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Article

Axiomatic of Fuzzy Complex Numbers

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Abstract: Fuzzy numbers are fuzzy subsets of the set of real numbers satisfying some additional conditions. Fuzzy numbers allow us to model very difficult uncertainties in a very easy way. Arithmetic operations on fuzzy numbers have also been developed, and are based mainly on the crucial Extension Principle. When operating with fuzzy numbers, the results of our calculations strongly depend on the shape of the membership functions of these numbers. Logically, less regular membership functions may lead to very complicated calculi. Moreover, fuzzy numbers with a simpler shape of membership functions often have more intuitive and more natural interpretations. But not only must we apply the concept and the use of fuzzy sets, and its particular case of fuzzy number, but also the new and interesting mathematical construct designed by Fuzzy Complex Numbers, which is much more than a correlate of Complex Numbers in Mathematical Analysis. The selected perspective attempts here that of advancing through axiomatic descriptions.

Keywords: mathematical analysis; number theory; non-classical logics; fuzzy logic; fuzzy sets; fuzzy measure theory; fuzzy systems; A.I.

1. Introduction

Historically, the extension of the set of numbers were developed from natural numbers to integers, then rational numbers, real numbers, and finally—in recent times—complex numbers. Similarly, the concept of set has been extended in a variety of ways. A fuzzy set is one such an extension of conventional (classical or crisp) sets. From a point of view of the extension of ranges of characteristic functions, that is, extensions from the discrete set {0,1} to the closed unit interval, [0,1], it looks like the extension from natural numbers to real numbers.

Complex fuzzy sets, which are proposed by Ramot *et al*. [1], are characterized by complex-valued characteristic functions. The extension looks like one from real numbers to complex numbers. We here take the complex fuzzy sets into consideration from an algebraic and logic's point of view. So, we will try to apply them (in the future) to design algorithms of quantum computers.

Fuzzy Logic (FL, by acronym) is also known as the *mathematical theory of Vagueness*, and also the theory of the "common-sense reasoning" that is mainly based on the use of NL (natural language).

It is possible to demonstrate that FL is a well-developed logical theory. It includes, amongst others, the theory of functional systems in FL, giving an explanation of what and how it can be represented by formulas of FL calculi. A more general interpretation of FL within other proper categories of fuzzy sets is also feasible (see the book [2]).

Fuzzy logic provides a systematic tool to incorporate human experience. It is based on three core concepts, namely, fuzzy sets, linguistic variables, and possibility distributions. Fuzzy set is used to characterize linguistic variables whose values can be described qualitatively using a linguistic expression and quantitatively using a membership function. Linguistic expressions are useful for communicating concepts and knowledge with human beings, whereas membership functions are useful for processing numeric input data. When a fuzzy set is assigned to a linguistic variable, it imposes an elastic constraint, called a possibility distribution, on the possible values of the variable.

Fuzzy logic is a very rigorous and mathematical discipline of increasing interest [3]. Fuzzy Reasoning is a straightforward formalism for encoding human knowledge or common sense in a numerical framework, and Fuzzy Inference Systems (FISs, by acronym) can approximate arbitrarily well any continuous function on a compact domain. FISs and neural networks (NNs) can approximate each other to any degree of accuracy. Since its first industrial application (in 1982), it has aroused general interest in the scientific community, and fuzzy logic has also been widely applied in data analysis, regression methods and prediction, as well as signal and image processing. Many integrated circuits have been designed for fuzzy logic.

It is well established that Propositional Logic is isomorphic to Set Theory under the appropriate correspondence between relative components of these two mathematical systems. Furthermore, both of these systems are isomorphic to a Boolean Algebra, which is a well-known mathematical system, defined by abstract entities and their axiomatic properties. The isomorphism between Boolean Algebra, Propositional Logic, and Set Theory guarantees that every theorem in any one of these theories has a counterpart in each of the other two theories. This isomorphism allows us to cover all these theories by developing only one of them. Consequently, we will not spend a lot of time reviewing Mathematical or Crisp Logic; but we spend some time on it in order to reach the comparable concept in Fuzzy Logic.

Fuzzy Complex Sets are given by a complex degree of membership, represented in polar coordinates, which is a combination of a degree of membership in a fuzzy set along with a crisp phase value that denotes the position within the set. The compound value carries more information than a traditional fuzzy set and enables efficient reasoning. We may present a new and generalized interpretation of a complex grade of membership, where a complex membership grade defines a complex fuzzy class. The new definition provides rich semantics that is not readily available through traditional fuzzy sets or complex fuzzy sets and is not limited to a compound of crisp cyclical data with fuzzy data. Furthermore, the two components of the complex fuzzy class carry fuzzy information [4].

A complex class is represented either in Cartesian or in polar coordinates where both axes induce fuzzy interpretation. Another novelty of this scheme is that it enables representing an infinite set of fuzzy sets.

This provides a new definition of fuzzy complex fuzzy classes along with the axiomatic definition of basic operations on complex fuzzy classes. In addition, coordinate transformation, as well as an extension from two-dimensional fuzzy classes to n-dimensional fuzzy classes are presented.

2. History of Complex Numbers

In the modern sense, the introduction of Complex Numbers is due to Girolamo Cardano (1501–1572). We can observe this in his book *Ars Magna*, published in 1545. Its apparition is seen as a tool for finding the root of a cubic equation.

Its use is not a pleasure, according to Cardano, but a "mental torture". Later (1572), Rafael Bombelli introduced the symbol *i*, establishing the operational rules of complex numbers.

In 1629, the Flemish amateur mathematician Albert Girard denominated complex numbers as "impossible solutions". Girard was of French origin and had emigrated due to religious prosecution.

It was René Descartes who spoke of "imaginary numbers" in 1637. After him, the use of complex numbers was somewhat generalized among mathematicians, amongst others the family Bernoulli (associated with Basel), Leonhard Euler, John Wallis, Caspar Wessel, Abraham De Moivre, Jean Robert Argand, Augustin Cauchy, Karl Friedrich Gauss, Bernhard Riemann, Caspar Wessel, William Rowan Hamilton, Hermann G. Grassmann, William K. Clifford, Lars V. Ahlfors, *etc*.

But the origins of this theory is much more arcane. In fact, the notion of complex numbers is intimately related with the attempt to prove the famous *Fundamental Theorem of Algebra* (FTA, by acronym), according to which

"Every n-th order polynomial has exactly n roots in the complex plane".

The oldest reference known to square roots of negative numbers may be by Heron of Alexandria, around the year 60 AD, when this great mathematician and inventor was calculating volumes of geometric bodies. Two centuries after the aforementioned Heron of Alexandria, Diophantus (about 275 AD) worked on a simple geometrical problem: Find the sides of a right-angled triangle, known its area and perimeter.

We must mention the attempts to find the roots of an arbitrary polynomial by Al-Khwarizmi (*ca*. 780–*ca*. 850 AD), thereby only admitting positive roots and therefore being a particular case of the FTA.

G. J. Toomer, quoted by B. L. Van der Waerden [5] said: "Under the caliph al-Ma'mun (reigned 813–833) al-Khwarizmi became a member of the "House of Wisdom" (Dar al-Hikma), a kind of academy of scientists set up at Baghdad, probably by Caliph Harun al-Rashid, but owing its preeminence to the interest of al-Ma'mun, a great patron of learning and scientific investigation. It was for al-Ma'mun that Al-Khwarizmi composed his astronomical treatise, and his Algebra also is dedicated to that ruler".

The methods of algebra known to the Arabs were introduced in Italy by the Latin translation of the algebra of al-Khwarizmi by Gerard of Cremona (1114–1187) and by the work of Leonardo da Pisa (more known as Fibonacci), who lived from 1170 to 1250.

There are indications that C.F. Gauss (1777–1855) had been in possession of the geometric representation of complex numbers since 1796. He introduced the term *complex number*, but it wrote that "if this subject has hitherto been considered from the wrong viewpoint and thus enveloped in mystery and surrounded by darkness, it is largely an unsuitable terminology which should be blamed. Had +1, −1 and p − 1, instead of being called positive, negative and imaginary (or worse still, impossible) unity, been given the names say, of direct, inverse and lateral unity, there would hardly have been any scope for such obscurity".

Augustin-Louis Cauchy (1789–1857) initiated the complex function theory in an 1814 memoir submitted to the French Académie des Sciences. The term analytic function was not mentioned in his memoir, but the concept is there.

Hermann G. Grassmann (1809–1877) also introduced the related field of Multidimensional Vector Calculus, and in 1848, James Cockle contributed the split-complex numbers. Combining split-complex numbers and quaternion algebra, W. K. Clifford introduced the so-called *Clifford Algebra*, as a new system of hyper-complex numbers.

Many years later, when generalizing complex numbers, we can mention the so-called Hyper-Complex Numbers, due to W. R. Hamilton (1805–1865). This scientist introduced a new mathematical construct, the so-called quaternions, in 1843.

Other members of this line of research of multi-complex numbers may be cited: *Twistors*, developed by Roger Penrose, in 1967; Mc Farlane's hyperbolic quaternions; *etc*.

In more recent times, James J. Buckley [4,6] created a new and very useful tool related to the new and very successful Fuzzy Set Theory of Lofti A. Zadeh [7]. They are the so-called *Fuzzy Complex Numbers*, which we will analyze here.

3. Essentials of Fuzzy Sets and Fuzzy Numbers

A fuzzy set, ***A***, on the Universe of Discourse ***U***, with images on the lattice L, may be defined [8–10] as an application

$$\boldsymbol{A: U \to L}$$

But many times it may be described, instead, by

$$\boldsymbol{m_A: U \to L}$$

Assigning a value, A (x) $\in$ L, to each element x $\in$U, *i.e.*, it may be considered as a collection of ordered pairs

$$\mathrm{A} = \{(x \mid \mathrm{m_A}\ (x))\}_{x\in\mathbf{U}}$$

But also can be expressed by

$$\mathrm{A} = \{x \mid \mathrm{A}\ (x)\text{: } x\in\mathbf{U}\}$$

Here, both notations, A (x) and m_A (x), may be called the membership function value for the element x into the fuzzy set A.

Often, the range of A, denoted by the lattice L, coincides with the closed unit interval of the field of real numbers, *i.e.*, $L = [0, 1] \subset \mathbf{R}$.

Note that we will use the same symbol for both fuzzy set, A, and membership function associated with A, in the same sense. Both couples,

$$(x \mid m_A(x)) \text{ or } (x \mid A(x))$$

has the same meaning, *i.e.*,

"The element x belongs to A with the membership degree, or equivalently the membership function value, equal either to $A(x)$ or $m_A(x)$".

This value expresses the degree of truth that the element x belongs to A.

The fuzzy set A is called *normal* if there is at least a real point, x^*, with

$$m_A(x^*) = 1$$

A *fuzzy set*, denoted by A, on **R**, will be *convex*, if for any real numbers, x, y, and any t, the following is true:

$$0 \leq t \leq 1$$

And it holds

$$m_A[t\,x + (1 - t)\,y] \geq \min\{m_A(x), m_A(y)\}$$

A *fuzzy number* (FN, by acronym) is indeed a fuzzy set, ***N*,** on the real line (**R**) that satisfies the above mentioned conditions of normality and convexity. *I.e.*, a fuzzy number is merely a particular case of a fuzzy set,

$$N: \mathbf{R} \to [0, 1]$$

Such that

(1) N is *normal*, *i.e.*, height of N = hgt (N) = 1.

(2) N^α, the *α-cut* of N, will be a closed interval, and it is true for all the values of α comprised between 0 and 1:

$$0 < \alpha \leq 1$$

But if all α-cuts are closed intervals then every fuzzy number must be a convex fuzzy set.

Observe that the converse is not necessarily true.

(3) The *support of N*, denoted by *supp (N)*, is bounded.

Illustrative examples of fuzzy numbers are the well-known types, such as

- the Fuzzy Trapezoidal Numbers,

and

- the *Fuzzy Triangular Number*, as the limiting case;
- bell-shaped membership function (Gaussian);
- L-R fuzzy numbers, *etc*.

A ***FGN*** (*Fuzzy Generalized Number*, by acronym), *F*, is any fuzzy subset of the real line whose membership function, m_F, verifies:

(1) $m_F(x) = 0$, if $-\infty < x \leq a$;
(2) $m_F(x) = L(x)$ strictly increases on [a, b];
(3) $m_F(x) = w$; if $b \leq x \leq c$;
(4) $m_F(x) = R(x)$ strictly decreases on [c, d];
(5) $m_F(x) = 0$, if $d \leq x < +\infty$.

The more usual way to denote a FGN may be

$$A = (a, b, c, d; w)$$

In the particular case when w = 1, we can express the FGN by

$$A = (a, b, c, d)_{LR}$$

Obviously, when the functions $L(x)$ and $R(x)$ correspond to straight lines, we have a *Fuzzy Trapezoidal Number*, being denoted by

$$(a, b, c, d)$$

In 1999, Chen and Hsieh [10] proposed a graded mean representation of *FGNs (Fuzzy Generalized Numbers, by acronym).*
In particular

Fuzzy Generalized Trapezoidal Numbers may be denoted by

$$(a, b, c, d; w)$$

And ***Fuzzy Generalized Triangular Numbers***, by

$$(a, b, d; w)$$

identifying both *b* and *c* values.
We need to introduce a new representational tool for various different reasons, which justify their necessity and convenience, instead of a simple two-element vector representation.

- It will be easiest for calculation.
- It is physically accurate.
- It is very useful in applications, by using complex algebra.

Let *U* be the universe of discourse. Then, we define a ***Fuzzy Complex Set*** (***FCS***, by acronym), also denoted by *A*, based on its membership function,

$$\mu_A(x) = r_A(x) \exp [i\, \phi_A(x)]$$

Observe that *i* represents the imaginary unit, whereby r_A and ϕ_A are both real-valued functions, with an important restriction on r_A such that

$$0 \leq r_A(x) \leq 1$$

Therefore, we can consider the precedent membership function, denoted by μ_A (x), as composed by two factors,

- *membership amplitude*, r_A (x)

and

- *membership phase*, ϕ_A (x).

As a particular case, we may consider the case of A with a null membership function, μ_A (x) = 0; for instance in the case where either r_A (x), or r_A (x) and ϕ_A (x) = 0; therefore, null phase and amplitude.

We then analyze the membership phase component in more detail, with respect to the fuzzy operations, union, intersection, complement, and so on.

Let *A* and *B* be two FCS. Then we can define

$$\mu_{A\cup B}(x) = [r_A(x) \blacktriangleright r_B(x)] \exp\{i\, \phi_{A\cup B}(x)\}$$

$\blacktriangleright$ here is some *T-conorm* operator, also called *S-norm.*

And similarly we can also define

$$\mu_{A\cap B}(x) = [r_A(x) \blacktriangleleft r_B(x)] \exp\{i\, \phi_{A\cap B}(x)\}$$

$\blacktriangleleft$ in this case being some T-norm operator.

Their respective phases remain without definition until now,

$$\phi_{A\cup B}(x) \text{ and } \phi_{A\cap B}(x)$$

To obtain such definitions, it would be convenient to introduce two auxiliary functions, *u* and *v*, which permits the specification of the Fuzzy Union and the Fuzzy Intersection of both given fuzzy sets, A and B:

In both cases, the domain will be the same fuzzy domain product, also sharing its range:

$$R_u = R_j = \{a \in \mathbf{C}: |a| = 1\}\ x\ \{b \in \mathbf{C}: |b| = 1\} \rightarrow \{d \in \mathbf{C}: |d| = 1\}$$

when **C** represent the field of complex numbers.

Axioms defining the u application

At least it must hold the following:

(1) u (a, 0) = a
(2) if |b| ≤ |d|, then |u(a, b)| ≤ |u(a, d)|
(3) u (a, b) = u (b, a)
(4) u (a, u (b, d)) = u (u (a, b), d)

The name of such axioms must be:

(1) *boundary conditions*;
(2) *monotonicity*;
(3) *commutativity*, and
(4) *associativity*, respectively.

In some cases, it will be convenient to dispose of certain additional axioms for u; as may be:

(5) u is a continuous function

(6) |u (a, a)| > |a|
(7) If |a| ≤ |c|, and |b| ≤ |d |, then |u (a, b)| ≤ u (c, d)

Such axioms are so-called

(5) *continuity*;
(6) *supidempotency* (with "p"); and
(7) *strict monotonicity*, respectively.

Furthermore [11], we need to analyze the following properties required for the phase intersection function, *v*

Axioms defining the v application

The *v* application must at least verify these conditions:

(1) v (a, 0) = a
(2) if |b| ≤ |d| then |v (a, b)| ≤ | v (a, d)|
(3) v (a, b) = v (b, a)
(4) v (a, v (b, d)) = v (v (a, b), d)

The adequate name of such axioms may be

(1) *boundary conditions*;
(2) *monotonicity*;
(3) *commutativity*;
(4) *associativity*, respectively.

In some cases it will be convenient to dispose of certain additional axioms for v, as may be

(5) *v* is a continuous function;
(6) |v (a, a)| < |a|;
(7) If |a| ≤ |c|, and |b| ≤ |d|, then |v (a, b)| ≤ v (c, d).

They are so-called

(5) *continuity*;
(6) *subidempotency* (with "b");
(7) *strict monotonicity*, respectively.

4. Lattice of Fuzzy Numbers

To expose the fundamental operations between fuzzy numbers [5,11–13], we firstly analyze the so-called ***Interval Operations***. So, we have

- *Addition*,

$$[a, b] (+) [c, d] = [a + c, b + d]$$

- *Difference*,

$$[a, b]\ (-)\ [c, d] = [a - d, b - c]$$

- *Product*,

$$[a, b]\ (*)\ [c, d] = [ac \wedge ad \wedge bc \wedge bd, ac \vee ad \vee bc \vee bd]$$

- *Division*,

$$[a, b]\ (:)\ [c, d] = [a{:}c \wedge a{:}d \wedge b{:}c \wedge b{:}d, a{:}c \vee a{:}d \vee b{:}c \vee b{:}d]$$

The last case may be defined this way, except when

$$c = d = 0$$

Two examples are either

$$[1, 2]\ (+)\ [3, 4] = [1 + 3, 2 + 4] = [4, 6]$$

or

$$[0, 1]\ (+)\ [-3, 5] = [0 + (-3), 1 + 5] = [-3, 6]$$

The difference of the same two fuzzy numbers will be either

$$[1, 2]\ (-)\ [3, 4] = [1 - 4, 2 - 3] = [-3, -1]$$

or

$$[0, 1]\ (-)\ [-3, 5] = [0 - 5, 1 - (-3)] = [0, 4]$$

In the first example for multiplication, it produces

$$[1, 2]\ (*)\ [3, 4] = [1 * 3 \wedge 1 * 4 \wedge 2 * 3 \wedge 2 * 4, 1 * 3 \vee 1 * 4 \vee 2 * 3 \vee 2 * 4]$$

$$= [3 \wedge 4 \wedge 6 \wedge 8, 3 \vee 4 \vee 6 \vee 8] = [3, 8]$$

In the case of division between both fuzzy numbers of this example, we have

$$[1, 2]\ (:)\ [3, 4] = [1{:}3 \wedge 1{:}4 \wedge 2{:}3 \wedge 0\ 2{:}4, 1{:}3 \vee 1{:}4 \vee 2{:}3 \vee 2{:}4] = [1{:}4, 2{:}3]$$

Properties of Interval Operations

(1) *Commutative*,

$$A\ (+)\ B = B\ (+)\ A$$

$$A\ (-)\ B = B\ (-)\ A$$

(2) *Associative*,

$$[A\ (+)\ B]\ (+)\ C = A\ (+)\ [B\ (+)\ C]$$

$$[A\ (-)\ B]\ (-)\ C = A\ (-)\ [B\ (-)\ C]$$

(3) *Identity*,

$$A = A\ (+)\ 0 = 0\ (+)\ A$$

$$A = A\ (*)\ 1 = 1\ (*)\ A$$

(4) *Subdistributive*,

$$A(*)\ [B\ (+)\ C] \rightarrow [A\ (*)\ B]\ (+)\ [A\ (*)\ C]$$

(5) *Inverse*,

$$0 = A\ (-)\ A,\ 1 = A\ (:)\ A$$

(6) *Monotonicity* for any operation.

Recall that a *Lattice* is a poset (*i.e.*, a partially ordered set) with an ordering relation.

Let $\perp \in \{(+), (-), (*), (:)\}$ with the restriction that for any such operations

$$0 \notin B^{\alpha},\ \text{being } 0 < \alpha \leq 1$$

Then, we obtain this fuzzy number

$$A\ (\perp)\ B = \cup_{\alpha \in (0,\ 1]}\ [A\ (\perp)\ B]^{\alpha}$$

Its generalization is indeed possible, because we dispose of the *Extension Principle*, for any arithmetic operation, ($\perp$). This may be expressed by this crucial result,

$$[A\ (\perp)\ B]\ (z) = \sup_{z\, =\, x\, (\perp)\, y}\ [\min\ \{A(x), B(y)\}]$$

We dispose in this case of *Meet* (*g. l. b.*), and *Join* (*l.u.b.*) operations.

So, we can describe the *Lattice of Fuzzy Numbers* by

$$\text{MIN}\ (A, B) = \sup z = \min_{(x,\ y)}\ \{\min\ [A(x), B\ (y)]\} = \text{MEET}\ (A, B)$$

Jointly with

$$\text{MAX}\ (A, B) = \sup z = \max_{(x,\ y)}\ \{\min\ [A(x), B\ (y)]\} = \text{JOIN}\ (A, B)$$

Therefore a *Distributive Lattice* holds that

$$\text{MIN}\ [A, \text{MAX}\ (B, C)] = \text{MAX}\ [\text{MIN}\ (A, B), \text{MIN}\ (A, C)]$$

$$\text{MAX}\ [A, \text{MIN}\ (B, C)] = \text{MIN}\ [\text{MAX}\ (A, B), \text{MAX}\ (A, C)]$$

Hence, we find that

$$< R_F, MIN, MAX >$$

is a distributive lattice, being into the R_F family, or collection, of all fuzzy sets.

5. Conclusions

Expressing adequate definitions and operational formulae in terms of Fuzzy Numbers [11–14] is currently a very inspired and useful mathematical insight for us. In fact, their applications extends to many different and very promising fields, such as Non-Classical Logics, Fuzzy Mathematical Analysis, Physics, Quantum Theory, Theoretical Computer Science, Artificial Intelligence, Automata Theory, Complex Networks, and so on.

This above introduced notion of FCNs (Fuzzy Complex Numbers, by acronym) is capable of representing and aggregating a variety of inexact knowledge and data in a unified manner [15–17]. Therefore, it is very useful for Computer Science in general, and for Knowledge Engineering in particular [18]. This ability may be reached by mathematically exploiting the framework to support the performance evaluation of classifiers. The effectiveness of the approach may be compared to the traditional fuzzy measure-based approach. Based on experimental results, the FCN-based performance evaluation is intuitively reliable and very consistent. It maintains the underlying semantics of different fuzzy evaluation measures. This ensures that the resulting ranking and hence selection process of choosing pattern classifiers is interpretable and explainable to the user. This may be essential in assisting the user to make informed decisions when given a challenging and many times very difficult classification task. Furthermore, we have attempted to contribute to provide further proof that Fuzzy Logic, and its application to generalize the Classical Set Theory, *i.e.*, the so-called Fuzzy Set Theory, is a well-developed logical theory, and also a very interesting and certainly useful new branch of new Mathematics [2]. Because the Fuzzy Logic and its associate Theory of Fuzzy Sets provides an adequate mathematical theory of Vagueness, it permits to reach more convenient solutions to many problems with uncertainty, such as general presenting challenges of our world.

For these reasons, the Complex Fuzzy Number must be a new and interesting tool in the continuous advance through the generalization of the (until now often too traditional and monotonic) fields of Science. For more complete and interesting information about these topics, please see the references [19–22].

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