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*Article*

# Generating Functions for $q$-Apostol Type Frobenius–Euler Numbers and Polynomials

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**Abstract:** The aim of this paper is to construct generating functions, related to nonnegative real parameters, for $q$-Eulerian type polynomials and numbers (or $q$-Apostol type Frobenius–Euler polynomials and numbers). We derive some identities for these polynomials and numbers based on the generating functions and functional equations. We also give multiplication formula for the generalized Apostol type Frobenius–Euler polynomials.

**Keywords:** Euler Numbers; Frobenius–Euler Numbers; Frobenius–Euler polynomials; $q$-Frobenius–Euler polynomials; $q$-series; Generating function

## 1. Introduction

Throughout this paper we assume that $q \in \mathbb{C}$, the set of complex numbers, with $|q| < 1$ and

$$[x] = [x : q] = \begin{cases} \frac{1-q^x}{1-q}, \text{ if } q \neq 1 \\ x, \text{ if } q = 1 \end{cases}$$

We use the following standard notions:

$\mathbb{N} = \{1, 2, \cdots\}$, $\mathbb{N}_0 = \{0, 1, 2, \cdots\} = \mathbb{N} \cup \{0\}$ and also, as usual $\mathbb{R}$ denotes the set of real number, $\mathbb{R}^+$ denotes the set of positive real number and $\mathbb{C}$ denotes the set of complex numbers.

In this section, we define $q$-Apostol type Frobenius–Euler polynomials and numbers, related to nonnegative real parameters. numbers).

**Definition 1** *Let $a$, $b \in \mathbb{R}^+$ ($a \neq b$) and $u \in \mathbb{C} \setminus \{1\}$. A $q$-Apostol type Frobenius–Euler numbers*

$$\mathcal{H}_n(u; a, b; \lambda; q) \;\; (\lambda, q \in \mathbb{C})$$

*are defined by means of the following generating function:*

$$F_{\lambda,q}(t;u,a,b) = \left(1-\frac{a^t}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n b^{[n]t} \tag{1}$$
$$= \sum_{n=0}^{\infty}\mathcal{H}_n(u;a,b;\lambda;q)\frac{t^n}{n!}$$

*where*

$$\left|\frac{\lambda}{u}\right| < 1$$

**Definition 2** *Let $a$, $b \in \mathbb{R}^+$ ($a \neq b$) and $u \in \mathbb{C}\diagdown\{1\}$. A $q$-Apostol type Frobenius–Euler polynomials*

$$\mathcal{H}_n(x;u;a,b;\lambda;q) \;\; (\lambda \in \mathbb{C})$$

*are defined by means of the following generating function:*

$$F_{\lambda,q}(x,t;u,a,b) = \left(1-\frac{a^t}{u}\right)\sum_{n=0}^{\infty}\left(\frac{\lambda}{u}\right)^n b^{[n+x]t} = \sum_{n=0}^{\infty}\mathcal{H}_n(x;u;a,b;\lambda;q)\frac{t^n}{n!} \tag{2}$$

*where*

$$\left|\frac{\lambda}{u}\right| < 1$$

*with, of course*

$$\mathcal{H}_n(0;u;a,b;\lambda;q) = \mathcal{H}_n(u;a,b;\lambda;q)$$

*where $\mathcal{H}_n(u;a,b;\lambda;q)$ denotes the $q$-Apostol type Frobenius–Euler numbers.*

By using the following well-known identity

$$[n+x] = [x] + q^x[n]$$

in Equation (2), we derive the following functional equation:

$$F_{\lambda,q}(x,t;u,a,b) = F_{\lambda,q}(q^x t;u,a,b)b^{t[x]} \tag{3}$$

By using Equation (3), we arrive at the following theorem:

**Theorem 1** *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_n(x;u;a,b;\lambda;q) = \sum_{k=0}^{n}\binom{n}{k} q^{kx}\left([x]\ln b\right)^{n-k}\mathcal{H}_k(u;a,b;\lambda;q)$$

By using Theorem 1, one can easily obtain the following result:

**Corollary 1** *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_n(x;u;a,b;\lambda;q) = \left([x]\ln b + q^x\mathcal{H}(u;a,b;\lambda;q)\right)^n \tag{4}$$

*with the usual convention of replacing $\mathcal{H}^j$ by $\mathcal{H}_j$.*

**Remark 1.** One can easily see that

$$\lim_{q\to 1} F_{\lambda,q}(x,t;u,a,b) = \frac{(a^t - u)\, b^{xt}}{\lambda b^t - u}$$
$$= \sum_{n=0}^{\infty} \mathcal{H}_n(x;u;a,b;\lambda)\frac{t^n}{n!}$$

(*cf.* [1]).

**Remark 2.** In their special case when $\lambda^r = 1$ ($\lambda \neq 1$) and $a = 1$ and $b = e$, the $q$-Apostol type Frobenius–Euler polynomials $\mathcal{H}_n(x;u;1,e;\lambda;q)$ are reduced to the twisted $q$-Frobenius–Euler polynomials (*cf.* [2]).

**Remark 3.** Substituting $a = 1$, $\lambda = 1$ and $b = e$ into Equation (2), then we get the generating function of $q$-Frobenius–Euler numbers and polynomials respectively:

$$\left(1 - \frac{1}{u}\right)\sum_{n=0}^{\infty} \frac{e^{[n]t}}{u^n} = \sum_{n=0}^{\infty} H_n(u,q)\frac{t^n}{n!} \tag{5}$$

and

$$\left(1 - \frac{1}{u}\right)\sum_{n=0}^{\infty} \frac{e^{[n+x]t}}{u^n} = \sum_{n=0}^{\infty} H_n(x,u,q)\frac{t^n}{n!}$$

where $| u |> 1$ (*cf.* [2,3]). Substituting $a = 1$, $\lambda = 1$ and $b = e$ into Equation (4), in this paper, we also generalize the Carlitz's $q$-Frobenius–Euler polynomials $H_k(x,u,q)$ as follows:

$$H_n(u,x,q) = (q^x H + [x])^n$$

(*cf.* [2,4–9]) and the Carlitz's $q$-Frobenius–Euler numbers $H_k(u,q)$, which can be determined inductively by (*cf.* [4,6–13]):

$$H_0(u,q) = 1$$

and for $n \geq 1$,

$$(qH + 1)^n - uH_n(u,q) = 0$$

(*cf.* [1–43]).

**Remark 4.** Substituting $u = -1$ in Equation (5), we have

$$2\sum_{n=0}^{\infty}(-1)^n e^{[n]t} = \sum_{n=0}^{\infty} E_n(q)\frac{t^n}{n!}$$

and also $H_n(x,-1,q) = E_n(x,q)$ (*cf.* [21]). If $q \to 1$, then Equation (5) reduces to the generating function for the classical Frobenius–Euler numbers $H_n(u)$:

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\frac{t^n}{n!}$$

(*cf.* [1–43]).

In [1], we give the following generating function, which we need in the next section:

$$\frac{t}{\lambda a^t - 1} a^{xt} = \sum_{n=0}^{\infty} Y_n(x; \lambda; a) \frac{t^n}{n!} \tag{6}$$

We also note that

$$Y_n(0; \lambda; a) = Y_n(\lambda; a)$$

If we substitute $x = 0$ and $a = 1$ into Equation (6), we see that

$$Y_n(\lambda; 1) = \frac{1}{\lambda - 1}$$

(*cf.* [1])

## 2. Identities

In this section, we derive some identities related to the $q$-Apostol type Frobenius–Euler numbers and polynomials, using generating functions.

We are now ready to give explicit formulas for computing the $q$-Apostol type Frobenius–Euler numbers and polynomials.

**Theorem 2** *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_n(u; a, b; \lambda; q) = u \left( \frac{\ln b}{1 - q} \right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{u - \lambda q^k}$$
$$+ \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{\left( \ln a + \frac{\ln b}{1-q} \right)^{n-k} \left( \frac{\ln b}{1-q} \right)^k}{u - \lambda q^k}$$

**Proof 1** *By using Equation (1), we get*

$$\sum_{n=0}^{\infty} \mathcal{H}_n(u; a, b; \lambda; q) \frac{t^n}{n!} = (1 - \frac{a^t}{u}) b^{\frac{t}{1-q}} \sum_{n=0}^{\infty} \left( \frac{\lambda}{u} \right)^n b^{-\frac{q^n}{1-q} t}$$

*From the above equation, we obtain*

$$\sum_{n=0}^{\infty} \mathcal{H}_n(u; a, b; \lambda; q) \frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty} \left( u \left( \frac{\ln b}{1 - q} \right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{u - \lambda q^k} \right) \frac{t^n}{n!}$$
$$+ \sum_{n=0}^{\infty} \left( \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{\left( \ln a + \frac{\ln b}{1-q} \right)^{n-k} \left( \frac{\ln b}{1-q} \right)^k}{u - \lambda q^k} \right) \frac{t^n}{n!}$$

*Therefore, equating the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we obtain the desired result.*

Substituting $n = 0$ into Theorem 2, we have

$$\mathcal{H}_0(u; a, b; \lambda; q) = \frac{u - 1}{u - \lambda}$$

By using Theorem 2, all the $q$-Apostol type Frobenius–Euler numbers are easily computed.

**Remark 5.** If we put $a = 1$, $u = -1$, $\lambda = 1$ and $b = e$, then Theorem 2 reduces to Theorem 1 in [21].

If we substitute $a = 1$, $\lambda = 1$ and $b = e$ into Theorem 2, then we obtain an explicit formula, for the $q$-Frobenius–Euler numbers $H_n(u; q) = \mathcal{H}_n(u; 1, e; 1; q)$, which is given by the following corollary:

**Corollary 2** *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_n(u; 1, e; 1; q) = (u - 1) \left( \frac{1}{1 - q} \right)^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{1}{u - q^k}$$

We give an explicit formula for the $q$-Apostol type Frobenius–Euler polynomials as follows:

**Theorem 3** *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{H}_n(x; u; a, b; \lambda; q) = u \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{q^{xk} \left( \frac{\ln b}{1-q} \right)^n}{u - \lambda q^k}$$
$$+ \sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} \frac{q^{xk} \left( \ln a + \frac{\ln b}{1-q} \right)^{n-k} \left( \frac{\ln b}{1-q} \right)^k}{u - \lambda q^k}$$

**Proof 2** *Proof of this theorem is the same as that of Theorem 2, so we omit it.*

**Remark 6.** If we put $a = 1$, $u = -1$, $\lambda = 1$ and $b = e$, then Theorem 3 reduces to Theorem 2 in [21].

By using Equation (4) and Theorem 3, we arrive at the following result:

**Corollary 3** *Let $n \in \mathbb{N}_0$. Then we have*

$$([x] \ln b + q^x \mathcal{H}(u; a, b; \lambda; q))^n$$
$$= \sum_{k=0}^{n} (-1)^k \frac{\binom{n}{k} u q^{xk} \left( \frac{\ln b}{1-q} \right)^n}{u - \lambda q^k} + \sum_{k=0}^{n} (-1)^{k+1} \frac{\binom{n}{k} q^{xk} \left( \ln a + \frac{\ln b}{1-q} \right)^{n-k} \left( \frac{\ln b}{1-q} \right)^k}{u - \lambda q^k}$$

### *2.1. Multiplication Formula*

Here we prove multiplication formula for the $q$-Apostol type Frobenius–Euler numbers and polynomials. This formula is very important to investigate fundamental properties of these polynomials.

**Theorem 4** *Let $n \in \mathbb{N}$. Then we have*

$$n [f]^{1-n} \mathcal{H}_{n-1}(fx; u; a, b; \lambda; q)$$
$$= -u \sum_{j=0}^{n} \binom{n}{j} Y_{n-j} \left( \frac{1}{u^{2f}}; a \right) \sum_{l=1}^{f} \frac{\lambda^l}{u^{l+f+1}} \mathcal{H}_j \left( x + \frac{l}{f}; u^f; a, b; \lambda^f; q^f \right)$$
$$+ \sum_{j=0}^{n} \binom{n}{j} Y_{n-j} \left( \frac{1}{[f]}; \frac{1}{u^{2f}}; a \right) \sum_{l=1}^{f} \frac{\lambda^l}{u^{l+f+1}} \mathcal{H}_j \left( x + \frac{l}{f}; u^f; a, b; \lambda^f; q^f \right)$$

**Proof 3** *Substituting* $n = l + mf$ *with* $m = 0, 1, \ldots, \infty$, $l = 1, 2, \ldots, f$ *into Equation (2), we get*

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b; \lambda; q) \frac{t^n}{n!} = (1 - \frac{a^t}{u}) \sum_{m=0}^{\infty} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^{l+mf} b^{[l+mf+x]t}$$

*From the above equation, we obtain*

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; a, b; \lambda; q) \frac{t^n}{n!} = \frac{1 - \frac{a^t}{u}}{1 - \frac{a^{[f]t}}{u^f}} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^l \sum_{m=0}^{\infty} \mathcal{H}_n \left(\frac{x+l}{f}; u^f; a, b; \lambda^f; q^f\right) \frac{[f]^n t^n}{n!}$$

*By using Equation (6) in the right side of the above equation, we have*

$$[f] \sum_{n=0}^{\infty} n \mathcal{H}_{n-1}(x; u; a, b; \lambda; q) \frac{t^n}{n!}$$

$$= -\frac{u}{u^{f+1}} \sum_{m=0}^{\infty} Y_n \left(\frac{1}{u^{2f}}; a\right) \frac{[f]^n t^n}{n!} \sum_{m=0}^{\infty} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^l \mathcal{H}_n \left(\frac{x+l}{f}; u^f; a, b; \lambda^f; q^f\right) \frac{[f]^n t^n}{n!}$$

$$+ \frac{1}{u^{f+1}} \sum_{m=0}^{\infty} Y_n \left(\frac{1}{[f]}; \frac{1}{u^{2f}}; a\right) \frac{[f]^n t^n}{n!} \sum_{m=0}^{\infty} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^l \mathcal{H}_n \left(\frac{x+l}{f}; u^f; a, b; \lambda^f; q^f\right) \frac{[f]^n t^n}{n!}$$

*Thus, by using the Cauchy product in the above equation and then equating the coefficients of* $\frac{t^n}{n!}$ *on both sides of the resulting equation, we get the desired result.*

If we put $a = 1$ in Equation (2), we simplify Theorem 4 as follows:

$$\sum_{n=0}^{\infty} \mathcal{H}_n(x; u; 1, b; \lambda; q) \frac{t^n}{n!} = \frac{u^f - u^{f-1}}{u^f - 1} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^l \sum_{m=0}^{\infty} \mathcal{H}_n \left(\frac{x+l}{f}; u^f; a, b; \lambda^f; q^f\right) \frac{[f]^n t^n}{n!}$$

Equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we obtain:

$$\mathcal{H}_n(x; u; 1, b; \lambda; q) = [f]^n \frac{u^f - u^{f-1}}{u^f - 1} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^l \mathcal{H}_n \left(\frac{x+l}{f}; u^f; 1, b; \lambda^f; q^f\right)$$

Replacing $x$ by $fx$ in the above equation, we get the following result:

**Corollary 4** *Let* $n \in \mathbb{N}_0$. *Then we have*

$$\mathcal{H}_n(fx; u; 1, b; \lambda; q) = [f]^n \frac{u^f - u^{f-1}}{u^f - 1} \sum_{l=1}^{f} \left(\frac{\lambda}{u}\right)^l \mathcal{H}_n \left(x + \frac{l}{f}; u^f; 1, b; \lambda^f; q^f\right) \tag{7}$$

**Remark 7.** If we put $u = -1$, $\lambda = 1$ and $b = e$, and $q \to 1$ in Equation (7), Equation (4,13) in [1].

**Remark 8.** If we put $u = -1$, $\lambda = 1$ and $b = e$, for odd $f$, in Equation (7), we get Theorem 4 in [21].

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**References**

1. Simsek, Y. Generating functions for generalized Stirling type numbers, Array type polynomials, Eulerian type polynomials and their application. **2011**, arxiv:1111.3848v1.
2. Simsek, Y. $q$-analogue of the twisted $l$-series and $q$-twisted Euler numbers. *J. Number Theory* **2005**, *110*, 267–278.
3. Satoh, J. A construction of $q$-analogue of Dedekind sums. *Nagoya Math. J.* **1992**, *127*, 129–143.
4. Carlitz, L. $q$-Bernoulli numbers and polynomials. *Duke Math. J.* **1948**, *15*, 987–1000.
5. Carlitz, L. $q$-Bernoulli and Eulerian numbers. *Trans. Am. Math. Soc.* **1954**, *76*, 332–350.
6. Choi, J.; Anderson, P.J.; Srivastava, H.M. Carlitz's $q$-Bernoulli and $q$-Euler numbers and polynomials and a class of generalized $q$-Hurwitz zeta functions. *Appl. Math. Comput.* **2009**, *215*, 1185–1208.
7. Choi, J.; Anderson, P.J.; Srivastava, H.M. Some $q$-extensions of the Apostol-Bernoulli and the Apostol-Euler polynomials of order $n$, and the multiple Hurwitz zeta function. *Appl. Math. Comput.* **2008**, *199*, 723–737.
8. Luo, Q.-M.; Srivastava, H.M. Some relationships between the Apostol-Bernoulli and Apostol-Euler polynomials. *Comput. Math. Appl.* **2005**, *10*, 631–642.
9. Satoh, J. $q$-analogue of Riemann's $\zeta$-function and $q$-Euler Numbers. *J. Number Theory* **1989**, *31*, 346–362.
10. Kim, T. On Euler-Barnes multiple zeta functions. *Russ. J. Math. Phys.* **2003**, *10*, 261–267.
11. Koblitz, N. On Carlitz's $q$-Bernoulli numbers. *J. Number Theory* **1982**, *14*, 332–339.
12. Simsek, Y.; Kim, T.; Park, D.W.; Ro, Y.S.; Jang, L.C.; Rim, S.-H. An explicit formula for the multiple Frobenius-Euler numbers and polynomials. *J. Algebra Number Theory Appl.* **2004**, *4*, 519–529.
13. Srivastava, H.M.; Kim, T.; Simsek, Y. $q$-Bernoulli numbers and polynomials associated with multiple $q$-zeta functions and basic $L$-series. *Russ. J. Math Phys.* **2005**, *12*, 241–268.
14. Jang, L.C.; Kim, T. $q$-analogue of Euler Barnes' numbers and polynomials. *Bull. Korean Math. Soc.* **2005**, *42*, 491–499.
15. Kim, T. On a $q$-analogue of the $p$-adic $\log$ gamma functions and related integrals. *J. Number Theory* **1999**, *76*, 320–329.
16. Kim, T. An invariant $p$-adic integral associated with Daehee numbers. *Integral Transform. Spec. Funct.* **2002**, *13*, 65–69.
17. Kim, T. On the analogs of Euler numbers and polynomials associated with $p$-adic $q$-integral on $Z_p$ at $q = -1$. *J. Math. Anal. Appl.* **2007**, *331*, 779–792.
18. Kim, T. On the $q$-extension of Euler and Genocchi numbers. *J. Math. Anal. Appl.* **2007**, *326*, 1458–1465.
19. Kim, T. A note on some formulae for the $q$-Euler numbers and polynomials. *Proc. Jangjeon Math. Soc.* **2006**, *9*, 227–232, arXiv:math/0608649.

20. Kim, T. The modified $q$-Euler numbers and polynomials. *Adv. Stud. Contemp. Math.* **2008**, *16*, 161–170, arXiv:math/0702523.
21. Kim, T. A note on the alternating sums of powers of consecutive $q$-integers. *Adv. Stud. Contemp. Math.* **2005**, *11*, 137–140, arXiv:math/0604227v1.
22. Kim, T.; Jang, L.C.; Park, H.K. A note on $q$-Euler and Genocchi numbers. *Proc. Jpn. Acad.* **2001**, *77*, 139–141.
23. Kim, T.; Rim, S.-H. On the twisted $q$-Euler numbers and polynomials associated with basic $q$-$l$-functions. *J. Math. Anal. Appl.* **2007**, *336*, 738–744.
24. Kim, T.; Rim, S.-H. A new Changhee $q$-Euler numbers and polynomials associated with $p$-adic $q$-integral. *Comput. Math. Appl.* **2007**, *54*, 484–489.
25. Kim, T.; Jang, L.C.; Rim, S.-H.; Pak, H.K. On the twisted $q$-zeta functions and $q$-Bernoulli polynomials. *Far East J. Appl. Math.* **2003**, *13*, 13–21.
26. Koblitz, N. A New proof of certain formulas for $p$-adic $L$-functions. *Duke Math. J.* **1979**, *46*, 455–468.
27. Rim, S.-H.; Kim, T. A note on $q$-Euler numbers associated with the basic $q$-zeta function. *Appl. Math. Lett.* **2007**, *20*, 366–369.
28. Ozden, H.; Simsek, Y. A new extension of $q$-Euler numbers and polynomials related to their interpolation functions. *Appl. Math. Lett.* **2008**, *21*, 934–939.
29. Ozden, H.; Simsek, Y.; Srivastava, H.M. A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials. *Comput. Math. Appl.* **2010**, *60*, 2779–2787.
30. Schempp, W. A contour integral representation of Euler-Frobenius polynomials. *J. Approx. Theory* **1981**, *31*, 272–278.
31. Schempp, W. Euler-Frobenius polynomials. *Numer. Methods Approx. Theory* **1983**, *7*, 131–138.
32. Shiratani, K. On Euler Numbers. *Mem. Fac. Sci. Kyushu Univ.* **1975**, *27*, 1–5.
33. Shiratani, K.; Yamamoto, S. On a $p$-adic interpolation function for the Euler numbers and its derivatives. *Mem. Fac. Sci. Kyushu Univ.* **1985**, *39*, 113–125.
34. Simsek, Y. Theorems on twisted $L$-functions and twisted Bernoulli numbers. *Adv. Stud. Contemp. Math.* **2005**, *11*, 205–218.
35. Simsek, Y. Twisted $(h, q)$-Bernoulli numbers and polynomials related to twisted $(h, q)$-zeta function and $L$-function. *J. Math. Anal. Appl.* **2006**, *324*, 790–804.
36. Simsek, Y. On $p$-adic twisted $q$-$L$-functions related to generalized twisted Bernoulli numbers. *Russ. J. Math. Phys.* **2006**, *13*, 327–339.
37. Simsek, Y.; Yurekli, O.; Kurt, V. On interpolation functions of the twisted generalized Frobenius-Euler numbers. *Adv. Stud. Contemp. Math.* **2007**, *15*, 187–194.
38. Simsek, Y.; Bayad, A.; Lokesha, V. $q$-Bernstein polynomials related to $q$-Frobenius-Euler polynomials, $l$-functions, and $q$-Stirling numbers. *Math. Meth. Appl. Sci.* **2012**, *35*, 877–884.
39. Srivastava, H.M. Some generalizations and basic (or $q$-) extensions of the Bernoulli, Euler and Genocchi polynomials. *Appl. Math. Inform. Sci.* **2011**, *5*, 390–444.
40. Srivastava, H.M.; Choi, J. *Zeta and $q$-Zeta Functions and Associated Series and Integrals*; Elsevier Science Publishers: Amsterdam, The Netherlands, 2012.

41. Srivastava, H.M.; Kurt, B.; Simsek, Y. Some families of Genocchi type polynomials and their interpolation functions. *Integral Transform. Spec. Funct.* **2012**, *23*, 919–938.
42. Tsumura, H. On a $p$-adic interpolation of the generalized Euler numbers and its applications. *Tokyo J. Math.* **1987**, *10*, 281–293.
43. Tsumura, H. A note on $q$-analogues of the Dirichlet series and $q$-Bernoulli numbers. *J. Number Theory* **1991**, *39*, 251–256.