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A Quadratic Mean Field Games Model for the Langevin Equation

Fabio Camilli

Dipartimento di Scienze di Base e Applicate per l'Ingegneria, Sapienza Università di Roma, Via Scarpa 16, 00161 Roma, Italy; fabio.camilli@uniroma1.it

Abstract: We consider a Mean Field Games model where the dynamics of the agents is given by a controlled Langevin equation and the cost is quadratic. An appropriate change of variables transforms the Mean Field Games system into a system of two coupled kinetic Fokker–Planck equations. We prove an existence result for the latter system, obtaining consequently existence of a solution for the Mean Field Games system.

Keywords: langevin equation; Mean Field Games system; kinetic Fokker–Planck equation; hypoelliptic operators

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1. Introduction

The Mean Field Games (MFG in short) theory concerns the study of differential games with a large number of rational, indistinguishable agents and the characterization of the corresponding Nash equilibria. In the original model introduced in [1,2], an agent can typically act on its velocity (or other first order dynamical quantities) via a control variable. Mean Field Games where agents control the acceleration have been recently proposed in [3–5].

A prototype of stochastic process involving acceleration is given by the Langevin diffusion process, which can be formally defined as

$$\ddot{X}(t) = -b(X(t)) + \sigma \dot{B}(t), \tag{1}$$

where $\ddot{X}$ is the second time derivative of the stochastic process X, B a Brownian motion and σ a positive parameter. The solution of (1) can be rewritten as a Markov process (X, V) solving

$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = -b(X(t)) + \sigma \dot{B}(t). \end{cases}$$

The probability density function of the previous process satisfies the kinetic Fokker–Planck equation

$$\partial_t p - \frac{\sigma^2}{2}\Delta_v p - b(x) \cdot D_v p + v \cdot D_x p = 0 \qquad \text{in} \quad (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d.$$

The previous equation, in the case $b \equiv 0$, was first studied by Kolmogorov [6] who provided an explicit formula for its fundamental solution. Then considered by Hörmander [7] as motivating example for the general theory of the hypoelliptic operators (see also [8–10]).

We consider a Mean Field Games model where the dynamics of the single agent is given by a controlled Langevin diffusion process, i.e.,

$$\begin{cases} \dot{X}(s) = V(s), & s \geq t \\ \dot{V}(s) = -b(X(s)) + \alpha(s) + \sigma\dot{B}(s) & s \geq t \\ X(t) = x,\ V(t) = v \end{cases} \tag{2}$$

for $(t, x, v) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. In (2), the control law $\alpha : [t, T] \to \mathbb{R}^d$, which is a progressively measurable process with respect to a fixed filtered probability space such that $\mathbb{E}[\int_t^T |\alpha(t)|^2 dt] < +\infty$, is chosen to *maximize* the functional

$$J(t, x, v; \alpha) = \mathbb{E}_{t,(x,v)}\Big\{\int_t^T \Big[f(X(s), V(s), m(s)) - \frac{1}{2}|\alpha(s)|^2\Big] ds \\ + u_T(X(T), V(T))\Big\},$$

where $m(s)$ is the distribution of the agents at time s. Let u the value function associated with the previous control problem, i.e.,

$$u(t, x, v) = \sup_{\alpha \in \mathcal{A}_t} \{J(t, x, v; \alpha)\}$$

where $\mathcal{A}_t$ is the the set of the control laws. Formally, the couple (u, m) satisfies the MFG system (see Section 4.1 in [3] for more details)

$$\begin{cases} \partial_t u + \frac{\sigma^2}{2}\Delta_v u - b(x) \cdot D_v u + v \cdot D_x u + \frac{1}{2}|D_v u|^2 = -f(x, v, m) \\ \partial_t m - \frac{\sigma^2}{2}\Delta_v m - b(x) \cdot D_v m + v \cdot D_x m + \mathrm{div}_v(m D_v u) = 0 \\ m(0, x, v) = m_0(x, v), \quad u(T, x, v) = u_T(x, v). \end{cases} \tag{3}$$

for $(t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$. The first equation is a backward Hamilton–Jacobi–Bellman equation, degenerate in the x-variable and with a quadratic Hamiltonian in the v variable, and the second equation is forward kinetic Fokker–Planck equation. In the standard setting, MFG systems with quadratic Hamiltonians has been extensively considered in literature both as a reference model for the general theory and also since, thanks to the Hopf-Cole change of variable, the nonlinear Hamilton-Jacobi-Bellman equation can be transformed into a linear equation, allowing to use all the tools developed for this type of problem (see for example [2,11–15]). Recently, a similar procedure has been used for ergodic hypoelliptic MFG with quadratic cost in [16] and for a flocking model involving kinetic equations in Section 4.7.3 of [17].

We study (3) by means of a change of variable introduced in [11,14] for the standard case. By defining the new unknowns $\phi = e^{u/\sigma^2}$ and $\psi = me^{-u/\sigma^2}$, the system (3) is transformed into a system of two kinetic Fokker–Planck equations

$$\begin{cases} \partial_t \phi + \frac{\sigma^2}{2}\Delta_v \phi - b(x) \cdot D_v \phi + v \cdot D_x \phi = -\frac{1}{\sigma^2} f(x, v, \psi\phi)\phi \\ \partial_t \psi - \frac{\sigma^2}{2}\Delta_v \psi - b(x) \cdot D_v \psi + v \cdot D_x \psi = \frac{1}{\sigma^2} f(x, v, \psi\phi)\psi \\ \psi(0, x, v) = \frac{m_0(x,v)}{\phi(0,x,v)}, \quad \phi(T, x, v) = e^{\frac{u_T(x,v)}{\sigma^2}}. \end{cases} \tag{4}$$

for $(t, x, v) \in (0, T) \times \mathbb{R}^d \times \mathbb{R}^d$. In the previous problem, the coupling between the two equations is only in the source terms. Following [14], we prove existence of a weak solution to (4) by showing the convergence of an iterative scheme defined, starting from $\psi^{(0)} \equiv 0$, by solving alternatively the backward problem

$$\begin{cases} \partial_t\phi^{(k+\frac{1}{2})} + \frac{\sigma^2}{2}\Delta_v\phi^{(k+\frac{1}{2})} - b(x)\cdot D_v\phi^{(k+\frac{1}{2})} + v\cdot D_x\phi^{(k+\frac{1}{2})} \\ \qquad\qquad\qquad\qquad = -\frac{1}{\sigma^2}f(\psi^{(k)}\phi^{(k+\frac{1}{2})})\phi^{(k+\frac{1}{2})} \\ \phi^{(k+\frac{1}{2})}(T,x,v) = e^{\frac{u_T(x,v)}{\sigma^2}}, \end{cases} \tag{5}$$

and the forward one

$$\begin{cases} \partial_t\psi^{(k+1)} - \frac{\sigma^2}{2}\Delta_v\psi^{(k+1)} - b(x)\cdot D_v\psi^{(k+1)} + v\cdot D_x\psi^{(k+1)} \\ \qquad\qquad\qquad\qquad = \frac{1}{\sigma^2}f(\psi^{(k+1)}\phi^{(k+\frac{1}{2})})\psi^{(k+1)} \\ \psi^{(k+1)}(0,x,v) = \frac{m_0(x,v)}{\phi^{(k+\frac{1}{2})}(0,x,v)}. \end{cases} \tag{6}$$

We show that the resulting sequence $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$, $k \in \mathbb{N}$, monotonically converges to the solution of (4). Hence, by the inverse change of variable (see again [11,14] for details)

$$u = \frac{\ln(\phi)}{\sigma^2}, \qquad m = \phi\psi, \tag{7}$$

we obtain a solution of the original problem (3). We have

Theorem 1. *The sequence $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$ defined by (5) and (6) converges in $L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ and a.e. to a weak solution (ϕ,ψ) of (4). Moreover, the couple (u,m) defined by (7) is a weak solution to (3).*

The main difficulty in the study of problems (3) and (4) is due both in the degeneracy of the second order operator with respect to x and in the unbounded dependence of the coefficients of the first order terms with respect to v. To overcome the previous difficulties we rely on the results for linear kinetic Fokker–Planck equations developed in [18]. We mention that existence of weak solutions for the standard MFG problem, possibly degenerate, has been studied in [19], but the results in this paper do not cover the present setting. The previous iterative procedure also suggests a monotone numerical method for the approximation of (4), hence for (3). Indeed, by approximating (5) and (6) by finite differences and solving alternatively the resulting discrete equations, we obtain an approximation of the sequence $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$. A corresponding procedure for the standard quadratic MFG system was studied in [14], where the convergence of the method is proved. We plan to study the properties of the previous numerical procedure in a future work.

2. Well Posedness of the Kinetic Fokker–Planck System

In this section, we study the existence of a solution to system (4). The proof of the result follows the strategy implemented in Section 2 of [14] for the case of a standard MFG system with quadratic Hamiltonian and relies on the results for linear kinetic Fokker–Planck equations in Appendix A of [18]. We remark the model here studied does not fit exactly the problem treated in [18] because of the presence of a zero order term in the Fokker-Planck equation. Hence some technical aspects should be analyzed in more detail, however the present paper is mainly intended to give some idea on the change of variabile for the kinetic MGF.

We fix the assumptions we will assume in the whole paper. The vector field $b:\mathbb{R}^d\to\mathbb{R}^d$ and the coupling cost $f:\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}\to\mathbb{R}$ are assumed to satisfy

$$b \in L^\infty(\mathbb{R}^d),$$
$$f \in L^\infty(\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}),\ f \le 0 \text{ and } f(x,v,\cdot) \text{ strictly decreasing}.$$

Moreover, the diffusion coefficient σ is positive and the initial and terminal data satisfy

$$\begin{gathered} m_0 \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d),\, m_0 \geq 0,\, \iint m_0(x,v)dxdv = 1, \\ \text{and } \exists\, R_0 > 0 \text{ s.t. } \operatorname{supp}\{m_0\} \subset \mathbb{R}^d \times B(0,R_0) \end{gathered} \tag{8}$$

and

$$\begin{gathered} u_T \in C^0(\mathbb{R}^d \times \mathbb{R}^d) \text{ and } \exists\, C_0, C_1 > 0 \text{ s.t. } \forall (x,v) \in \mathbb{R}^d \times \mathbb{R}^d \\ -C_0(|v|^2 + |x|) - C_0 \leq u_T(x,v) \leq -C_1(|v|^2 + |x|) + C_1. \end{gathered} \tag{9}$$

Note that (9) implies that $e^{u_T/\sigma^2} \in L^\infty(\mathbb{R}^d \times \mathbb{R}^d) \cap L^2(\mathbb{R}^d \times \mathbb{R}^d)$. We denote with $(\cdot,\cdot)$ the scalar product in $L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ and with $\langle \cdot,\cdot \rangle$ the pairing between $\mathcal{X} = L^2([0,T] \times \mathbb{R}^d_x; H^1(\mathbb{R}^d_v))$ and its dual $\mathcal{X}' = L^2([0,T] \times \mathbb{R}^d_x; H^{-1}(\mathbb{R}^d_v))$. We define the following functional space

$$\mathcal{Y} = \left\{ g \in L^2([0,T] \times \mathbb{R}^d_x, H^1(\mathbb{R}^d_v)), \partial_t g + v \cdot D_x g \in L^2([0,T] \times \mathbb{R}^d_x, H^{-1}(\mathbb{R}^d_v)) \right\}$$

and we set $\mathcal{Y}_0 = \{g \in \mathcal{Y} : g \geq 0\}$. If $g \in \mathcal{Y}$, then it admits (continuous) trace values $g(0,x,v), g(T,x,v) \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ (see [18, Lemma A.1]) and therefore the initial/terminal conditions for (4) are well defined in L^2 sense. We first prove the well posedness of problems (5) and (6).

Proposition 2. *We have*

(i) *For any $\psi \in \mathcal{Y}_0$, there exists a unique solution $\phi \in \mathcal{Y}_0$ to*

$$\begin{cases} \partial_t \phi + \frac{\sigma^2}{2} \Delta_v \phi - b(x) \cdot D_v \phi + v \cdot D_x \phi = -\frac{1}{\sigma^2} f(x,v,\psi\phi)\phi \\ \phi(T,x,v) = e^{\frac{u_T(x,v)}{\sigma^2}}. \end{cases} \tag{10}$$

Moreover, $\phi \in L^\infty([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ and, for any $R > 0$, there exist $\delta_R \in \mathbb{R}$ and $\rho > 0$ such that

$$\phi(t,x,v) \geq C_R := e^{\frac{1}{\sigma^2}(\delta_R - \rho T)} \quad \forall t \in [0,T],\, (x,v) \in B(0,R) \subset \mathbb{R}^d \times \mathbb{R}^d. \tag{11}$$

(ii) *Let $\Phi : \mathcal{Y}_0 \to \mathcal{Y}_0$ be the map which associates to ψ the unique solution of (10). Then, if $\psi_2 \leq \psi_1$, we have $\Phi(\psi_2) \geq \Phi(\psi_1)$.*

Proof. We first prove existence of a solution to the nonlinear problem (10) by a fixed point argument exploiting the results for the corresponding linear problem proved in [18]. Fixed $\psi \in \mathcal{Y}_0$, consider the map $F = F(\varphi)$ from $L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ into itself that associates with φ the weak solution $\phi \in L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ of the linear problem

$$\begin{cases} \partial_t \phi + \frac{\sigma^2}{2} \Delta_v \phi - b(x) \cdot D_v \phi + v \cdot D_x \phi = -\frac{1}{\sigma^2} f(\psi\varphi)\phi \\ \phi(T,x,v) = e^{\frac{u_T(x,v)}{\sigma^2}}. \end{cases} \tag{12}$$

By Prop. A.2 of [18], ϕ belongs to $\mathcal{Y}$ and it coincides with the unique solution of (12) in this space. Moreover, the following estimate

$$\|\phi\|_{L^2([0,T] \times \mathbb{R}^d_x; H^1(\mathbb{R}^d_v))} + \|\partial_t \phi + v \cdot D_x \phi\|_{L^2([0,T] \times \mathbb{R}^d_x; H^{-1}(\mathbb{R}^d_v))} \leq C \tag{13}$$

holds for some constant C which depends only on $\|e^{u_T/\sigma^2}\|_{L^2}$, $\|f\|_{L^\infty}$ and σ. Hence F maps B_C, the closed ball of radius C of $L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$, into itself.

To show that the map F is continuous on B_C, consider $\{\varphi_n\}_{n \in \mathbb{N}}$, $\varphi \in L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ such that $\|\varphi_n - \varphi\|_{L^2} \to 0$ and set $\phi_n = F(\varphi_n)$. Then $\phi_n \in \mathcal{Y}$, and, by the estimate (13), we get that, up to a subsequence, there exists $\overline{\phi} \in \mathcal{Y}$ such that $\phi_n \to \overline{\phi}$, $D_v \phi_n \to D_v \overline{\phi}$

in $L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$, $\partial_t\phi_n + v\cdot D_x\phi_n \to \partial_t\overline{\phi}_n + v\cdot D_x\overline{\phi}_n$ in $L^2([0,T]\times\mathbb{R}^d_x;H^{-1}(\mathbb{R}^d_v))$. Moreover, $\varphi_n \to \varphi$ almost everywhere. By the definition of weak solution to (12), we have that

$$\langle\partial_t\phi_n + v\cdot D_x\phi_n, w\rangle - \frac{\sigma^2}{2}(D_v\phi_n, D_v w) - (b\cdot D_v\phi_n, w) = (-\frac{1}{\sigma^2}\phi_n f(\varphi_n\psi), w), \tag{14}$$

for any $w\in\mathcal{D}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$, the space of infinite differentiable functions with compact support in $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$. Employing weak convergence for left hand side of (14) and the Dominated Convergence Theorem for the right hand one, we get for $n\to\infty$

$$\langle\partial_t\overline{\phi} + v\cdot D_x\overline{\phi}, w\rangle - \frac{\sigma^2}{2}(D_v\overline{\phi}, D_v w) - (b\cdot D_v\overline{\phi}, w) = (-\overline{\phi}f(\varphi\psi), w)$$

for any $w\in\mathcal{D}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$. Hence $\overline{\phi} = F(\varphi)$ and $F(\varphi_n)\to F(\varphi)$ for $n\to\infty$ in $L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$. The compactness of the map F in $L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ follows by the compactness of the set of the solutions to (12), see Theorem 1.2 of [20]. We conclude, by Schauder's Theorem, that there exists a fixed-point of the map F in L^2, hence in $\mathcal{Y}$, and therefore a solution to the nonlinear parabolic Equation (10).

Observe that, if ϕ is a solution of (10), then $\tilde{\phi} = e^{\lambda t}\phi$ is a solution of

$$\partial_t\tilde{\phi} + \frac{\sigma^2}{2}\Delta_v\tilde{\phi} - b(x)\cdot D_v\tilde{\phi} + v\cdot D_x\tilde{\phi} - \lambda\tilde{\phi} = -\frac{1}{\sigma^2}f(e^{-\lambda t}\psi\tilde{\phi})\tilde{\phi} \tag{15}$$

with the corresponding final condition. In the following, we assume that $\lambda > 0$. To show that ϕ is non-negative, we will exploit the following property (see Lemma A.3 of [18]): given $\phi\in\mathcal{Y}$ and defined $\phi^\pm = \max(\pm\phi, 0)$, then $\phi^\pm\in\mathcal{X}$ and

$$\langle\partial_t\phi + v\cdot D_x\phi, \phi^-\rangle = \frac{1}{2}\Big(\iint|\phi(0,x,v)^-|^2dxdv - \iint|\phi(T,x,v)^-|^2dxdv\Big). \tag{16}$$

Let ϕ be a solution of (15), multiply the equation by ϕ^- and integrate. Then, since $\phi(T,x,v)$ is non-negative, by (16) we get

$$\begin{aligned}-\frac{1}{\sigma^2}(\phi f(e^{\lambda t}\phi\psi),\phi^-) = \langle\partial_t\phi + v\cdot D_x\phi,\phi^-\rangle - \\ \frac{\sigma^2}{2}(D_v\phi, D_v\phi^-) - (b\cdot D_v\phi,\phi^-) - \lambda(\phi,\phi^-) = \\ \frac{1}{2}\iint|\phi(0,x,v)^-|^2dxdv + \frac{\sigma^2}{2}(D_v\phi^-, D_v\phi^-) + \lambda(\phi^-,\phi^-) \geq \\ \lambda(\phi^-,\phi^-),\end{aligned}$$

where it has been exploited that, by integration by parts, $(b\cdot D_v\phi,\phi^-) = 0$. Since $f\leq 0$ and therefore

$$-(\phi f(e^{\lambda t}\phi\psi),\phi^-) = (\phi^- f(e^{\lambda t}\phi\psi),\phi^-) \leq 0,$$

we get $(\phi^-,\phi^-)\equiv 0$, hence $\phi\geq 0$.

To prove the uniqueness of the solution to (10), consider two solutions ϕ_1, ϕ_2 of (15) and set $\overline{\phi} = \phi_1 - \phi_2$. Multiplying the equation for $\overline{\phi}$ by $\overline{\phi}$, integrating and using $\overline{\phi}(x,v,T) = 0$, we get

$$\begin{aligned}-\frac{1}{\sigma^2}(f(e^{-\lambda t}\psi\phi_1)\phi_1 - f(e^{-\lambda t}\psi\phi_2)\phi_2, \phi_1-\phi_2) = \langle\partial_t\overline{\phi} + v\cdot D_x\overline{\phi},\overline{\phi}\rangle - \\ \frac{\sigma^2}{2}(D_v\overline{\phi}, D_v\overline{\phi}) - (b\cdot D_v\overline{\phi},\overline{\phi}) - \lambda(\overline{\phi},\overline{\phi}) = \\ -\frac{1}{2}\iint|\overline{\phi}(x,v,0)|^2dxdv - \frac{\sigma^2}{2}(D_v\overline{\phi}, D_v\overline{\phi}) - \lambda(\overline{\phi},\overline{\phi}) \leq -\lambda(\phi_1-\phi_2,\phi_1-\phi_2)\end{aligned} \tag{17}$$

and, by the strict monotonicity of f, we conclude that $\phi_1 = \phi_2$.

To prove that ϕ is bounded from above, we observe that the function $\overline{\phi}(t,x,v) = e^{C_1+(T-t)\|f\|_\infty/\sigma^2}$, where C_1 as in (9), is a supersolution of the linear problem (12) for any $\varphi \in L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$, i.e., $\phi(T,x,v) \geq e^{u_T(x,v)/\sigma^2}$ and

$$\partial_t\overline{\phi} + \frac{\sigma^2}{2}\Delta_v\overline{\phi} - b(x)\cdot D_v\overline{\phi} + v\cdot D_x\overline{\phi} \leq -\frac{1}{\sigma^2}f(\psi\varphi)\overline{\phi}.$$

By the Maximum Principle (see Prop. A.3 (i) in [18]), we get that $\overline{\phi} \geq \phi$, where ϕ is the solution of (12). Since the previous property holds for any $\varphi \in L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$, we conclude that $\overline{\phi} \geq \phi$, where ϕ is the solution of the nonlinear problem (10).

A similar argument show that $\underline{\phi}(x,v,t) = e^{(-C_0(|v|^2+|x|+1)-\rho(T-t))/\sigma^2}$, where C_0 as in (9) and ρ sufficiently large, is a subsolution of (12) for any $\varphi \in L^2([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$. Indeed, replacing $\underline{\phi}$ in the equation, we get that the inequality

$$\begin{aligned}\partial_t\underline{\phi} + \frac{\sigma^2}{2}\Delta_v\underline{\phi} - b(x)\cdot D_v\underline{\phi} + v\cdot D_x\underline{\phi} =\\ = \frac{\underline{\phi}}{\sigma^2}\left(\rho - C_0 d\sigma^2 + 2C_0^2\sigma^2|v|^2 + 2C_0 b(x)\cdot v - C_0 v\cdot\frac{x}{|x|}\right) \geq\\ -\frac{1}{\sigma^2}f(\psi\varphi)\underline{\phi}\end{aligned}$$

is satisfied for ρ large enough and, moreover, $\underline{\phi} \leq e^{u_T(x,v)/\sigma^2}$. Hence $\underline{\phi} \leq \phi$, where ϕ is the solution of the nonlinear problem (10), and, from this estimate, we deduce (11).

We finally prove the monotonicity of the map Φ. Set $\phi_i = \Phi(\psi_i)$, $i = 1,2$, and consider the equation satisfied by $\overline{\phi} = e^{\lambda t}\phi_1 - e^{\lambda t}\phi_2$, multiply it by $\overline{\phi}^+$ and integrate. Performing a computation similar to (17), we get

$$-\frac{1}{\sigma^2}(f(\phi_1\psi_1)\phi_1 - f(\phi_2\psi_2)\phi_2, \overline{\phi}^+) \leq -\lambda(\overline{\phi}^+, \overline{\phi}^+).$$

Since, by monotonicity of f and non-negativity of ϕ_i, we have

$$\begin{aligned}-(f(\phi_1\psi_1)\phi_1 - f(\phi_2\psi_2)\phi_2, \overline{\phi}^+) = -(f(\phi_1\psi_1)(\phi_1-\phi_2), \overline{\phi}^+) -\\ ((f(\phi_1\psi_1) - f(\phi_2\psi_2))\phi_2, \overline{\phi}^+) \geq 0,\end{aligned}$$

we get $(\overline{\phi}^+, \overline{\phi}^+) = 0$ and therefore $\phi_1 \leq \phi_2$. $\square$

We set

$$\mathcal{Y}_R = \{\phi \in \mathcal{Y}_0 : \phi \geq C_R \quad \forall (x,v) \in B(0,R),\, t \in [0,T]\},$$

where C_R is defined as in (11).

Proposition 3. *Given $R > R_0$, where R_0 as in (8), we have*

(i) For any $\phi \in \mathcal{Y}_R$, there exists a unique solution $\psi \in \mathcal{Y}_0$ to

$$\begin{cases} \partial_t\psi - \frac{\sigma^2}{2}\Delta_v\psi - b(x)\cdot D_v\psi + v\cdot D_x\psi = \frac{1}{\sigma^2}f(x,v,\psi\phi)\psi \\ \psi(0,x,v) = \frac{m_0(x,v)}{\phi(0,x,v)}. \end{cases} \tag{18}$$

Moreover

$$\psi(x,v,t) \leq \frac{\|m_0\|_{L^\infty}}{C_R} \qquad \forall t \in [0,T],\ (x,v) \in \mathbb{R}^d\times\mathbb{R}^d, \tag{19}$$

where C_R as in (11).

(ii) *Let $\Psi : \mathcal{Y}_R \to \mathcal{Y}_0$ be the map which associates with $\phi \in \mathcal{Y}_R$ the unique solution of* (18). *Then, if $\phi_2 \leq \phi_1$, we have $\Psi(\phi_2) \geq \Psi(\phi_1)$.*

Proof. First observe that, since $R > R_0$, then $\psi(0, x, v)$ is well defined for $\phi \in \mathcal{Y}_R$. The proof of the first part of (i) is very similar to the one of the corresponding result in Proposition 2, hence we only prove the bound (19). If ψ is a solution of (18), then $\tilde{\psi} = e^{-\lambda t}\psi$ is a solution of

$$\partial_t \tilde{\psi} - \frac{\sigma^2}{2}\Delta_v \tilde{\psi} - b(x) \cdot D_v \tilde{\psi} + v \cdot D_x \psi + \lambda \tilde{\psi} = \frac{1}{\sigma^2} f(x, v, e^{\lambda t}\tilde{\psi}\phi)\psi. \tag{20}$$

Let ψ be a solution of (20), set $\bar{\psi} = \psi - e^{-\lambda t}\|m_0\|_{L^\infty}/C_R$ and observe that $\bar{\psi}(0) \leq 0$. Multiply the equation for $\bar{\psi}$ by $\bar{\psi}^+$ and integrate to obtain

$$\begin{aligned}
(\psi f(e^{\lambda t}\psi\phi), \bar{\psi}^+) &= \\
\langle \partial_t \bar{\psi} + v \cdot D_x \bar{\psi}, \bar{\psi}^+ \rangle + \frac{1}{\sigma^2}(D_v \bar{\psi}, D_v \bar{\psi}^+) - (b(x) D_v \bar{\psi}, \bar{\psi}^+) + \lambda(\bar{\psi}, \bar{\psi}^+) &\geq \\
\iint |\bar{\psi}^+(x, v, T)|^2 dx dv + \lambda(\bar{\psi}^+, \bar{\psi}^+) &\geq \lambda(\bar{\psi}^+, \bar{\psi}^+).
\end{aligned}$$

Since $\psi \geq 0$ and $f \leq 0$, we have

$$(\psi f(e^{\lambda t}\psi\phi), \bar{\psi}^+) \leq 0$$

and therefore $\bar{\psi}^+ \equiv 0$. Hence the upper bound (19).

Now we prove *(ii)*. Set $\psi_i = \Psi(\phi_i)$, $i = 1, 2$, and $\bar{\psi} = e^{-\lambda t}\psi_1 - e^{-\lambda t}\psi_2$. Multiply the equation satisfied by $\bar{\psi}$ by $\bar{\psi}^+$ and integrate. Since, by monotonicity and negativity of f, we have

$$\begin{aligned}
(f(e^{\lambda t}\phi_1\psi_1)\psi_1 - f(e^{\lambda t}\phi_2\psi_2)\psi_2, \bar{\psi}^+) = (f(e^{\lambda t}\phi_1\psi_1)(\psi_1 - \psi_2), \bar{\psi}^+) +& \\
(\psi_2(f(e^{-\lambda t}\phi_1\psi_1) - f(e^{-\lambda t}\phi_2\psi_2)), \bar{\psi}^+) \leq 0.&
\end{aligned}$$

Then

$$\begin{aligned}
0 \geq \langle \partial_t \bar{\psi} + v \cdot D_x \bar{\psi}, \bar{\psi}^+ \rangle + \frac{1}{\sigma^2}(D_v \bar{\psi}, D_v \bar{\psi}^+) - (b(x) D_v \bar{\psi}, \bar{\psi}^+) + \lambda(\bar{\psi}, \bar{\psi}^+) &\geq \\
\iint |\bar{\psi}^+(x, v, T)|^2 dx dv + \lambda(\bar{\psi}^+, \bar{\psi}^+) &\geq \lambda(\bar{\psi}^+, \bar{\psi}^+).
\end{aligned}$$

Hence $\bar{\psi}^+ \equiv 0$ and therefore $\psi_1 \leq \psi_2$. □

Proof of Theorem 1. Given $\psi^{(0)} \equiv 0$, consider the sequence $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$, $k \in \mathbb{N}$, defined in (5) and (6). It can rewritten as

$$\begin{cases} \phi^{(k+\frac{1}{2})} = \Phi(\psi^{(k)}) \\ \psi^{(k+1)} = \Psi(\phi^{(k+\frac{1}{2})}) \end{cases} \tag{21}$$

where the maps Φ, Ψ are as in Propositions 2 and, respectively 3. Observe that, by (11), we have $\phi^{(k+\frac{1}{2})} \in \mathcal{Y}_R$ for $R > R_0$ and $\psi^{(k+1)} \geq 0$ for any k. Hence the sequence $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$ is well defined. We first prove by induction the monotonicity of the components of $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$. By non-negativity of solutions to (18), we have $\psi^{(1)} = \Phi(\phi^{(\frac{1}{2})}) \geq 0$ and therefore $\psi^{(1)} \geq \psi^{(0)}$. Moreover, by the monotonicity of Φ, $\phi^{(\frac{3}{2})} = \Phi(\psi^{(1)}) \leq \Phi(\psi^{(0)}) = \phi^{(\frac{1}{2})}$. Now assume that $\psi^{(k+1)} \geq \psi^{(k)}$. Then

$$\phi^{(k+\frac{3}{2})} = \Phi(\psi^{(k+1)}) \leq \Phi(\psi^{(k)}) = \phi^{(k+\frac{1}{2})}$$

and

$$\psi^{(k+2)} = \Psi(\phi^{(k+\frac{3}{2})}) \geq \Psi(\phi^{(k+\frac{1}{2})}) = \psi^{(k+1)},$$

therefore the monotonicity of two sequences.

Since $\phi^{(k+\frac{1}{2})} \geq 0$ and, by (19), for $k \to \infty$, the sequence $\psi^{(k+1)} \leq \|m_0\|_{L^\infty}/C_R$, $(\phi^{(k+\frac{1}{2})}, \psi^{(k+1)})$ converges a.e. and in $L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ to a couple (ϕ, ψ). Taking into account the estimate (13), the a.e. convergence of the two sequences and repeating an argument similar to the one employed for the continuity of the map F in Proposition 2, we get that the couple (ϕ, ψ) satisfies, in weak sense, the first two equations in (4). The terminal condition for ϕ is obviously satisfied, while the initial condition for ψ, in L^2 sense, follows by convergence of $\phi^{(k+\frac{1}{2})}(0)$ to $\phi(0)$.

We now consider the couple (u, m) given by the change of variable in (7). We first observe that, by Theorem 1.5 of [10], we have $\partial_t\phi + v \cdot D_x\phi$, $D_v\phi$, $\Delta_v\phi \in L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ and a corresponding regularity for ψ. Taking into account the boundedness of ϕ and the estimate in (11), we have that u, $\partial_t u + v \cdot D_x u$, $D_v u$, $\Delta_v u \in L^2_{loc}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$. Hence we can write the equation for u in weak form, i.e.,

$$(\partial_t u + v \cdot D_x u, w) - \frac{\sigma^2}{2}(D_v u, D_v w) - (b \cdot D_v u, w) + \frac{1}{2}(|D_v u|^2, w) = -(f(m), w),$$

for any $w \in \mathcal{D}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$, with final datum in trace sense. In a similar way, since m, $\partial_t m + v \cdot D_x m$, $D_v m$, $\Delta_v m \in L^2_{loc}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ and m is locally bounded, we can rewrite also the equation for m in weak form, i.e.,

$$(\partial_t m + v \cdot D_x m, w) + \frac{\sigma^2}{2}(D_v m, D_v w) - (b \cdot D_v m, w) - (m D_v u, Dw) = 0,$$

for any $w \in \mathcal{D}([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ with the initial datum in trace sense. $\square$

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References

1. Huang, M.; Caines, P.E.; Malhame, R.P. Large-population cost-coupled LQG problems with non uniform agents: Individual-mass behaviour and decentralized ϵ-Nash equilibria. *IEEE Trans. Autom. Control* **2007**, 52, 1560–1571. [CrossRef]
2. Lasry, J.-M.; Lions, P.-L. Mean field games. *Jpn. J. Math.* **2007**, *2*, 229–260. [CrossRef]
3. Achdou, Y.; Mannucci, P.; Marchi, C.; Tchou, N. Deterministic mean field games with control on the acceleration. *Nodea Nonlinear Differ. Eq. Appl.* **2020**, *27*, 33. [CrossRef]
4. Bardi, M.; Cardaliaguet, P. Convergence of some Mean Field Games systems to aggregation and flocking models. *arXiv* **2004**, arXiv:2004.04403.
5. Cannarsa, P.; Mendico, C. Mild and weak solutions of Mean Field Games problem for linear control systems. *Minimax Theory Appl.* **2020**, *5*, 221–250.
6. Kolmogoroff, A. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. Math.* **1934**, *35*, 116–117. [CrossRef]
7. Hörmander, L. Hypoelliptic second order differential equations. *Acta Math.* **1967**, *119*, 147–171. [CrossRef]
8. Lanconelli, E.; Polidoro, S. On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino* **1994**, *52*, 29–63
9. Armstrong, S.; Mourrat, J.-C. Variational methods for the kinetic Fokker-Planck equation. *arXiv* **1902**, arXiv:1902.04037.
10. Bouchut, F. Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl.* **2002**, *81*, 1135–1159. [CrossRef]
11. Guéant, O.; Lasry, J.; Lions, P. Mean field games and applications. In *Paris-Princeton Lectures on Mathematical Finance 2010*; Lecture Notes in Math; Springer: Berlin, Germany, 2011; Volume 2003; pp. 205–266.
12. Gomes, D.A.; Mitake, H. Existence for stationary mean-field games with congestion and quadratic Hamiltonians. *Nodea Nonlinear Differ. Eq. Appl.* **2015**, *22*, 1897–1910. [CrossRef]
13. Gomes, D.A.; Pimentel, E.A.; Voskanyan, V. *Regularity Theory for Mean-Field Game Systems*; Springer Briefs in Mathematics; Springer: Berlin, Germany, 2016.

14. Guéant, O. Mean field games equations with quadratic Hamiltonian: A specific approach. *Math. Models Methods Appl. Sci.* **2012**, *22*, 37. [CrossRef]
15. Ullmo, D.; Swiecicki, I.; Gobron, T. Quadratic mean field games. *Phys. Rep.* **2019**, *799*, 1–35. [CrossRef]
16. Feleqi, E.; Gomes, D.; Tada, T. Hypoelliptic mean field games—A case study. *Minimax Theory Appl.* **2020**, *5*, 305–326.
17. Carmona, R.; Delarue, F. *Probabilistic Theory of Mean Field Games with Applications. I Mean Field FBSDEs, Control, and Games*; Probability Theory and Stochastic Modelling, 83; Springer: Cham, Switzerland, 2018.
18. Degond, P. Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in 1 and 2 space dimensions. *Ann. Sci. École Norm. Sup.* **1986**, *19*, 519–542. [CrossRef]
19. Cardaliaguet, P.; Graber, P.J.; Porretta, A.; Tonon, D. Second order mean field games with degenerate diffusion and local coupling. *Nodea Nonlinear Differ. Eq. Appl.* **2015**, *22*, 1287–1317. [CrossRef]
20. Camellini, F.; Eleuteri, M.; Polidoro, S. A compactness result for the Sobolev embedding via potential theory. *arXiv* **1806**, arXiv:1806.03606.