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*Article*

# Quasitriangular Structure of Myhill–Nerode Bialgebras

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**Abstract:** In computer science the Myhill–Nerode Theorem states that a set $L$ of words in a finite alphabet is accepted by a finite automaton if and only if the equivalence relation $\sim_L$, defined as $x \sim_L y$ if and only if $xz \in L$ exactly when $yz \in L, \forall z$, has finite index. The Myhill–Nerode Theorem can be generalized to an algebraic setting giving rise to a collection of bialgebras which we call Myhill–Nerode bialgebras. In this paper we investigate the quasitriangular structure of Myhill–Nerode bialgebras.

**Keywords:** algebra; coalgebra; bialgebra; Myhill–Nerode theorem; Myhill–Nerode bialgebra; quasitriangular structure

## 1. Introduction

Let $\Sigma_0$ be a finite alphabet and let $\hat{\Sigma}_0$ denote the set of words formed from the letters in $\Sigma_0$. Let $L \subseteq \hat{\Sigma}_0$ be a language, and let $\sim_L$ be the equivalence relation defined as $x \sim_L y$ if and only if $xz \in L$ exactly when $yz \in L, \forall z \in \hat{\Sigma}_0$. The Myhill–Nerode Theorem of computer science states that $L$ is accepted by a finite automaton if and only if $\sim_L$ has finite index (cf. [1, 1, Chapter III, §9, Proposition 9.2], [2, §3.4, Theorem 3.9]). In [3, Theorem 5.4] the authors generalize the Myhill–Nerode theorem to an algebraic setting in which a finiteness condition involving the action of a semigroup on a certain function plays the role of the finiteness of the index of $\sim_L$, while a bialgebra plays the role of the finite automaton which accepts the language. We call these bialgebras *Myhill–Nerode bialgebras*.

The purpose of this paper is to investigate the quasitriangular structure of Myhill–Nerode bialgebras.

By construction, a Myhill–Nerode bialgebra $B$ is cocommutative and finite dimensional over its base field. Thus $B$ admits (at least) the trivial quasitriangular structure $(B, 1 \otimes 1)$. We ask: does $B$ (or its linear dual $B^*$) have any non-trivial quasitriangular structures?

Towards a solution to this problem, we construct a class of commutative Myhill–Nerode bialgebras and give a complete account of the quasitriangular structure of one of them. We begin with some background information regarding algebras, coalgebras, and bialgebras.

## 2. Algebras, Coalgebras and Bialgebras

Let $K$ be an arbitrary field of characteristic $0$ and let $A$ be a vector space over $K$ with scalar product $ra$ for all $r \in K$, $a \in A$. Scalar product defines two maps $s_1 : K \otimes A \to A$ with $r \otimes a \mapsto ra$ and $s_2 : A \otimes K \to A$ with $a \otimes r \mapsto ra$, for $a \in A$, $r \in K$. Let $I_A : A \to A$ denote the identity map. A $K$-*algebra* is a triple $(A, m_A, \eta_A)$ where $m_A : A \otimes A \to A$ is a $K$-linear map which satisfies

$$m_A(I_A \otimes m_A)(a \otimes b \otimes c) = m_A(m_A \otimes I_A)(a \otimes b \otimes c) \tag{1}$$

and $\eta_A : K \to A$ is a $K$-linear map for which

$$m_A(I_A \otimes \eta_A)(a \otimes r) = ra = m_A(\eta_A \otimes I_A)(r \otimes a) \tag{2}$$

for all $r \in K$, $a, b, c \in A$. The map $m_A$ is the *multiplication map* of $A$ and $\eta_A$ is the *unit map* of $A$. Condition (1) is the *associative property* and Condition (2) is the *unit property*.

We write $m_A(a \otimes b)$ as $ab$. The element $1_A = \eta_A(1_K)$ is the unique element of $A$ for which $a1_A = a = 1_Aa$ for all $a \in A$. Let $A, B$ be algebras. An *algebra homomorphism* from $A$ to $B$ is a $K$-linear map $\phi : A \to B$ such that $\phi(m_A(a_1 \otimes a_2)) = m_B(\phi(a_1) \otimes \phi(a_2))$ for all $a_1$, $a_2 \in A$, and $\phi(1_A) = 1_B$. In particular, for $A$ to be a subalgebra of $B$ we require $1_A = 1_B$.

For any two vector spaces $V$, $W$ let $\tau : V \otimes W \to W \otimes V$ denote the *twist map* defined as $\tau(a \otimes b) = b \otimes a$, for $a \in V$, $b \in W$. For $K$-algebras $A, B$, we have that $A \otimes B$ is a $K$-algebra with multiplication

$$m_{A\otimes B} : (A \otimes B) \otimes (A \otimes B) \to A \otimes B$$

defined by

$$\begin{aligned} m_{A\otimes B}((a \otimes b) \otimes (c \otimes d)) &= (m_A \otimes m_B)(I_A \otimes \tau \otimes I_B)(a \otimes (b \otimes c) \otimes d) \\ &= (m_A \otimes m_B)((a \otimes c) \otimes (b \otimes d)) = ac \otimes bd \end{aligned}$$

for $a, c \in A$, $b, d \in B$. The unit map $\eta_{A\otimes B} : K \to A \otimes B$ given as

$$\eta_{A\otimes B}(r) = \eta_A(r) \otimes 1_B$$

for $r \in K$.

Let $C$ be a $K$-vector space. A $K$-*coalgebra* is a triple $(C, \Delta_C, \epsilon_C)$ in which $\Delta_C : C \to C \otimes C$ is $K$-linear and satisfies

$$(I_C \otimes \Delta_C)\Delta_C(c) = (\Delta_C \otimes I_C)\Delta_C(c) \tag{3}$$

and $\epsilon_C : C \to K$ is $K$-linear with

$$s_1(\epsilon_C \otimes I_C)\Delta_C(c) = c = s_2(I_C \otimes \epsilon_C)\Delta_C(c) \tag{4}$$

for all $c \in C$. The maps $\Delta_C$ and $\epsilon_C$ are the *comultiplication* and *counit* maps, respectively, of the coalgebra $C$. Condition (3) is the *coassociative property* and Condition (4) is the *counit property*.

We use the notation of M. Sweedler [4, §1.2] to write

$$\Delta_C(c) = \sum_{(c)} c_{(1)} \otimes c_{(2)}$$

Note that Condition (4) implies that

$$\sum_{(c)} \epsilon_C(c_{(1)})c_{(2)} = c = \sum_{(c)} \epsilon_C(c_{(2)})c_{(1)} \tag{5}$$

Let $C$ be a $K$-coalgebra. A nonzero element $c$ of $C$ for which $\Delta_C(c) = c \otimes c$ is a *grouplike element* of $C$. If $c$ is grouplike, then

$$\begin{aligned} c &= s_1(\epsilon_C \otimes I_C)\Delta_C(c) \\ &= s_1(\epsilon_C \otimes I_C)(c \otimes c) = \epsilon_C(c)c \end{aligned}$$

and so, $\epsilon_C(c) = 1$. The grouplike elements of $C$ are linearly independent [4, Proposition 3.2.1].

Let $C, D$ be coalgebras. A $K$-linear map $\phi : C \to D$ is a *coalgebra homomorphism* if $(\phi\otimes\phi)\Delta_C(c) = \Delta_D(\phi(c))$ and $\epsilon_C(c) = \epsilon_D(\phi(c))$ for all $c \in C$. The tensor product $C \otimes D$ of two coalgebras is again a coalgebra with comultiplication map

$$\Delta_{C\otimes D} : C \otimes D \to (C \otimes D) \otimes (C \otimes D)$$

defined by

$$\begin{aligned} \Delta_{C\otimes D}(c \otimes d) &= (I_C \otimes \tau \otimes I_D)(\Delta_C \otimes \Delta_D)(c \otimes d) \\ &= (I_C \otimes \tau \otimes I_D)(\Delta_C(c) \otimes \Delta_D(d)) \\ &= (I_C \otimes \tau \otimes I_D)(\sum_{(c),(d)} c_{(1)} \otimes c_{(2)} \otimes d_{(1)} \otimes d_{(2)}) \\ &= \sum_{(c),(d)} c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)} \end{aligned}$$

for $c \in C$, $d \in D$. The counit map $\epsilon_{C\otimes D} : C \otimes D \to K$ is defined as

$$\epsilon_{C\otimes D}(c \otimes d) = \epsilon_C(c)\epsilon_D(d)$$

for $c \in C$, $d \in D$.

A *$K$-bialgebra* is a $K$-vector space $B$ together with maps $m_B$, $\eta_B$, $\Delta_B$, $\epsilon_B$ for which $(B, m_B, \eta_B)$ is a $K$-algebra and $(B, \Delta_B, \epsilon_B)$ is a $K$-coalgebra and for which $\Delta_B$ and $\epsilon_B$ are algebra homomorphisms. Let $B, B'$ be bialgebras. A $K$-linear map $\phi : B \to B'$ is a *bialgebra homomorphism* if $\phi$ is both an algebra and coalgebra homomorphism.

A *$K$-Hopf algebra* is a bialgebra $H$ together with an additional $K$-linear map $\sigma_H : H \to H$ that satisfies

$$m_H(I_H \otimes \sigma_H)\Delta_H(h) = \epsilon_H(h)1_H = m_H(\sigma_H \otimes I_H)\Delta_H(h) \tag{6}$$

for all $h \in H$. The map $\sigma_H$ is the *coinverse* (or *antipode*) map and property Condition (6) is the *coinverse* (or *antipode*) *property*. Though we will not consider Hopf algebras here, more details on the subject can be found in [5–8].

An important example of a $K$-bialgebra is given as follows. Let $G$ be a semigroup with unity, $1$. Let $KG$ denote the semigroup algebra. Then $KG$ is a bialgebra with comultiplication map

$$\Delta_{KG} : KG \to KG \otimes KG$$

defined by $x \mapsto x \otimes x$, for all $x \in G$, and counit map $\epsilon_{KG} : KG \to K$ given by $x \mapsto 1$, for all $x \in G$. The bialgebra $KG$ is the *semigroup bialgebra on* $G$.

Let $B$ be a bialgebra, and let $A$ be an algebra which is a left $B$-module with action denoted by "$\cdot$". Suppose that

$$b \cdot (aa') = \sum_{(b)} (b_{(1)} \cdot a)(b_{(2)} \cdot a')$$

and

$$b \cdot 1_A = \epsilon_B(b) 1_A$$

for all $a, a' \in A$, $b \in B$. Then $A$ is a *left* $B$*-module algebra*. A $K$-linear map $\phi : A \to A'$ is a *left* $B$*-module algebra homomorphism* if $\phi$ is both an algebra and a left $B$-module homomorphism.

Let $C$ be a coalgebra and a right $B$-module with action denoted by "$\cdot$". Suppose that for all $c \in C$, $b \in B$,

$$\Delta_C(c \cdot b) = \sum_{(c),(b)} c_{(1)} \cdot b_{(1)} \otimes c_{(2)} \cdot b_{(2)}$$

and

$$\epsilon_C(c \cdot b) = \epsilon_C(c)\epsilon_B(b)$$

Then $C$ is a *right* $B$*-module coalgebra*. A $K$-linear map $\phi : C \to C'$ is a *right* $B$*-module coalgebra homomorphism* if $\phi$ is both a coalgebra and a right $B$-module homomorphism.

Let $C$ be a coalgebra and let $C^* = \mathrm{Hom}_K(C, K)$ denote the linear dual of $C$. Then the coalgebra structure of $C$ induces an algebra structure on $C^*$.

**Proposition 2.1** *If $C$ is a coalgebra, then $C^*$ is an algebra.*

*Proof.* Recall that $C$ is a triple $(C, \Delta_C, \epsilon_C)$ where $\Delta_C : C \to C \otimes C$ is $K$-linear and satisfies the coassociativity property, and $\epsilon_C : C \to K$ is $K$-linear and satisfies the counit property. The dual map of $\Delta_C$ is a $K$-linear map

$$\Delta_C^* : (C \otimes C)^* \to C^*$$

Since $C^* \otimes C^* \subseteq (C \otimes C)^*$, we define the multiplication map of $C^*$, denoted as $m_{C^*}$, to be the restriction of $\Delta_C^*$ to $C^* \otimes C^*$. For $f, g \in C^*$, $c \in C$,

$$(fg)(c) = m_{C^*}(f \otimes g)(c) = \Delta_C^*(f \otimes g)(c) = (f \otimes g)(\Delta_C(c)) = \sum_{(c)} f(c_{(1)})g(c_{(2)})$$

The coassociatively property of $\Delta_C$ yields the associative property of $m_{C^*}$. Indeed, for $f, g, h \in C^*$, $c \in C$,

$$\begin{aligned}
m_{C^*}(I_{C^*} \otimes m_{C^*})(f \otimes g \otimes h)(c) &= \Delta_C^*(I_{C^*} \otimes \Delta_C^*)(f \otimes g \otimes h)(c) \\
&= \Delta_C^*(f \otimes \Delta_C^*(g \otimes h))(c) \\
&= (f \otimes \Delta_C^*(g \otimes h))\Delta_C(c) \\
&= \sum_{(c)} f(c_{(1)})\Delta_C^*(g \otimes h)(c_{(2)}) \\
&= \sum_{(c)} f(c_{(1)})(g \otimes h)\Delta_C(c_{(2)}) \\
&= (f \otimes g \otimes h)(\sum_{(c)} c_{(1)} \otimes \Delta_C(c_{(2)})) \\
&= (f \otimes g \otimes h)(\sum_{(c)} \Delta_C(c_{(1)}) \otimes c_{(2)}) \quad \text{by Condition (3)} \\
&= \sum_{(c)}(f \otimes g)\Delta_C(c_{(1)}) \otimes h(c_{(2)}) \\
&= \sum_{(c)} \Delta_C^*(f \otimes g)(c_{(1)}) \otimes h(c_{(2)}) \\
&= (\Delta_C^*(f \otimes g) \otimes h)\Delta_C(c) \\
&= \Delta_C^*(\Delta_C^*(f \otimes g) \otimes h)(c) \\
&= \Delta_C^*(\Delta_C^* \otimes I_{C^*})(f \otimes g \otimes h)(c) \\
&= m_{C^*}(m_{C^*} \otimes I_{C^*})(f \otimes g \otimes h)(c)
\end{aligned}$$

In addition, the counit map of $C$ dualizes to yield

$$\epsilon_C^* : K := K^* \to C^*$$

defined as $\epsilon_C^*(k)(c) = k(\epsilon(c)) = k\epsilon(c)$. Thus we define the unit map $\eta_{C^*}$ to be $\epsilon_C^*$. One can show that the counit property of $\epsilon_C$ implies the unit property for $\eta_{C^*}$. To this end, for $f \in C^*$, $r \in K$, $c \in C$,

$$\begin{aligned}
m_{C^*}(I_{C^*} \otimes \eta_{C^*})(f \otimes r)(c) &= \Delta_C^*(I_{C^*} \otimes \epsilon_C^*)(f \otimes r)(c) \\
&= \Delta_C^*(f \otimes \epsilon_C^*(r))(c) \\
&= (f \otimes \epsilon_C^*(r))(\Delta_C(c)) \\
&= \sum_{(c)} f(c_{(1)})\epsilon_C^*(r)(c_{(2)}) \\
&= \sum_{(c)} f(c_{(1)})r(\epsilon_C(c_{(2)})) \\
&= r\sum_{(c)} f(c_{(1)})\epsilon_C(c_{(2)}) \\
&= r\sum_{(c)} \epsilon_C(c_{(2)})f(c_{(1)}) \\
&= r\sum_{(c)} f(\epsilon_C(c_{(2)})c_{(1)}) \\
&= rf(\sum_{(c)} \epsilon_C(c_{(2)})c_{(1)}) \\
&= rf(c) \quad \text{by Condition (5)}
\end{aligned}$$

In a similar manner, one obtains

$$m_{C^*}(\eta_{C^*} \otimes I_{C^*})(r \otimes f) = rf$$

Thus $(C^*, m_{C^*}, \eta_{C^*})$ is an algebra. Note that $\eta_{C^*}(1_K)(c) = \epsilon_C(c), \forall c$, and so, $\epsilon_C$ is the unique element of $C^*$ for which $\epsilon_C f = f = f\epsilon_C$ for all $f \in C^*$. $\diamond$

Let $(A, m_A, \eta_A)$ be a $K$-algebra. Then one may wonder if $A^*$ is a $K$-coalgebra. The multiplication map $m_A : A \otimes A \to A$ dualizes to yield $m_A^* : A^* \to (A \otimes A)^*$. Unfortunately, if $A$ is infinite dimensional over $K$, then $A^* \otimes A^*$ is a proper subset of $(A \otimes A)^*$, and hence $m_A^*$ may not induce the required comultiplication map $A^* \to A^* \otimes A^*$.

There is still however a $K$-coalgebra arising via duality from the algebra $A$. An ideal $I$ of $A$ is *cofinite* if $\dim(A/I) < \infty$. The *finite dual* $A^\circ$ of $A$ is defined as

$$A^\circ = \{f \in A^* : \ f(I) = 0 \text{ for some cofinite ideal } I \text{ of } A\}$$

Note that $A^\circ$ is the largest subspace $W$ of $A^*$ for which $m_A^*(W) \subseteq W \otimes W$.

**Proposition 2.2** *If $A$ is an algebra, then $A^\circ$ is a coalgebra.*

*Proof.* The proof is similar to the method used in Proposition 2.1. We restrict the map $m_A^*$ to $A^\circ$ to yield the $K$-linear map $m_A^* : A^\circ \to (A \otimes A)^*$. Now by [4, Proposition 6.0.3], $m_A^*(A^\circ) \subseteq A^\circ \otimes A^\circ$. Let $\Delta_{A^\circ}$ denote the restriction of $m_A^*$ to $A^\circ$. We show that $\Delta_{A^\circ}$ satisfies the coassociative condition. For $f \in A^\circ$, $a, b, c \in A$, we have

$$\begin{aligned}
(I \otimes \Delta_{A^\circ})\Delta_{A^\circ}(f)(a \otimes b \otimes c) &= (I \otimes m_A^*)m_A^*(f)(a \otimes b \otimes c) \\
&= m_A^*(f)((I \otimes m_A)(a \otimes b \otimes c)) \\
&= m_A^*(f)(a \otimes bc) \\
&= f(m_A(a \otimes bc)) \\
&= f(a(bc)) \\
&= f((ab)c) \\
&= f(m_A(ab \otimes c)) \\
&= m_A^*(f)(ab \otimes c) \\
&= m_A^*(f)((m_A \otimes I)(a \otimes b \otimes c)) \\
&= (m_A^* \otimes I)m_A^*(f)(a \otimes b \otimes c) \\
&= (\Delta_{A^\circ} \otimes I)\Delta_{A^\circ}(f)(a \otimes b \otimes c)
\end{aligned}$$

For the counit map of $A^\circ$, we consider the dual map $\eta_A^* : A^* \to K^* := K$. Now $\eta_A^*$ restricts to a map $\eta_A^* : A^\circ \to K$. We let $\epsilon_{A^\circ}$ denote the restriction of $\eta_A^*$ to $A^\circ$. For $f \in A^\circ$, $r \in K$,

$$\epsilon_{A^\circ}(f)(r) = f(\eta_A(r)) = f(r1_A) = rf(1_A) = f(1_A)(r)$$

and so, $\epsilon_{A^\circ}(f) = f(1_A)$. We show that $\epsilon_{A^\circ}$ satisfies the counit property. First let $s_1 : K \otimes A^\circ \to A^\circ$ be defined by the scalar multiplication of $A^\circ$. For $f \in A^\circ$, $r \in K$, $a \in A$,

$$\begin{aligned} s_1((\epsilon_{A^\circ} \otimes I)\Delta_{A^\circ}(f))(a) &= s_1((\eta_A^* \otimes I)m_A^*(f))(a) \\ &= (\eta_A^* \otimes I)m_A^*(f)(s_1^*(a)) \\ &= (\eta_A^* \otimes I)m_A^*(f)(1 \otimes a) \\ &= m_A^*(f)((\eta_A \otimes I)(1 \otimes a)) \\ &= f(m_A(\eta_A \otimes I)(1 \otimes a)) \\ &= f(a) \end{aligned}$$

In a similar manner, one obtains

$$s_2((I \otimes \epsilon_{A^\circ})\Delta_{A^\circ}(f))(a) = f(a)$$

where $s_2 : A^\circ \otimes K \to A^\circ$ is given by scalar multiplication. Thus $A^\circ$ is a coalgebra.

$\diamond$

**Proposition 2.3** *If $B$ is a bialgebra, then $B^\circ$ is a bialgebra.*

*Proof.* As a coalgebra, $B$ is a triple $(B, \Delta_B, \epsilon_B)$. By Proposition 2.1, $B^*$ is an algebra with maps $m_{B^*} = \Delta_B^*$ and $\eta_{B^*} = \epsilon_B^*$. Let $m_{B^\circ}$ denote the restriction of $m_{B^*}$ to $B^\circ \otimes B^\circ$, and let $\eta_{B^\circ}$ denote the restriction of $\eta_{B^*}$ to $B^\circ$. Then the triple $(B^\circ, m_{B^\circ}, \eta_{B^\circ})$ is a $K$-algebra.

As an algebra, $B$ is a triple $(B, m_B, \eta_B)$. By Proposition 2.2, $B^\circ$ is a coalgebra with maps $\Delta_{B^\circ}$ and $\epsilon_{B^\circ}$. It remains to show that $\Delta_{B^\circ}$ and $\epsilon_{B^\circ}$ are algebra homomorphisms. First observe that for $f, g \in B^\circ$, $a, b \in B$ one has

$$(fg)(a) = m_{B^\circ}(f \otimes g)(a) = \Delta_B^*(f \otimes g)(a) = (f \otimes g)\Delta_B(a)$$

and

$$\Delta_{B^\circ}(f)(a \otimes b) = m_B^*(f)(a \otimes b) = f(m_B(a \otimes b)) = f(ab)$$

We have

$$\begin{aligned} \Delta_{B^\circ}(fg)(a \otimes b) &= (fg)(ab) \\ &= (f \otimes g)(\Delta_B(ab)) \\ &= (f \otimes g)(\Delta_B(a)\Delta_B(b)) \\ &= (f \otimes g)(m_{B\otimes B}(\Delta_B(a) \otimes \Delta_B(b)) \\ &= m_{B\otimes B}^*(f \otimes g)(\Delta_B(a) \otimes \Delta_B(b)) \\ &= (I \otimes \tau \otimes I)(\Delta_{B^\circ} \otimes \Delta_{B^\circ})(f \otimes g)(\Delta_B(a) \otimes \Delta_B(b)) \\ &= (\Delta_{B^\circ}(f) \otimes \Delta_{B^\circ}(g))(I \otimes \tau \otimes I)(\Delta_B \otimes \Delta_B)(a \otimes b) \\ &= (\Delta_{B^\circ}(f) \otimes \Delta_{B^\circ}(g))(\Delta_{B\otimes B}(a \otimes b)) \\ &= \Delta_{B\otimes B}^*(\Delta_{B^\circ}(f) \otimes \Delta_{B^\circ}(g))(a \otimes b) \\ &= m_{B^\circ\otimes B^\circ}(\Delta_{B^\circ}(f) \otimes \Delta_{B^\circ}(g))(a \otimes b) \\ &= (\Delta_{B^\circ}(f)\Delta_{B^\circ}(g))(a \otimes b) \end{aligned}$$

and so $\Delta_{B^\circ}$ is an algebra map. We next show that $\epsilon_{B^\circ}$ is an algebra map. For $f, g \in B^\circ$,

$$\epsilon_{B^\circ}(f) = \epsilon_{B^\circ}(f)(1) = f(\eta_B(1)) = f(1_B)$$

Thus

$$\begin{aligned} \epsilon_{B^\circ}(fg) &= (fg)(1_B) \\ &= f(1_B)g(1_B) \\ &= \epsilon_{B^\circ}(f)\epsilon_{B^\circ}(g) \end{aligned}$$

and so, $\epsilon_{B^\circ}$ is an algebra map.

$\diamond$

**Proposition 2.4** *Suppose that $B$ is a bialgebra that is finite dimensional over $K$. Then $B^*$ is a bialgebra.*

*Proof.* If $\dim(B) < \infty$, then $B^\circ = B^*$. The result then follows from Proposition 2.3.

$\diamond$

Let $G = \{x_1, x_2, \ldots, x_n\}$ be a finite semigroup with unity element $1_{KG} = x_1$, and let $KG$ denote the semigroup bialgebra. By Proposition 2.4 $KG^*$ is a bialgebra of dimension $n$ over $K$. Let $\{e_1, e_2, \ldots, e_n\}$ be the dual basis for $KG^*$ defined as $e_i(x_j) = \delta_{i,j}$.

**Proposition 2.5** *The comultiplication map $\Delta_{KG^*} : KG^* \to KG^* \otimes KG^*$ is given as*

$$\Delta_{KG^*}(e_i) = \sum_{x_i = x_j x_k} e_j \otimes e_k$$

*and the counit map $\epsilon_{KG^*} : KG^* \to K$ is defined as $\epsilon_{KG^*}(e_i) = e_i(x_1) = \delta_{i,1}$.*

*Proof,* See [7, (1.3.7)]. $\diamond$

Let $B$ be a $K$-bialgebra. Then $B$ is *cocommutative* if

$$\tau(\Delta_B(b)) = \Delta_B(b)$$

for all $b \in B$.

**Proposition 2.6** *If $B$ is cocommutative, then $B^\circ$ is a commutative algebra. If $B$ is a commutative algebra, then $B^\circ$ is cocommutative.*

*Proof.* See [7, Lemma 1.2.2, Proposition 1.2.4].

$\diamond$

## 3. Quasitriangular Bialgebras

Let $B$ be a bialgebra and let $B \otimes B$ be the tensor product algebra. Let $U(B \otimes B)$ denote the group of units in $B \otimes B$ and let $R \in U(B \otimes B)$. The pair $(B, R)$ is *almost cocommutative* if the element $R$ satisfies

$$\tau(\Delta_B(b)) = R\Delta_B(b)R^{-1} \tag{7}$$

for all $b \in B$.

If the bialgebra $B$ is cocommutative, then the pair $(B, 1\otimes 1)$ is almost cocommutative since $R = 1\otimes 1$ satisfies Condition (7). However, if $B$ is commutative and non-cocommutative, then $(B, R)$ cannot be almost cocommutative for any $R \in U(B \otimes B)$ since Condition (7) in this case reduces to the condition for cocommutativity.

Write $R = \sum_{i=1}^{n} a_i \otimes b_i \in U(B \otimes B)$. Let

$$R^{12} = \sum_{i=1}^{n} a_i \otimes b_i \otimes 1 \in B \otimes B \otimes B$$

$$R^{13} = \sum_{i=1}^{n} a_i \otimes 1 \otimes b_i \in B \otimes B \otimes B$$

$$R^{23} = \sum_{i=1}^{n} 1 \otimes a_i \otimes b_i \in B \otimes B \otimes B$$

The pair $(B, R)$ is *quasitriangular* if $(B, R)$ is almost cocommutative and the following conditions hold

$$(\Delta_B \otimes I)R = R^{13}R^{23} \tag{8}$$

$$(I \otimes \Delta_B)R = R^{13}R^{12} \tag{9}$$

Clearly, if $B$ is cocommutative then $(B, 1 \otimes 1)$ is quasitriangular.

Let $B$ be a bialgebra. A *quasitriangular structure* is an element $R \in U(B \otimes B)$ so that $(B, R)$ is quasitriangular. Let $(B, R)$ and $(B', R')$ be quasitriangular bialgebras. Then $(B, R)$, $(B', R')$ are *isomorphic as quasitriangular bialgebras* if there exists a bialgebra isomorphism $\phi : B \to B'$ for which $R' = (\phi \otimes \phi)(R)$. Two quasitriangular structures $R, R'$ on a bialgebra $B$ are *equivalent quasitriangular structures* if $(B, R) \cong (B, R')$ as quasitriangular bialgebras.

The following proposition shows that every bialgebra isomorphism $\phi : B \to B'$ with $B$ quasitriangular extends to an isomorphism of quasitriangular bialgebras.

**Proposition 3.1** *Suppose $(B, R)$ is quasitriangular and suppose that $\phi : B \to B'$ is an isomorphism of $K$-bialgebras. Let $R' = (\phi \otimes \phi)(R)$. Then $(B', R')$ is quasitriangular.*

*Proof.* Note that $(\phi \otimes \phi)(R^{-1}) = ((\phi \otimes \phi)(R))^{-1}$. Let $b' \in B'$. Then there exists $b \in B$ for which $\phi(b) = b'$. Now

$$\begin{aligned}
\tau\Delta_{B'}(b') &= \tau\Delta_{B'}(\phi(b)) \\
&= \tau(\phi \otimes \phi)\Delta_B(b) \\
&= (\phi \otimes \phi)\tau\Delta_B(b) \\
&= (\phi \otimes \phi)(R\Delta_B(b)R^{-1}) \\
&= (\phi \otimes \phi)(R)(\phi \otimes \phi)\Delta_B(b)(\phi \otimes \phi)(R^{-1}) \\
&= (\phi \otimes \phi)(R)\Delta_{B'}(\phi(b))((\phi \otimes \phi)(R))^{-1} \\
&= (\phi \otimes \phi)(R)\Delta_{B'}(b')((\phi \otimes \phi)(R))^{-1} \\
&= R'\Delta_{B'}(b')(R')^{-1}
\end{aligned}$$

and so, $(B, R')$ is almost cocommutative. Moreover,

$$
\begin{aligned}
(\Delta_{B'} \otimes I)(R') &= (\Delta_{B'} \otimes I)(\phi \otimes \phi)(R) \\
&= (\Delta_{B'} \otimes I)(\sum_{i=1}^{n} \phi(a_i) \otimes \phi(b_i)) \\
&= \sum_{i=1}^{n} \Delta_{B'}\phi(a_i) \otimes \phi(b_i)) \\
&= \sum_{i=1}^{n} (\phi \otimes \phi)\Delta_B(a_i) \otimes \phi(b_i)) \\
&= (\phi \otimes \phi \otimes \phi)(\sum_{i=1}^{n} \Delta_B(a_i) \otimes b_i) \\
&= (\phi \otimes \phi \otimes \phi)(\Delta_B \otimes I)(R) \\
&= (\phi \otimes \phi \otimes \phi)(R^{13}R^{23}) \\
&= (\phi \otimes \phi \otimes \phi)((\sum_{i=1}^{n} a_i \otimes 1 \otimes b_i)(\sum_{i=1}^{n} 1 \otimes a_i \otimes b_i)) \\
&= (\sum_{i=1}^{n} \phi(a_i) \otimes 1 \otimes \phi(b_i))(\sum_{i=1}^{n} 1 \otimes \phi(a_i) \otimes \phi(b_i)) \\
&= ((\phi \otimes \phi)(R))^{13}((\phi \otimes \phi)(R))^{23} \\
&= (R')^{13}(R')^{23}
\end{aligned}
$$

In a similar manner one shows that

$$(I \otimes \Delta_{B'})(R') = (R')^{13}(R')^{12}$$

Thus $(B', R')$ is quasitriangular. $\diamond$

Quasitriangular bialgebras are important since they give rise to solutions of the equation

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{10}$$

which is known as the *quantum Yang–Baxter equation (QYBE)*. The QYBE was first introduced in statistical mechanics, see [9]. An element $R \in B \otimes B$ which satisfies (10) is a *solution to the QYBE.*

Certainly, the QYBE admits the trivial solution $R = 1 \otimes 1$, and of course, if $B$ is commutative, then any $R \in B \otimes B$ is a solution to the QYBE. For $B$ non-commutative, it is of great interest to find non-trivial solutions $R \in B \otimes B$ to the QYBE. We have the following result due to V. G. Drinfeld [10].

**Proposition 3.2** *(Drinfeld) Suppose $(B, R)$ is quasitriangular. Then $R$ is a solution to the QYBE.*

*Proof.* One has

$$\begin{aligned}
R^{12}R^{13}R^{23} &= R^{12}(\Delta \otimes I)(R) \quad \text{by (8)} \\
&= (R \otimes 1)(\sum_{i=1}^{n} \Delta(a_i) \otimes b_i) \\
&= \sum_{i=1}^{n} R\Delta(a_i) \otimes b_i \\
&= \sum_{i=1}^{n} \tau\Delta(a_i)R \otimes b_i \quad \text{by (7)} \\
&= (\sum_{i=1}^{n} \tau\Delta(a_i) \otimes b_i)(R \otimes 1) \\
&= (\tau\Delta \otimes I)(R)R^{12} \\
&= (\tau \otimes I)(\Delta \otimes I)(R)R^{12} \\
&= (\tau \otimes I)(R^{13}R^{23})R^{12} \quad \text{by (8)} \\
&= R^{23}R^{13}R^{12}
\end{aligned}$$

$\diamond$

The following proposition provides necessary conditions on $R \in U(B \otimes B)$ in order for $(B, R)$ to be quasitriangular.

**Proposition 3.3** *Suppose* $(B, R)$ *is quasitriangular. Then*

*(i)* $s_1(\epsilon \otimes I)(R) = 1$,

*(ii)* $s_2(I \otimes \epsilon)(R) = 1$.

*Proof.* For (i) one has

$$\begin{aligned}
(s_1 \otimes I)(\epsilon \otimes I \otimes I)(\Delta \otimes I)(R) &= (s_1 \otimes I)(\epsilon \otimes I \otimes I)(\sum_{i=1}^{n} \Delta(a_i) \otimes b_i) \\
&= (s_1 \otimes I)(\sum_{i=1}^{n} (\epsilon \otimes I)\Delta(a_i) \otimes b_i) \\
&= \sum_{i=1}^{n} s_1(\epsilon \otimes I)\Delta(a_i) \otimes b_i \\
&= \sum_{i=1} a_i \otimes b_i \\
&= R
\end{aligned}$$

In view of Condition (8)

$$\begin{aligned}
R &= (s_1 \otimes I)(\epsilon \otimes I \otimes I)(R^{13}R^{23}) \\
&= (s_1 \otimes I)(\epsilon \otimes I \otimes I)(R^{13})(s_1 \otimes I)(\epsilon \otimes I \otimes I)(R^{23}) \\
&= (s_1 \otimes I)(\epsilon \otimes I \otimes I)(\sum_{i=1}^{n} a_i \otimes 1 \otimes b_i)(s_1 \otimes I)(\epsilon \otimes I \otimes I)(\sum_{i=1}^{n} 1 \otimes a_i \otimes b_i) \\
&= (\sum_{i=1}^{n} \epsilon(a_i)1 \otimes b_i)(\sum_{i=1}^{n} a_i \otimes b_i) \\
&= (\sum_{i=1}^{n} 1 \otimes \epsilon(a_i)b_i)R
\end{aligned}$$

Thus

$$1 \otimes \sum_{i=1} \epsilon(a_i)b_i = 1 \otimes 1$$

and consequently,

$$1 = s_1(\sum_{i=1}^{n} \epsilon(a_i) \otimes b_i) = s_1(\epsilon \otimes I)(R)$$

A similar argument is used to prove (ii).

$\diamond$

## 4. Myhill–Nerode Bialgebras

In this section we review the main result of [3] in which the authors give a bialgebra version of the Myhill–Nerode Therorem. Let $G$ be a semigroup with unity, $1$ and let $H = KG$ be the semigroup bialgebra. There is a right $H$-module structure on $H^*$ defined as

$$(p \leftharpoonup x)(y) = p(xy)$$

for all $x, y \in H$, $p \in H^*$. For $x \in H$, $p \in H^*$, the element $p \leftharpoonup x$ is the *right translate of* $p$ *by* $x$.

**Proposition 4.1** *([3, Proposition 5.4].) Let $G$ be a semigroup with $1$, let $H = KG$ denote the semigroup bialgebra. Let $p \in H^*$. Then the following are equivalent.*

*(i) The set $\{p \leftharpoonup x : x \in G\}$ of right translates is finite.*

*(ii) There exists a finite dimensional bialgebra $B$, a bialgebra homomorphism $\Psi : H \to B$, and an element $f \in B^*$ so that $p(h) = f(\Psi(h))$ for all $h \in H$.*

(Note: The bialgebras of (ii) are defined to be *Myhill–Nerode bialgebras*.)

*Proof.* $(i) \implies (ii)$. Let $Q = \{p \leftharpoonup x : x \in G\}$ be the finite set of right translates. For each $u \in G$, we define a right operator $r_u : Q \to Q$ by the rule

$$(p \leftharpoonup x)r_u = (p \leftharpoonup x) \leftharpoonup u = p \leftharpoonup xu$$

Observe that the set $\{r_u : u \in G\}$ is finite with $|\{r_u : u \in G\}| \leq |Q|^{|Q|}$. The set $\{r_u : u \in G\}$ is a semigroup with unity, $1 = r_1$ under composition of operators. Indeed,

$$(p \leftharpoonup x)(r_u r_v) = (p \leftharpoonup xu)r_v = p \leftharpoonup xuv = (p \leftharpoonup x)r_{uv}$$

Thus $r_u r_v = r_{uv}$, for all $u, v \in G$. Let $B$ denote the semigroup bialgebra on $\{r_u : u \in G\}$. Let $\Psi : H \to B$ be the $K$-linear map defined by $\Psi(u) = r_u$. Then

$$\Psi(uv) = r_{uv} = r_u r_v = \Psi(u)\Psi(v)$$

and

$$\begin{aligned}\Delta_B(\Psi(u)) &= \Delta_B(r_u)\\ &= r_u \otimes r_u\\ &= \Psi(u)\otimes\Psi(v)\\ &= (\Psi\otimes\Psi)(u\otimes u)\\ &= (\Psi\otimes\Psi)\Delta_H(u)\end{aligned}$$

and so, $\Psi$ is a homomorphism of bialgebras.

Let $f \in B^*$ be defined by

$$\begin{aligned}f(r_u) &= ((p \leftharpoonup 1)r_u)(1)\\ &= (p \leftharpoonup u)(1)\\ &= p(u)\end{aligned}$$

Then $p(h) = f(\Psi(h))$, for all $h \in H$, as required.

$(ii) \implies (i)$. Suppose there exists a finite dimensional bialgebra $B$, a bialgebra homomorphism $\Psi : H \to B$, and an element $f \in B^*$ so that $p(h) = f(\Psi(h))$ for all $h \in H$. Define a right $H$-module action $\cdot$ on $B$ as

$$b \cdot h = b\Psi(h)$$

for all $b \in B$, $h \in H$. Then for $b \in B$, $x \in G$,

$$\begin{aligned}\Delta_B(b\cdot x) &= \Delta_B(b\Psi(x))\\ &= \Delta_B(b)\Delta_B(\Psi(x))\\ &= (\sum_{(b)} b_{(1)}\otimes b_{(2)})(\Psi\otimes\Psi)\Delta_H(x)\\ &= (\sum_{(b)} b_{(1)}\otimes b_{(2)})(\Psi(x)\otimes\Psi(x))\\ &= \sum_{(b)} b_{(1)}\Psi(x)\otimes b_{(2)}\Psi(x)\\ &= \sum_{(b)} b_{(1)}\cdot x\otimes b_{(2)}\cdot x\end{aligned}$$

and

$$\epsilon_B(b\cdot x) = \epsilon_B(b\Psi(x)) = \epsilon_B(b)\epsilon_B(\Psi(x)) = \epsilon_B(b)\epsilon_H(x)$$

Thus $B$ is a right $H$-module coalgebra.

Now, let $Q$ be the collection of grouplike elements of $B$. Since $Q$ is a linearly independent subset of $B$ and $B$ is finite dimensional, $Q$ is finite. Since $B$ is a right $H$-module coalgebra with action "$\cdot$",

$$\Delta_B(q\cdot x) = q\cdot x\otimes q\cdot x$$

for $q \in Q$, $x \in G$. Thus $\cdot$ restricts to give an action (also denoted by "$\cdot$") of $G$ on $Q$. Now for $x, y \in G$,

$$\begin{aligned}(p \leftharpoonup x)(y) &= p(xy) \\ &= f(\Psi(xy)) \\ &= f(\Psi(x)\Psi(y)) \\ &= f((1_B\Psi(x))\Psi(y)) \\ &= f((1_B \cdot x) \cdot y) \end{aligned} \tag{11}$$

Let

$$S = \{q \in Q : \ q = 1_B \cdot x \text{ for some } x \in G\}$$

In view of Condition (11) there exists a function

$$\varrho : S \to \{p \leftharpoonup x : \ x \in G\}$$

defined as

$$\varrho(1_B \cdot x)(y) = f((1_B \cdot x) \cdot y) = (p \leftharpoonup x)(y)$$

Since $\varrho$ is surjective and $S$ is finite, $\{p \leftharpoonup x : \ x \in G\}$ is finite.

$\diamond$

We illustrate the connection between Proposition 4.1 and the usual Myhill–Nerode Theorem. Let $\hat{\Sigma}_0$ denote the set of words in a finite alphabet $\Sigma_0$. Let $L \subseteq \hat{\Sigma}_0$ be a language. Suppose that the equivalence relation $\sim_L$ (as in the Introduction) has finite index. Then the usual Myhill–Nerode Theorem says that there exists a finite automaton which accepts $L$. We show how to construct this finite automaton using Proposition 4.1.

Consider $G = \hat{\Sigma}_0$ as a semigroup with unity where the semigroup operation is concatenation and the unity element is the empty word. Let $H = KG$ denote the semigroup bialgebra. Then the characteristic function of $L$ extends to an element $p \in H^*$. Since $\sim_L$ has finite index, the set of right translates $\{p \leftharpoonup x : \ x \in G\}$ is finite [3, Proposition 2.3]. Now Proposition 4.1 (i)$\Longrightarrow$ (ii) applies to show that there exists a finite dimensional bialgebra $B$, a bialgebra homomorphism $\Psi : H \to B$ and an element $f \in B^*$ so that $p(h) = f(\Psi(h))$, for all $h \in H$.

This bialgebra determines a finite automaton $\langle Q, \Sigma, \delta, q_0, F\rangle$, where $Q$ is the finite set of states, $\Sigma$ is the input alphabet, $\delta$ is the transition function, $q_0$ is the initial state, and $F$ is the set of final states (see [2, Chapter 2] for details on finite automata.)

For the states of the automata, we let $Q$ be the (finite) set of grouplike elements of $B$. For the input alphabet, we choose $\Sigma = \Sigma_0$. As we have seen, the right $H$-module structure of $B$ restricts to an action "$\cdot$" of $G$ on $Q$, and so we define the transition function $\delta : Q \times \Sigma_0 \to Q$ by the rule $\delta(q, x) = q \cdot x$, for $q \in Q$, $x \in \Sigma_0$. The initial state is $q_0 = 1_B$, and the set of final states $F$ is the subset of $Q$ of the form $1_B \cdot x$, $x \in G$ for which

$$p(x) = f(\Psi(x)) = f1_B\Psi(x)) = f(1_B \cdot x) = 1$$

By construction, the finite automaton $\langle Q, \Sigma_0, \delta, 1_B, F\rangle$ accepts $L$.

## 5. Quasitriangular Structure of Myhill–Nerode Bialgebras

In this section we use Proposition 4.1 to construct a collection of Myhill–Nerode bialgebras. We then compute the quasitriangular structure of one of these bialgebras.

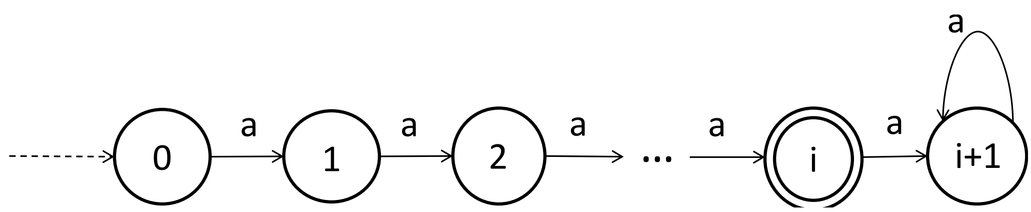
Let $\Sigma_0 = \{a\}$ be the alphabet on a single letter $a$. Let $\hat{\Sigma}_0 = \{1, a, aa, aaa, \dots\}$ denote the collection of all words of finite length formed from $\Sigma_0$. Here $1$ denotes the empty word of length $0$. For convenience, we shall write

$$a^i = \underbrace{aaa\cdots a}_{i \text{ times}},$$

for $i \geq 0$.

Fix an integer $i \geq 0$ and let $L_i = \{a^i\} \subseteq \hat{\Sigma}_0$. Then the language $L_i$ is accepted by the finite automaton given in Figure 1.

**Figure 1.** Finite automaton accepting $L_i = \{a^i\}$, accepting state is $i$.




By the usual Myhill–Nerode Theorem, the equivalence relation $\sim_{L_i}$, defined as $x \sim_{L_i} y$ if and only if $xz \in L_i$ exactly when $yz \in L_i, \forall z$, has finite index. If $p_i : \hat{\Sigma}_0 \to \{0,1\} \subseteq K$ is the characteristic function of $L_i$, then $\sim_{L_i}$ is equivalent to the relation $\sim_{p_i}$ defined as: $x \sim_{p_i} y$ if and only if $p_i(xz) = p_i(yz), \forall z \in \hat{\Sigma}_0$. Let $[x]_{p_i}$ denote the equivalence class of $x$ under $\sim_{p_i}$. The Myhill–Nerode theorem now says that the set $\{[x]_{p_i} : x \in \hat{\Sigma}_0\}$ is finite.

Now we consider $G = \hat{\Sigma}_0$ as a semigroup with unity $1$ with concatenation as the binary operation. Let $H = KG$ be the semigroup bialgebra. The characteristic function $p_i$ of $L_i$ extends to an element of $H^*$. By [3, Proposition 2.3], the set of right translates $\{p_i \leftharpoonup x : x \in G\}$ is finite. Thus by Proposition 4.1, there exists a finite dimensional bialgebra $B_i$, a bialgebra homomorphism $\Psi : H \to B_i$, and an element $f_i \in B_i^*$ so that $p_i(h) = f_i(\Psi(h))$ for all $h \in H$.

In what follows, we give the bialgebra structure of the collection $\{B_i : i \geq 0\}$ and compute the quasitriangular structure of the bialgebra $B_0$.

For $i \geq 0$, the finite set of right translates of $p_i \in H^*$ is

$$Q_i = \{p_i \leftharpoonup 1, p_i \leftharpoonup a, p_i \leftharpoonup a^2, \dots, p_i \leftharpoonup a^i, p_i \leftharpoonup a^{i+1}\}$$

One finds that the set of right operators on $Q_i$ is $\{r_1, r_a, r_{a^2}, \dots, r_{a^i}, r_{a^{i+1}}\}$. Under composition, the set of right operators is a semigroup with unity $r_1$. We have, for $0 \leq m, n \leq i+1$,

$$r_{a^m} r_{a^n} = \begin{cases} r_{a^{m+n}} & \text{if } 0 \leq m+n \leq i+1 \\ r_{a^{i+1}} & \text{if } m+n > i+1 \end{cases}$$

By construction, $B_i$ is the semigroup bialgebra on $\{r_1, r_a, r_{a^2}, \dots, r_{a^i}, r_{a^{i+1}}\}$.

*5.1. Quasitriangular Structure of $B_0$*

In the case $i = 0$, $B_0$ is the semigroup bialgebra on $\{r_1, r_a\}$ with algebra structure defined by $r_1r_1 = r_1$, $r_1r_a = r_a$, $r_ar_1 = r_a$, $r_ar_a = r_a$. Let $\{e_0, e_1\}$ be the dual basis defined as $e_0(r_1) = 1$, $e_0(r_a) = 0$, $e_1(r_1) = 0$, $e_1(r_a) = 1$. Then $\{e_0, e_1\}$ is the set of minimal idempotents for $B_0^*$. Comultiplication on $B_0^*$ is given as

$$\Delta_{B_0^*}(e_0) = e_0 \otimes e_0$$

$$\Delta_{B_0^*}(e_1) = e_0 \otimes e_1 + e_1 \otimes e_0 + e_1 \otimes e_1$$

and the counit map is defined by

$$\epsilon_{B_0^*}(e_0) = 1, \quad \epsilon_{B_0^*}(e_1) = 0$$

**Proposition 5.1** *Let $B_0$ be the $K$-bialgebra as above. Then there is exactly one quasitriangular structure on $B_0$, namely, $R = 1_{B_0} \otimes 1_{B_0}$.*

*Proof.* Certainly, $1 \otimes 1 = 1_{B_0} \otimes 1_{B_0}$ is a quasitriangular structure for $B_0$. We claim that $1 \otimes 1$ is the only quasitriangular structure. Observe that there is bialgebra isomorphism $\phi : B_0 \to B_0^*$ defined as $\phi(r_1) = e_0 + e_1$, $\phi(r_a) = e_0$. Thus if $(B_0, R)$ is quasitriangular, then $(B_0^*, R')$, $R' = (\phi \otimes \phi)(R)$, is quasitriangular by Proposition 3.1. So, we first compute all of the quasitriangular structures of $B_0^*$. To this end, suppose that $(B_0^*, R')$ is quasitriangular for some element $R' \in B_0^* \otimes B_0^*$. Since

$$B_0^* \otimes B_0^* = K(e_0 \otimes e_0) \oplus K(e_0 \otimes e_1) \oplus K(e_1 \otimes e_0) \oplus K(e_1 \otimes e_1)$$

$$R' = w(e_0 \otimes e_0) + x(e_0 \otimes e_1) + y(e_1 \otimes e_0) + z(e_1 \otimes e_1)$$

for $w, x, y, z \in K$. By Proposition 3.3(i),

$$\begin{aligned} 1_{B_0^*} &= e_0 + e_1 \\ &= s_1(\epsilon \otimes I)(w(e_0 \otimes e_0) + x(e_0 \otimes e_1) + y(e_1 \otimes e_0) + z(e_1 \otimes e_1)) \\ &= we_0 + xe_1 \end{aligned}$$

and so, $w = x = 1$. From Proposition 3.3(ii), one also has $y = 1$. Thus

$$R' = e_0 \otimes e_0 + e_0 \otimes e_1 + e_1 \otimes e_0 + z(e_1 \otimes e_1)$$

for $z \in K$. Now,

$$\begin{aligned} (\Delta \otimes I)(R') &= (\Delta \otimes I)(e_0 \otimes e_0 + e_0 \otimes e_1 + e_1 \otimes e_0 + z(e_1 \otimes e_1)) \\ &= (e_0 \otimes e_0) \otimes e_0 + (e_0 \otimes e_0) \otimes e_1 + (e_0 \otimes e_1 + e_1 \otimes e_0 + e_1 \otimes e_1) \otimes e_0 \\ &+ z((e_0 \otimes e_1 + e_1 \otimes e_0 + e_1 \otimes e_1) \otimes e_1) \\ &= e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0 \\ &+ z(e_0 \otimes e_1 \otimes e_1) + z(e_1 \otimes e_0 \otimes e_1) + z(e_1 \otimes e_1 \otimes e_1) \end{aligned} \tag{12}$$

Moreover,

$$\begin{aligned}
(R')^{13}(R')^{23} &= (e_0 \otimes (e_0+e_1) \otimes e_0 + e_0 \otimes (e_0+e_1) \otimes e_1 + e_1 \otimes (e_0+e_1) \otimes e_0 \\
&+ z(e_1 \otimes (e_0+e_1) \otimes e_1)) \cdot ((e_0+e_1) \otimes e_0 \otimes e_0 + (e_0+e_1) \otimes e_0 \otimes e_1 \\
&+ (e_0+e_1) \otimes e_1 \otimes e_0 + z((e_0+e_1) \otimes e_1 \otimes e_1)) \\
&= (e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1 + e_0 \otimes e_1 \otimes e_1 \\
&+ e_1 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0 + z(e_1 \otimes e_0 \otimes e_1) \\
&+ z(e_1 \otimes \otimes e_1 \otimes e_1)) \cdot (e_0 \otimes e_0 \otimes e_0 + e_1 \otimes e_0 \otimes e_0 + e_0 \otimes e_0 \otimes e_1 + e_1 \otimes e_1 \otimes e_0 \\
&+ e_0 \otimes e_1 \otimes e_0 + e_1 \otimes e_1 \otimes e_0 + z(e_0 \otimes e_1 \otimes e_1) + z(e_1 \otimes \otimes e_1 \otimes e_1)) \\
&= e_0 \otimes e_0 \otimes e_0 + e_0 \otimes e_1 \otimes e_0 + e_0 \otimes e_0 \otimes e_1 + z(e_0 \otimes e_1 \otimes e_1) \\
&+ e_1 \otimes e_0 \otimes e_0 + e_1 \otimes e_1 \otimes e_0 + z(e_1 \otimes e_0 \otimes e_1) + z^2(e_1 \otimes e_1 \otimes e_1) \qquad (13)
\end{aligned}$$

Equations 12 and 13 yield the relation $z^2 = z$. Thus either $z = 0$ or $z = 1$. If $z = 0$, then $R'$ is not a unit in $B_0^* \otimes B_0^*$. Thus

$$R' = e_0 \otimes e_0 + e_0 \otimes e_1 + e_1 \otimes e_0 + e_1 \otimes e_1 = 1 \otimes 1$$

is the only quasitriangular structure for $B_0^*$.

Consequently, if $(B_0, R)$ is quasitriangular, then $(\phi \otimes \phi)(R) = 1_{B_0^*} \otimes 1_{B_0^*}$. It follows that $R = 1_{B_0} \otimes 1_{B_0}$.

◇

### *5.2. Questions for Future Research*

Though the Myhill–Nerode bialgebra $B_0$ has only the trivial quasitriangular structure, it remains to compute the quasitriangular structure of $B_i$ for $i \geq 1$. Moreover, the linear dual $B_i^*$ is a commutative, cocommutative $K$-bialgebra and it would be of interest to find its quasitriangular structure. Unlike the $i = 0$ case, we may have $B_i \ncong B_i^*$ (for instance, $B_1 \ncong B_1^*$) and so this is indeed a separate problem.

Suppose that $L$ is a language of words built from the alphabet $\Sigma_0 = \{a, b\}$. If $L$ is accepted by a finite automaton, then by Proposition 4.1, $L$ gives rise to a Myhill–Nerode bialgebra $B$ (see for example, [3, §6].) By construction, $B$ is a cocommutative $K$-bialgebra and hence $B$ has at least the trivial quasitriangular structure. Are there any other structures? Note that $B^*$ is a commutative $K$-algebra. For which $R$ (if any) is $(B^*, R)$ quasitriangular?

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