,\pi]$, we have

$$|q^{-1}(h(z))| = |q^{-1}(h(e^{it}))| = \left| \left(1 + \frac{\beta\gamma e^{it}}{(1+e^{it})^{1-\gamma}}\right)^{\frac{1}{\gamma}} - 1 \right|.$$

The minimum of $|q^{-1}(h(e^{it}))|$ is obtained at $t = 0$. Thus

$$|q^{-1}(h(e^{it}))| \geq \left(1 + \frac{\beta\gamma}{2^{1-\gamma}}\right)^{\frac{1}{\gamma}} - 1 \geq 1,$$

provided $\beta \geq \frac{2-2^{1-\gamma}}{\gamma}$. Thus $h(D) \supset q(D)$. It follows that $q(z) \prec h(z)$, and the conclusion (4) is proved.

Now, we define the function ϕ by

$$\phi(z) = \frac{zf'(z)}{pf(z)},$$

then ϕ is analytic in D and $\phi(0) = 1$. By a simple calculation, we have

$$z\phi'(z) = \frac{zf'(z)}{pf(z)} + \frac{z^2 f''(z)}{pf(z)} - \frac{1}{p}\left(\frac{zf'(z)}{f(z)}\right)^2. \tag{5}$$

From (3)–(5), we obtain

$$\frac{zf'(z)}{pf(z)} \prec (1+z)^{\gamma}.$$

The proof of the theorem is completed. □

For $\gamma = \frac{1}{2}$ and $p = 1$, we have the following result, obtained in [2].

Corollary 1. *Let $\beta_0 = 2(2-\sqrt{2}) \approx 1.17$ and $f \in A$ with $f(z) \neq 0$ when $z \neq 0$. If f satisfies the subordination*

$$1 + \beta\left[\frac{zf'(z)}{f(z)} + \frac{z^2 f''(z)}{f(z)} - \left(\frac{zf'(z)}{f(z)}\right)^2\right] \prec \sqrt{1+z}, \quad \beta \geq \beta_0,$$

then $f \in SL$ or $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli. The lower bound β_0 is sharp.

Theorem 2. *Let $0 < \gamma \leq 1$, $\beta_0 = \frac{2(2^{\gamma}-1)}{\gamma}$ and $f \in A(p)$ with $f(z)f'(z) \neq 0$ when $z \neq 0$. If f satisfies the subordination*

$$1 + \beta\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec (1+z)^{\gamma}, \quad \beta \geq \beta_0, \tag{6}$$

then $\frac{zf'(z)}{pf(z)} \prec (1+z)^{\gamma}$. The lower bound β_0 is sharp.

Proof. We first derive the following conclusion:

$$1 + \beta\frac{z\phi'(z)}{\phi(z)} \prec (1+z)^{\gamma} \Rightarrow \phi(z) \prec (1+z)^{\gamma}, \tag{7}$$

where ϕ is analytic in D with $\phi(0) = 1$, $\beta \geq \beta_0$ and the lower bound β_0 is the best possible.

Let $q(z) = (1+z)^{\gamma}$ with $q(0) = 1$. We consider the subordination

$$1 + \frac{\beta z\phi'(z)}{\phi(z)} \prec 1 + \frac{\beta zq'(z)}{q(z)}.$$

This shows that

$$\frac{\beta z q'(z)}{q(z)} = \frac{\beta\gamma z}{1+z}$$

is starlike in D. By Lemma 1, we know that $\phi(z) \prec q(z)$.

Now, we define the function h by

$$h(z) := 1 + \frac{\beta z q'(z)}{q(z)} = 1 + \frac{\beta\gamma z}{1+z} \quad (z \in D).$$

Since

$$h(D) = \left\{ w : \mathrm{Re} w < 1 + \frac{\beta\gamma}{2} \right\}$$

and

$$q(D) \subset \{ w : \mathrm{Re} w < 2^{\gamma} \},$$

this shows that $q(D) \subset h(D)$ if $2^{\gamma} \leq 1 + \frac{\beta\gamma}{2}$. Hence, $q(z) \prec h(z)$ for $\beta \geq \frac{2(2^{\gamma}-1)}{\gamma}$, and conclusion (7) is proved.

Define the function ϕ by

$$\phi(z) = \frac{z f'(z)}{p f(z)},$$

then, ϕ is analytic in D and $\phi(0) = 1$. A simple calculation shows that

$$\frac{z\phi'(z)}{\phi(z)} = 1 + \frac{z f''(z)}{f(z)} - \frac{z f'(z)}{f(z)}. \tag{8}$$

From (6)–(8), we obtain

$$\frac{z f'(z)}{p f(z)} \prec (1+z)^{\gamma}.$$

Now, we complete the proof of Theorem 2. □

For $\gamma = \frac{1}{2}$ and $p = 1$, we obtain the following result, given in [2].

Corollary 2. *Let $\beta_0 = 4(\sqrt{2}-1) \approx 1.65$ and $f \in A$ with $f(z)f'(z) \neq 0$ when $z \neq 0$. If f satisfies the subordination*

$$1 + \beta\left(1 + \frac{z f''(z)}{f'(z)} - \frac{z f'(z)}{f(z)}\right) \prec \sqrt{1+z}, \quad \beta \geq \beta_0,$$

then $f \in SL$ or $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli. The lower bound β_0 is sharp.

Theorem 3. *Let $0 < \gamma \leq 1$, $\beta_0 = \frac{2^{1+\gamma}(2^{\gamma}-1)}{\gamma}$ and $f \in A(p)$ with $f(z)f'(z) \neq 0$ when $z \neq 0$. If f satisfies the subordination*

$$1 + \beta p \frac{f(z)}{z f'(z)}\left(\frac{z f''(z)}{f'(z)} + 1 - \frac{z f'(z)}{f(z)}\right) \prec (1+z)^{\gamma}, \quad \beta \geq \beta_0, \tag{9}$$

then $\frac{z f'(z)}{p f(z)} \prec (1+z)^{\gamma}$. The lower bound β_0 is sharp.

Proof. We first prove the following conclusion:

$$1 + \beta\frac{z\phi'(z)}{\phi(z)} \prec (1+z)^{\gamma} \Rightarrow \phi(z) \prec (1+z)^{\gamma}, \tag{10}$$

where ϕ is analytic in D with $\phi(0) = 1$, $\beta \geq \beta_0$ and the lower bound β_0 is the best possible.

Let $q(z) = (1+z)^{\gamma}$ with $q(0) = 1$. Then, q is a convex function in D. Define the function Q by

$$Q(z) := \frac{zq'(z)}{q^2(z)} = \frac{\gamma z}{(1+z)^{1+\gamma}}.$$

This shows that

$$\mathrm{Re}\frac{zQ'(z)}{Q(z)} = \mathrm{Re}\frac{1-\gamma z}{1+z} > 0.$$

Therefore, Q is starlike in D. By using Lemma 1, we obtain the subordination relation

$$1 + \frac{\beta z\phi'(z)}{\phi^2(z)} \prec 1 + \frac{\beta zq'(z)}{q^2(z)} \Rightarrow \phi(z) \prec q(z).$$

Further, we define h by

$$h(z) := 1 + \frac{\beta zq'(z)}{q^2(z)} = 1 + \frac{\beta\gamma z}{(1+z)^{1+\gamma}} \quad (z \in D).$$

Since $q^{-1}(w) = w^{\frac{1}{\gamma}} - 1$, it follows that

$$q^{-1}(h(z)) = \left(1 + \frac{\beta\gamma z}{(1+z)^{1+\gamma}}\right)^{\frac{1}{\gamma}} - 1.$$

For $z = e^{it}$, $t \in [-\pi, \pi]$, we have

$$|q^{-1}(h(z))| = |q^{-1}(h(e^{it}))| = \left|\left(1 + \frac{\beta\gamma e^{it}}{(1+e^{it})^{1+\gamma}}\right)^{\frac{1}{\gamma}} - 1\right|.$$

The minimum of $|q^{-1}(h(e^{it}))|$ is obtained at $t = 0$. Thus

$$|q^{-1}(h(e^{it}))| \geq \left(1 + \frac{\beta\gamma}{2^{1+\gamma}}\right)^{\frac{1}{\gamma}} - 1 \geq 1$$

for $\beta \geq \frac{2^{1+\gamma}(2^{\gamma}-1)}{\gamma}$. Hence $q(z) \prec h(z)$ and the conclusion (10) is proved.

Now, we define the function ϕ by

$$\phi(z) = \frac{zf'(z)}{pf(z)},$$

then ϕ is analytic in D and $\phi(0) = 1$. By a simple calculation, we have

$$\frac{z\phi'(z)}{\phi(z)} = \frac{pf(z)}{zf'(z)}\left(\frac{zf''(z)}{f(z)} + 1 - \frac{zf'(z)}{pf(z)}\right). \tag{11}$$

From (9)–(11), we obtain

$$\frac{zf'(z)}{pf(z)} \prec (1+z)^{\gamma}.$$

This completes the proof of Theorem 3. □

For $\gamma = \frac{1}{2}$ and $p = 1$, we derive the result obtained in [2].

Corollary 3. *Let $\beta_0 = 4\sqrt{2}(\sqrt{2}-1) \approx 2.34$ and $f \in A$ with $f(z)f'(z) \neq 0$ when $z \neq 0$. If f satisfies the subordination*

$$1 + \beta\frac{f(z)}{zf'(z)}\left(\frac{zf''(z)}{f'(z)} + 1 - \frac{zf'(z)}{f(z)}\right) \prec \sqrt{1+z}, \quad \beta \geq \beta_0,$$

then $f \in SL$ or $zf'(z)/f(z)$ lies in the region bounded by the right-half of the lemniscate of Bernoulli. The lower bound β_0 is sharp.

Theorem 4. *Let $0 < \gamma \leq 1$ and $f \in A(p)$ with $f(z)f'(z) \neq 0$ when $z \neq 0$. If f satisfies the subordination*

$$\frac{zf'(z)}{pf(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}-\alpha\left(1-\frac{1}{p}\right)\frac{zf'(z)}{f(z)}\right) \prec (1+z)^{\gamma}, \quad 0<\alpha\leq 1, \tag{12}$$

then $\frac{zf'(z)}{pf(z)} \prec (1+z)^{\gamma}$.

Proof. We first prove the following conclusion:

$$(1-\alpha)\phi(z)+\alpha\phi^2(z)+\alpha z\phi'(z) \prec (1+z)^{\gamma} \Rightarrow \phi(z) \prec (1+z)^{\gamma} \tag{13}$$

for $0 < \alpha \leq 1$.

Let $q(z) = (1+z)^{\gamma}$ with $q(0) = 1$. Additionally, let θ and φ be given by $\theta(w) := (1-\alpha)w + \alpha w^2$ and $\varphi(w) := \alpha$. Then, θ and φ are analytic in D with $\varphi(w) \neq 0$. Define Q and h by

$$Q(z) := zq'(z)\varphi(q(z)) = \alpha zq'(z),$$

and

$$\begin{aligned} h(z) &:= \theta(q(z)) + Q(z) = (1-\alpha)q(z) + \alpha q^2(z) + \alpha zq'(z) \\ &= \frac{\alpha\gamma z + (1-\alpha)(1+z) + \alpha(1+z)^{1+\gamma}}{(1+z)^{1-\gamma}}. \end{aligned}$$

Since q is convex, the function Q is univalent starlike in D. In view of $\mathrm{Re}q(z) > 0$, this shows that

$$\mathrm{Re}\frac{zh'(z)}{Q(z)} = \frac{1}{\alpha}\mathrm{Re}\left[(1-\alpha)+2\alpha q(z)+\alpha\left(1+\frac{zq''(z)}{q'(z)}\right)\right] > 0 \quad (z \in D)$$

for $0 < \alpha \leq 1$. From Lemma 2, we have $\phi(z) \prec q(z)$.

Now, we find conditions on α for $q(z) \prec h(z)$. It follows that

$$\left|\left[h(e^{it})\right]^{\frac{1}{\gamma}} - 1\right| \geq |h^{\frac{1}{\gamma}}(1) - 1| > 1$$

for $z = e^{it}$, $t \in [-\pi, \pi]$, if

$$h(1) = \frac{\gamma + 2(2^{\gamma}-1)}{2^{1-\gamma}}\alpha + 2^{\gamma} > 2^{\gamma}$$

for $\alpha > 0$. Hence, the proof of the conclusion (13) is completed.

Define the function ϕ by

$$\phi(z) = \frac{zf'(z)}{pf(z)},$$

then ϕ is analytic in D and $\phi(0) = 1$. A calculation shows that

$$\phi(z) + \frac{z\phi'(z)}{\phi(z)} = 1 + \frac{zf''(z)}{f'(z)} - \left(1-\frac{1}{p}\right)\frac{zf'(z)}{f(z)}. \tag{14}$$

Clearly

$$\frac{zf'(z)}{pf(z)}\left(1+\alpha\frac{zf''(z)}{f'(z)}-\alpha\left(1-\frac{1}{p}\right)\frac{zf'(z)}{f(z)}\right)$$
$$=(1-\alpha)\phi(z)+\alpha\phi^2(z)+\alpha z\phi'(z). \tag{15}$$

From (12)–(15) we have

$$\frac{zf'(z)}{pf(z)}\prec(1+z)^{\gamma}.$$

Thus we complete the proof of Theorem 4. □

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