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Communication

Introduction to the Yang-Baxter Equation with Open Problems

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Abstract: The Yang-Baxter equation first appeared in theoretical physics, in a paper by the Nobel laureate C. N. Yang, and in statistical mechanics, in R. J. Baxter's work. Later, it turned out that this equation plays a crucial role in: quantum groups, knot theory, braided categories, analysis of integrable systems, quantum mechanics, non-commutative descent theory, quantum computing, non-commutative geometry, *etc*. Many scientists have found solutions for the Yang-Baxter equation, obtaining qualitative results (using the axioms of various algebraic structures) or quantitative results (usually using computer calculations). However, the full classification of its solutions remains an open problem. In this paper, we present the (set-theoretical) Yang-Baxter equation, we sketch the proof of a new theorem, we state some problems, and discuss about directions for future research.

Keywords: Yang-Baxter equation; set-theoretical Yang-Baxter equation; algebra structures; Hopf algebras; quantum groups; relations on sets

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1. Introduction

The Yang-Baxter equation first appeared in theoretical physics, in a paper by Yang [1], and in the work of Baxter in Statistical Mechanics [2,3]. It turned out to be one of the basic equations in mathematical physics, and more precisely for introducing the theory of quantum groups. It also plays a crucial role in: Knot theory, braided categories, non-commutative descent theory, quantum computing, non-commutative geometry, *etc*. Many scientists have used the axioms of various algebraic structures (quasi-triangular Hopf algebras, Yetter-Drinfeld categories, quandles, group actions, Lie (super)algebras, (co)algebra structures, Jordan triples, Boolean algebras, relations on sets, *etc*.) or computer calculations (and Grobner bases) in order to produce solutions for the Yang-Baxter equation. However, the full classification of its solutions remains an open problem. At present the study of solutions of the Yang-Baxter equation attracts the attention of a broad circle of scientists (including mathematicians). Some suggested references related to our paper could be References [4,5,6,7,8], *etc*.

In this paper we present qualitative results concerning the (set-theoretical) Yang-Baxter equation. We first consider the solutions arising from relations. For any relation on a given set we construct a map. We give necessary and sufficient conditions for this map to be a solution to the set-theoretical Yang-Baxter equation. In Section 3 we give other examples of solutions for the Yang-Baxter equation, we present some of their applications, and we sketch the proof of a theorem which resembles Kaplansky's tenth conjecture about the classification of finite dimensional Hopf algebras. Finally, we conclude with a short section about directions for future research.

2. Preliminaries

Let V be a vector space over a field k, which is algebraically closed and of characteristic zero.

Definition. A linear automorphism R of $V \otimes V$ is a solution of the Yang-Baxter equation (sometimes called the braid relation), if the equality

$$(R \otimes id) \circ (id \otimes R) \circ (R \otimes id) = (id \otimes R) \circ (R \otimes id) \circ (id \otimes R) \tag{1}$$

holds in the automorphism group of $V \otimes V \otimes V$.

Definition. R is a solution of the quantum Yang-Baxter equation (QYBE) if

$$R_{12} \circ R_{13} \circ R_{23} = R_{23} \circ R_{13} \circ R_{12} \tag{2}$$

where R_{ij} means R acting on the i-th and j-th component.

Let T be the twist map, $T(v \otimes w) = w \otimes v$. Then R satisfies (1) if and only if R o T satisfies (2) if and only if T o R satisfies (2).

Finding all solutions of the Yang-Baxter equation is a difficult task far from being resolved. Nevertheless many solutions of these equations have been found during the last 30 years and the related algebraic structures have been studied.

Reference [9] posed the problem of studying set-theoretical solutions of the Yang-Baxter equation. Specifically, we consider a set X and S: X × X → X × X, and we consider the equation (1) as an equality of maps from X × X × X to X × X × X:

$$(S \times id) \circ (id \times S) \circ (S \times id) = (id \times S) \circ (S \times id) \circ (id \times S) \tag{3}$$

We call (3) the set-theoretical Yang-Baxter equation.

It is obvious that from a solution to (3), one could obtain a solution to (1), by considering the vector space generated by the set X, and linearly extending the map S.

A lesser known example of solutions for the set-theoretical Yang-Baxter equation is the following. Given a binary relation R on X (*i.e.*, R is a subset of X × X), we define a map

$$S: X \times X \to X \times X$$

$$S(u, v) = \begin{cases} (u, v) & \text{if } (u, v) \in R \\ (v, u) & \text{if } (u, v) \notin R \end{cases} \tag{4}$$

Theorem 1. (*D. Hobby and F. F. Nichita*, *[10]*). Assume R is a reflexive relation. The function S derived from the relation R satisfies (3) if and only if $R \cup R^{op}$ is an equivalence relation on X and the complement relation of R is a strict partial order on each class of $R \cup R^{op}$ (where R^{op} is the opposite relation of R).

Remark. The above theorem generalizes the twist map if R is an equivalence relation. For a vector space, if we give an equivalence relation on a basis of the space, we can construct a generalization of the twist map using formula (4).

Remark. [10] compared the above solutions with the solutions from Boolean algebras. More precisely, the function $(a,b) \to (a \wedge b, a \vee b)$ is a solution for the set-theoretical Yang-Baxter equation.

3. Main Results and Discussion

Let A be a k-algebra, and $x, y \in$ k-$\{0\}$. We define the k-linear map

$$\varphi = \varphi^A : A \otimes A \longrightarrow A \otimes A, \quad \varphi(a \otimes b) = xab \otimes 1 + y1 \otimes ab - xa \otimes b$$

Then, according to [11], φ is an invertible solution to (1). Such an operator is called a Yang-Baxter operator, or, simply, a YB operator.

Remarks.

(i) The above operator is connected to the theory of entwining structures and corings (see [12]).

(ii) Using the method of [13], the operator φ leads to the Alexander polynomial of knots (see [14]).

(iii) Other generalizations and properties for φ were presented in [15]: solutions for the initial Yang's equation (see [1]) and a possible vertex model in statistical mechanics.

Definition. Two YB operators (V, R) and (W, Q) are called *isomorphic* if there exists $f : V \longrightarrow W$ such that $Q \circ (f \otimes f) = (f \otimes f) \circ R$.

Remark. Reference [11] showed that for non-isomorphic algebra structures, the associated YB operators are non-isomorphic. It follows that, for any finite dimension vector space, the number of non-isomorphic classes of algebras structures on that vector space is less than the number of non-isomorphic YB operators on the same vector space.

Conjecture. The YB operators from algebra structures can be obtained from some kind of universal R-matrix (a universal R-matrix is related to the quasi-triangular structures presented in [6]).

The Kaplansky's tenth conjecture about the classification of finite dimensional Hopf algebras was proved in negative by references [16–18]. We present a similar result for Yang-Baxter operators below.

Theorem 2. There exist finite dimensional vector spaces for which there are infinitely many non-isomorphic Yang-Baxter operators.

Proof. The idea of the proof is shown below. The omitted technical details will be included in another paper. We take an arbitrary Hopf algebra from [17]. (We know that there are infinitely many non-isomorphic finite dimensional Hopf algebras.) Because this Hopf algebra can be viewed as an entwining structure, we associate a WXZ system as in [12]. But X is invertible, since its inverse could be obtained by using the antipode of this arbitrary Hopf algebra. We now construct a new Yang-Baxter operator using a remark from [12], because X is invertible:

$$R:(H\oplus H)\otimes(H\oplus H)\longrightarrow(H\oplus H)\otimes(H\oplus H),\quad R=\varphi^{H}\bigoplus\nolimits_{X\circ T}\psi^{H}.$$

According to [11], for non-isomorphic algebra (respective coalgebras) structures, $\varphi\ (resp.\ \psi)$ are non-isomorphic operators. It follows that for non-isomorphic Hopf algebras we obtain non-isomorphic YB operators.

4. Conclusions and Directions for Future Research

The author of [19] constructed solutions to the Yang-Baxter equation from algebra and coalgebra structures showing that the Yang-Baxter equation captures the fundamental piece of information encapsulated in the algebra and coalgebra structures. The solutions for the set-theoretical Yang-Baxter equation presented in Section 2 were shown in an attempt to relate the equivalence relations and the order relations.

Some directions for future research are: Finding the smallest dimension of a vector space for which Theorem 2 holds, and the study of solutions derived from relations for other non-linear equations from Quantum Group Theory. For example, these equations might be: Pentagonal equation, Long equation, Frobenius-separability equation (see [4]) or Yang-Baxter systems (which were introduced in [20]).

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References and Notes

1. Yang, C.N. Some exact results for the many-body problem in one dimension with repulsive delta-function interaction. *Phys. Rev. Lett*. **1967**, *19*, 1312–1315.
2. Baxter, R.J. Partition function for the eight-vertex lattice model. *Ann. Phys.* **1972**, *70*, 193–228.
3. Baxter, R.J. *Exactly Solved Models in Statistical Mechanics*; Academic Press: London, UK, 1982.
4. Caenepeel, S.; Militaru, G.; Zhu, S. Frobenius and separable functors for generalized Hopf modules and Nonlinear equations. In *Lecture Notes in Mathematics 1787*; Springer: Berlin, Germany, 2002.
5. Gateva-Ivanova, T. A combinatorial approach to the set-theoretical solutions of the Yang-Baxter equation. *J. Math. Phys.* **2004**, *45*, 3828–3859.
6. Kassel, C. Quantum groups. In *Graduate Texts in Mathematics*; Springer Verlag: Berlin, Germany, 1995.
7. Lambe, L.; Radford, D. Introduction to the quantum Yang-Baxter equation and quantum groups: An algebraic approach. In *Mathematics and Its Applications 423*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1997.
8. Nichita, F.F. *Non-Linear Equations*, *Quantum Groups*, *and Duality Theorems*; VDM Verlag: Saarbrücken, Germany, 2009.

9. Drinfeld, V.G. On some unsolved problems in quantum group theory. In *Quantum Groups, Lecture Notes in Mathematics*, Kulish, P.P., Ed.; Springer-Verlag: Berlin, Germany, 1992; Volume 1510, pp. 1–8.
10. Hobby, D.; Nichita, F.F. Solutions of the set-theoretical Yang-Baxter equation derived from relations. *Acta Universitatis Apulensis*, **2012**, in press.
11. Dascalescu, S.; Nichita, F.F. Yang-Baxter operators arising from (co)algebra structures. *Commun. Algebra* **1999**, *27*, 5833–5845.
12. Brzezinski, T.; Nichita, F.F. Yang-baxter systems and entwining structures. *Commun. Algebra* **2005**, *33*, 1083–1093.
13. Turaev, V. The Yang-Baxter equation and invariants of links. *Invent. Math.* **1988**, *92*, 527–553.
14. Massuyeau, G.; Nichita, F.F. Yang-Baxter operators arising from algebra structures and the Alexander polynomial of knots. *Commun. Algebra* **2005**, *33*, 2375–2385.
15. Nichita, F.F.; Parashar, D. Spectral-parameter dependent Yang-Baxter operators and Yang-Baxter systems from algebra structures. *Commun. Algebra* **2006**, *34*, 2713–2726.
16. Andruskiewitsch, N.; Schneider, H.J. Lifting of quantum linear spaces and pointed Hopf algebras of order p3. *J. Algebra* **1998**, *209*, 658–691.
17. Beattie, M.; Dascalescu, S.; Grünenfelder, L. On the number of types of finite dimensional Hopf algebras. *Invent. Math.* **1999**, *136*, 1–7.
18. Gelaki, S. Pointed Hopf algebras and Kaplansky's 10-th conjecture. *J. Algebra* **1998**, *209*, 635–657.
19. Nichita, F.F. Self-inverse Yang-Baxter operators from (co)algebra structures. *J. Algebra* **1999**, *218*, 738–759.
20. Hlavaty, L.; Snobl, L. Solutions of a Yang-Baxter system. 1998, arXiv:math.QA/9811016v2. arXiv.org e-Print archive. http://arxiv.org/pdf/math/9811016.pdf (accessed on 24 April 2012).