Why Cerenkov Radiation May Not Occur, Even When It Is Allowed by Lorentz-Violating Kinematics

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Abstract: In a Lorentz-violating quantum field theory, the energy-momentum relations for the field quanta are typically modified. This affects the kinematics, and processes that are normally forbidden may become allowed. One reaction that clearly becomes kinematically possible when photons' phase speeds are less than 1 is vacuum Cerenkov radiation. However, in spite of expectations, and in defiance of phase space estimates, a electromagnetic Chern–Simons theory with a timelike Lorentz violation coefficient does not feature any energy losses through Cerenkov emission. There is an unexpected cancelation, made possible by the existence of unstable long-wavelength modes of the field. The fact that the theory possesses a more limited form of gauge symmetry than conventional electrodynamics also plays a role.

1. Introduction

Symmetry plays a critical role in our understanding of physics, especially the physics of particles and fields. In fact, much of the history of modern physics has involved coming to understand symmetries that initially appeared to hold exactly, yet which may actually be violated in a subtle fashion. Isospin, parity (P), charge conjugation (C), and time reversal (T) symmetries are all examples of where we have been able to learn a great deal about the fundamental interactions of nature by studying why the symmetries do not quite hold.

There are further symmetries of physics which—so far as we presently know—truly do hold exactly. Prominent examples are Lorentz and CPT symmetries. Both of these are spacetime symmetries, and they are actually closely related; CPT violation in a theory with a well-defined *S*-matrix also requires Lorentz violation [1]. These symmetries describe spatial isotropy, relativistic boost invariance, and the consequences of having a hermitian Hamiltonian—all of which are basic to our understanding of relativistic quantum field theory and general relativity. However, there still exists the intriguing possibility that such symmetries may also be (very weakly) violated in nature. The ultimate quantum gravity theory that integrates theories of gravitation and particle physics may behave differently than its low-energy limits. In fact, many schematic theories that have been proposed to describe quantum gravity actually appear to have regimes in which the Lorentz and CPT symmetries may be broken. Conversely, if any evidence were ever uncovered experimentally that those symmetries do not hold exactly, that would be a discovery of the greatest fundamental importance, and it would provide new insights about the shape of physics at the deepest level.

Even without any experimental indication that violations of Lorentz or CPT symmetries are possible, studying exotic theories in which these symmetries do not hold can reveal a great deal about the general structure of quantum field theory. Theories with unusual traits can be powerful theoretical laboratories for understanding what kinds of phenomena might occur in the field theory framework.

Whatever the motivation for studying Lorentz and CPT symmetry violations, the most natural tools for that study are found in effective field theory (EFT). The framework of EFT allows us to incorporate a small amount of symmetry breaking on top of the well-developed theories that we already understand. A single EFT describing both Lorentz and CPT violation has been the subject of extensive study over the past two decades. This theory is known as the standard model extension (SME), and has an action that is constructed from all operators that may be built up from standard model fields [2,3]. With no requirement that the operators appearing in the action be Lorentz scalars, the number of possibilities is much larger than in the standard model. The most general formulation of the SME contains an infinite tower of operators, since the operators may be nonrenormalizable, with canonical dimensions greater than (mass)^4. For many practical calculations (and for parameterizing most experimental bounds), it suffices to use only a finite subset of the theory known as the minimal SME. The minimal SME action contains those operators which are local, power counting renormalizable, and invariant under the standard model's $SU(3)_c \times SU(2)_L \times U(1)_Y$ gauge group.

When the minimal SME is used as a theoretical laboratory for studying the general characteristics of quantum theories, some Lorentz-violating modifications are more intrinsically interesting than others. Some operators have more peculiar properties than others, and perhaps the most unique term is a type of Chern–Simons term. The Lorentz-violating electromagnetic Chern–Simons term modifies the propagation of left- and right-circularly polarized waves differently (this is tied to the fact that the Chern–Simons theory has broken P and CPT symmetries). The difference between the two modes' dispersion relations leads to vacuum birefringence; the plane of polarization for a linearly polarized wave will rotate as the wave propagates. Moreover, at extremely long wavelengths, the frequency for one of the helicity modes may become imaginary, which is indicative of a vacuum instability. Among others, these unconventional features have made the Chern–Simons theory especially interesting as a tool for sampling how exotic quantum field theories can behave.

If there is a Chern–Simons term that affects real, physical photons, it must be extremely tiny. The birefringence that arises from the left–right asymmetry in wave propagation speeds has been searched for and not found, even for electromagnetic waves that have propagated across cosmological distances [4–7]. The best bounds on the timelike Chern–Simons term discussed in this paper come from these birefringence measurements and are at the 10^{-43} GeV level. This is but one example of the many strong constraints on Lorentz-violating terms coming from astrophysical observations [8,9].

Nevertheless, the Chern–Simons theory is still of great theoretical interest, because it has some highly unusual features—features seen in few other quantum field theories. Another of the most notable features is that the Chern–Simons Lagrange density is not actually gauge invariant on its own. This might naively seem like a serious problem; the unitarity of a quantum theory with vector excitations relies crucially on the presence of gauge symmetry, and formally, the lack of gauge invariance would seem to exclude the Chern–Simons term from the minimal SME. However, the Chern–Simons theory does possess just enough gauge invariance to overcome these difficulties. Under a gauge transformation, the Lagrange density changes, but only by a total spacetime derivative. This makes the action fully gauge invariant (for gauge transformations that tend to the identity at infinity); and since the equations of motion are derived from variations of the integrated action, those equations are gauge invariant, involving only the fields $\vec{E}$ and $\vec{B}$, not the gauge-dependent potentials.

However, since the theory possesses less gauge invariance than normal, radiative corrections in the Chern–Simons theory have provoked a great deal of controversy [10–15] in the past. The troublesome radiative corrections could arise from diagrams with triangular fermion loops, which are very similar in form to those that appear in the study of the chiral anomaly and neutral pion decay. However, the unique result for the axial vector anomaly is derived from a gauge invariance condition, and that condition does not apply in the Chern–Simons theory. Because of the theory's weakened implementation of gauge symmetry, it turns out that the Chern–Simons term

may be generated by radiative corrections, such that the result is definitely finite but otherwise completely undetermined!

The radiative correction problem was quite controversial in the earliest years of the twenty-first century. More recently, another very peculiar property of the theory has attracted attention (although with less associated controversy). This is related to the possibility of vacuum Cerenkov radiation. Standard Cerenkov radiation occurs when charges move very rapidly inside a material. In matter, the phase speed of light is typically less than 1, so it is possible for a charged particle to outpace photons in a medium. The result is Cerenkov radiation, which is very much like the burst of sound emitted in a sonic boom when an object moves faster than the speed of sound in a fluid.

In a Lorentz-violating theory, the phase speed of light in vacuum need not be uniformly equal to 1 either. The speed may depend on the wavelength, polarization, and (if rotational isotropy is broken) on the propagation direction. Because the Chern–Simons term slows down one circular polarization mode of radiation, vacuum Cerenkov radiation would be expected to occur in the theory. In fact, any charge in uniform motion (no matter how small its speed v is) will outpace some long-wavelength electromagnetic waves. So the natural expectation would be that any charge moving with a constant velocity will emit radiation in the Chern–Simons theory. However, this turns out not to be case, for a rather subtle reason.

This paper aims to provide a careful conceptual explanation of how there can be no vacuum Cerenkov radiation in the timelike Chern–Simons theory. While many previous analyses have studied the radiation emitted just one mode at a time, our approach will be through a direct analysis of the electric and magnetic field profiles and how they contribute to the modified energy-memory tensor. Section 2 describes the structure of the theory. Then, Section 3 looks carefully at how the fields carry energy in this theory, and we outline a demonstration that a charge in uniform motion does not radiate. Finally, Section 4 provides a concluding conceptual discussion of why exactly the net radiation vanishes, and what questions remain unanswered.

2. The Chern–Simons Term in Electrodynamics

The Lagrange density for the electromagnetic sector of the minimal SME is

$$\mathcal{L}_{CS} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} k_F^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} + \frac{1}{2} k_{AF}^{\mu} \epsilon_{\mu\nu\rho\sigma} F^{\nu\rho} A^{\sigma} - j^{\mu} A_{\mu}. \tag{1}$$

This contains the usual Maxwell terms describing photon propagation and the minimal interaction with charged matter. However, it also includes two additional operators with tensor-valued coefficients. These are the sources of the electromagnetic Lorentz violation. The k_F term is even under CPT and has been extensively studied, but we shall not consider it here; our focus is on the Chern–Simons term parameterized by k_{AF}, which is odd under CPT. If Lorentz symmetry is broken spontaneously, a term like k_{AF} would be proportional to the vacuum expectation value of a vector-valued field. This kind of spontaneous breaking (as opposed to explicit symmetry breaking in the underlying Lagrangian) is the only kind of Lorentz violation that is consistent with dynamical geometric gravity [16].

The unique behavior we will be discussing is characteristic of the timelike Chern–Simons theory only, so it will be convenient to work in coordinates in which the k_{AF} vector is purely timelike: $k_{AF}^{\mu} = (k, \vec{0})$. In this case, the k_{AF} term is proportional to $\vec{A} \cdot \vec{B}$. Under a gauge transformation with parameter Λ, the change in $\mathcal{L}_{CS}$ is proportional to $(\vec{\nabla}\Lambda) \cdot \vec{B}$, which is total derivative because $\vec{\nabla} \cdot \vec{B} = 0$.

The modified Maxwell's equations can be derived straightforwardly from the Chern–Simons action. They are

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{2}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{3}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{4}$$

$$\vec{\nabla} \times \vec{B} = \frac{\partial \vec{E}}{\partial t} + 2k\vec{B} + \vec{J}. \tag{5}$$

As expected, these equations involve only the electric and magentic fields—not the potentials directly. Moreover, when k_{AF} is purely timelike, only the Ampere–Maxwell law is modified.

Plugging a plane wave ansatz into the vacuum Maxwell's equations gives the dispersion relation. Because of the theory's P and CPT violation, there are different dispersion relations for right- and left-circularly polarized waves. The relations are $\omega_\pm^2 = p(p \mp 2k)$. Clearly, at very long wavelengths where $p < |2k|$, one of the two frequencies becomes imaginary. An imaginary ω signals an instability in the theory. As we shall shortly see, the field energy is not bounded below, and there are runaway solutions with exponentially growing field amplitudes, which push the system toward ever more negative-energy field configurations.

In the very first analysis of this Chern–Simons theory [4], the proposed solution was simply to eliminate these modes from the theory. It is possible to excise the runaway solutions by using a nonstandard Green's function. If this Green's function is used, the problematic modes are never excited; however, this comes at a high price—acausality. Charges will begin to radiate before they actually start to move! This raises real questions about how to treat the Chern–Simons theory so as to obtain reasonable results. Our analysis of Cerenkov radiation will be specifically formulated so that the results are independent of how the Green's functions for the theory are selected. Moreover, those results will show that the apparently unstable modes actually play a crucial role in energy transport within this theory.

The initial interest in Cerenkov radiation in this theory arose from the observation that (even setting aside the modes with $\omega^2 < 0$) there are propagating modes with arbitrarily small phase speeds. Any charge q will be moving faster than some of these modes. Radiation into these slow modes then becomes kinematically possible, and so it was a real surprise when it was found that the net radiation rate from a uniformly moving charge is always vanishing in this theory.

In many cases, an analysis of the phase space available for a process is suitable for obtaining fairly reliable estimates of the rate at which the process occurs. This is particularly the case for processes that are forbidden in a normal Lorentz-invariant theory. The reason for this is fairly straightforward. If a process becomes kinematically allowed only above some threshold, the available phase space typically increases quite rapidly just beyond that threshold—growing as a fairly high power of the energy. In contrast, the matrix element tends to be slowly varying, and treating it as a constant is a fairly reasonable approximation.

However, this is not always the case. In some instances, the matrix element for a process may be rapidly varying or nearly zero, and a phase space estimate will not capture even the qualitative energy dependence of the reaction rate. For example, Ref. [17] considered fermionic forms of isotropic Lorentz violation—operators with dimension (mass)^5, drawn from the non-minimal (nonrenormalizable) SME [18]. The operators took the form

$$\mathcal{L}_\zeta = \bar{\psi} \left[i\gamma^\mu D_\mu - m + \frac{1}{2M}(\zeta_1 \gamma_0 + \zeta_2 \gamma_0 \gamma_5)(D_0)^2 \right] \psi, \tag{6}$$

expressed in terms of some large mass scale M. The expression for $\mathcal{L}_\zeta$ includes both the usual Dirac terms and two possible Lorentz-violating terms. Using somewhat indirect arguments, Ref. [17] calculated the rate for the photon decay process $\gamma \rightarrow e^+ + e^-$—pair creation from a photon in vacuum—in the presence of ζ_1. The kinematics of this processes are quite similar to the kinematics of vacuum Cerenkov radiation, with one relativistic quantum transforming into two.

Like Cerenkov radiation, pair creation from a single photon is normally forbidden in vacuum; an initial state, with a given momentum p and a normal relativistic energy-momentum relation, never has sufficient energy to create the outgoing states. However, both processes occur quite readily in matter, where there are additional atoms capable of taking up some of the momentum. In fact, the two processes—pair creation and Cerenkov radiation—are the most important quantum electrodynamics (QED) energy loss mechanisms for ultrarelativistic particles in a material medium. Together, for example, they are responsible for the air showers produced by TeV cosmic ray photons impinging on the atmosphere.

The presence of $\mathcal{L}_\zeta$ changes the dispersion relation for electrons and positrons at very high energies. As a result, the photon decay process becomes allowed above a certain threshold. The decay rate above threshold for ζ_1-type Lorentz violation is well predicted by an estimate of the phase space alone. However, for the seemingly very similar ζ_2 form of Lorentz violation, the phase space analysis fails completely. In [8], the rate for $\gamma \to e^+ + e^-$ was determined through a direct calculation of the matrix element for the process, and it was found to have an entirely different energy dependence. In particular, the photon decay arising from ζ_2 is strongly suppressed near the threshold, relative to a simple phase space estimate.

The reason for the discrepancy is tied fundamentally to the presence of the chirality operator γ_5 accompanying ζ_2. For ultrarelativistic spin-$\frac{1}{2}$ particles, plane-wave helicity eigenstates are very nearly eigenstates of γ_5. The eigenvalue is the product of the helicity s and the fermion number f. Thus, an electron and a positron with the same helicity have their energies shifted in opposite directions. The primary photon is a spin-1 state, with no helicity-0 component, and the process proceeds almost entirely through the normal QED vertex, so angular momentum is conserved in the process. That means that the electron and positron in the final state must also be in a total spin-1 state. At threshold, all the particles in the decay are collinear, so the two daughter particles must have identical spins. However, the process is not allowed for this spin configuration, because the ζ_2 term shifts one particle's energy upward and the other's energy down by an equal amount (whereas, for the process to occur, the total energy of both daughter particles must be shifted downward). The spin structure of the matrix element thus causes the rate to vanish at threshold, and this illustrates that estimates based only on phase space availability will sometimes fail.

Even more strangely, the notion of a phase space estimate itself may fail entirely when a theory is unstable. This is what we will find occurs for the Chern–Simons theory. Forming a phase space estimate of the rate of Cerenkov emission into a single mode of a Lorentz-violating photon sector is fairly straightforward [19]. Any given mode of the electromagnetic field behaves essentially according to standard Maxwell electrodynamics, only with a phase speed that generally differs from 1. The rate of radiation into that mode can thus be calculated using the standard formulas developed for dealing Cerenkov radiation in matter.

The radiation is emitted along the surface of a cone. The Cerenkov angle (the angle between the direction the charge q is moving and the directions in which the radiation is emitted) is

$$\theta_C = \cos^{-1}\left[\frac{\omega(p)}{vp}\right], \tag{7}$$

for waves with angular wave number p. θ_C controls the phase space available for the process, and the power emitted per unit p is $dP/dp = \frac{q^2}{4\pi}\sin^2\theta_C\omega$. In the absence of birefringence, this should be a satisfactory phase space estimate of the rate in a Lorentz-violating theory as well.

The birefringent case complicates things only a bit further. Ordinary Cerenkov radiation is linearly polarized in the plane defined by the direction $\hat{v}$ that the charge is moving and $\hat{p}$, the direction of the photon emission. Denoting the corresponding polarization vector by $\hat{e}_{(0)}(\vec{p})$, we need only calculate the overlap between the $\hat{e}_{(0)}$ mode which is naturally excited by the superluminal motion and the actual normal mode of propagation. If the mode for which the Cerenkov radiation is allowed

has a different polarization vector $\hat{e}_{(i)}$, the power is simply multiplied by $\left|\hat{e}_{(i)} \cdot \hat{e}_{(0)}\right|^2$. This overlap formula is especially simple in the timelike Chern–Simons theory, where the propagation modes are always circular, and thus $\left|\hat{e}_{(i)} \cdot \hat{e}_{(0)}\right|^2 = \frac{1}{2}$.

This method can be applied to the Chern–Simons theory, but the results are of very limited usefulness. The phase space estimate is simply not possible for the modes with $p < |2k|$, because they do not possess real frequencies. However, it is possible to analyze the modes in the range $|2k| < p < |2k|/(1 - v^2)$, which are the modes for which $0 < \omega/p < v$—those for which Cerenkov radiation is naturally expected. Integrating dP/dp over this range gives [20]

$$P = \frac{k^2 q^2 v^3}{30\pi} + \mathcal{O}(v^5), \tag{8}$$

when $v \ll 1$. The overall $k^2 v^3$ dependence is fairly simple to understand; the length of the p-space region covered by the integration is $\approx 2kv^2$, and the energy of a typical photon in this region is $\mathcal{O}(kv)$.

A similar calculation for the energy flow yields a fairly accurate result when the Chern–Simons term is spacelike [21]. However, the spacelike theory is fundamentally different from the timelike; when k_{AF} is a spacelike vector, there is no true instability—no modes that have unavoidable negative frequencies. This difference is crucial, and we shall see that for the timelike case, the phase space estimate of P is not very useful, because of the way the unstable modes also carry energy.

3. Vanishing of the Cerenkov Rate

To get a clear understanding of the energetics in a modified electromagnetic theory, we need to examine the theory's energy-momentum tensor. For an arbitrary k_{AF}, the tensor has the form [4]

$$\Theta^{\mu\nu} = -F^{\mu\alpha} F^{\nu}{}_{\alpha} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - \frac{1}{2} k^{\nu}_{AF} \epsilon^{\mu\alpha\beta\gamma} F_{\beta\gamma} A_{\alpha}. \tag{9}$$

For the case we are studying, in which k_{AF} is purely timelike, the energy density ($\mathcal{E} = \Theta^{00}$), momentum density ($\mathcal{P}_j = \Theta^{0j}$), and energy flux ($S_j = \Theta^{j0}$) are

$$\mathcal{E} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 - k\vec{B} \cdot \vec{A} \tag{10}$$

$$\vec{\mathcal{P}} = \vec{E} \times \vec{B} \tag{11}$$

$$\vec{S} = \vec{E} \times \vec{B} - kA_0\vec{B} + k\vec{A} \times \vec{E}. \tag{12}$$

The fact that the energy-momentum tensor is not symmetric, $\Theta^{\mu\nu} \neq \Theta^{\nu\mu}$, is indicative of the theory's Lorentz noninvariance. All theories with physically observable Lorentz violation have this feature.

However, some of the other peculiar features of this $\Theta^{\mu\nu}$ are particular to the Chern–Simons theory. The energy density $\mathcal{E}$, like the Lagrange density $\mathcal{L}$, fails to be gauge invariant; however, the total energy, integrated over all space, is once again gauge invariant. Moreover, as we previously noted, the energy is not bounded below. The runaway solutions with $p < |2k|$ are precisely those for which the unbounded $-k\vec{B} \cdot \vec{A}$ term can be made arbitrarily negative by increasing the field amplitude, without the positive $\frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2$ growing more quickly and keeping the total energy positive. The capacity of the theory to carry negative energy will be key to understanding the overall absence of Cerenkov radiation.

To study the possible Cerenkov radiation, we will look at the field profiles produced at time $t = 0$ by a charge q moving along the prescribed trajectory $\vec{r}(t) = vt\hat{z}$. To isolate the Cerenkov radiation, we will consider only a charge that remains in steady motion over all time. Obviously, a real particle must be accelerated to reach this speed; moreover, if the charge does lose energy to Cerenkov emission, there will be recoil. However, any radiation that is directly a result of these kinds of accelerative effects is not really Cerenkov radiation.

If the charge is in completely uniform motion, its electric and magnetic fields will also be moving along in the z-direction with speed v. This determines the time dependences of the fields in terms of their spatial profiles, and thus allows us to eliminate all of the time derivatives that appear in Maxwell's equations. The electric field has the form $\vec{E}(\vec{r}, t) = \vec{E}(\vec{r} - v\hat{z}t, 0)$ (and similarly for $\vec{B}$). So, any time derivative $\frac{\partial}{\partial t}$ acting on $\vec{E}$ or $\vec{B}$ may be replaced with $-v\frac{\partial}{\partial z}$. By ensuring that the fields automatically have this form, we also avoid having to deal with the question of whether or not we ought to use the acausal Green's function.

We will express the fields as power series in both k and v,

$$\vec{E} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \vec{E}^{(m,n)} \tag{13}$$

$$\vec{B} = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} \vec{B}^{(m,n)}. \tag{14}$$

Each term with superscripts (m, n) is $\mathcal{O}(k^m v^n)$. The usual terms are the $\vec{E}^{(0,n)}$ with n even and $\vec{B}^{(0,n)}$ with n odd. Using the modified Ampere–Maxwell law and the other vacuum Maxwell's equations, we may determine the fields iteratively, according to

$$\vec{\nabla} \times \vec{E}^{(m,n)} = v\frac{\partial \vec{B}^{(m,n-1)}}{\partial z} \tag{15}$$

$$\vec{\nabla} \times \vec{B}^{(m,n)} = -v\frac{\partial \vec{E}^{(m,n-1)}}{\partial z} + 2k\vec{B}^{(m-1,n)} \tag{16}$$

$$\vec{\nabla} \cdot \vec{E}^{(m,n)} = \vec{\nabla} \cdot \vec{B}^{(m,n)} = 0. \tag{17}$$

The starting points are the Lorentz-invariant fields $\vec{E}^{(0,n)}$ and $\vec{B}^{(0,n)}$. Then, successively solving these equations provides terms with increasing $m + n$. It is possible to solve these equations explicitly, and this has been done [20,22] for the low-order fields. The lowest order k-dependent field is

$$\vec{B}^{(1,1)} = \frac{kqv}{4\pi r}\left(2\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) \tag{18}$$

(with no additional δ-function singularity at $r = 0$ [23]). However, a more general analysis—based solely on symmetry considerations, rather than any particular field configurations—is also possible.

Table 1. z-parity values for the field components, with $+$ and $-$ denoting even and odd parity, respectively.

Field	$\hat{r}$	$\hat{\theta}$	$\hat{\phi}$
		z-Parity	
$\vec{E}$	$+$	$-$	$-$
$\vec{B}$	$-$	$+$	$+$

The key to demonstrating that there are no net energy losses through Cerenkov radiation is understanding how the various field terms behave under an inversion along the z-axis. A straightforward analysis [22] of (15)–(17) reveals that each component of $\vec{E}$ or $\vec{B}$ is either odd or even under this reflection. We call these the z-parities of the field components. They are listed in Table 1.

To calculate the Cerenkov losses, we must evaluate the net outward energy flow over a sphere at infinity, $P = R^2 \int d\Omega\, \vec{S} \cdot \hat{r}$, as $R \to \infty$. While neither $\mathcal{E}$ nor $\vec{S}$ is gauge invariant, the total energy and likewise the integral of $\vec{S} \cdot \hat{r}$ over the sphere at infinity are gauge invariant. However, since $\vec{S}$ depends on the potentials explicitly, we also need to know the z-parities of those potentials. To determine these, we may chose any convenient gauge, knowing that the final integral will not depend on the gauge choice. In the Coulomb gauge, the components of the vector potential $\vec{A}$ have

the same z-parities as the components of $\vec{B}$, as listed in Table 1; and in this gauge, the scalar potential A_0 has exactly the same k-independent form as in the Lorentz-invariant theory.

The final step in the calculation is to note that each term in the integrand $\vec{S} \cdot \hat{r}$ is either vanishing or has odd z-parity. For example, up to $\mathcal{O}(k^2 v^2)$, the radial component of the flux as $r \to \infty$ is

$$\vec{S} \cdot \hat{r} \approx -\frac{k^2 q^2 v}{8\pi^2 r^2} \cos\theta. \tag{19}$$

As an odd function of $\cos\theta$, this expression does indeed have odd z-parity. The integration of a $\vec{S} \cdot \hat{r}$ of this sort over a whole sphere then automatically gives zero, and there is no net outflow of energy from the moving charge.

4. Concluding Remarks

The result that P is identically zero could be seen as quite puzzling. On general principles, it seemed that there should at least be radiation emitted into the slow modes with $|2k| < p < |2k|/(1-v^2)$. Moreover, the expression for $\vec{S}$ does not actually vanish on the sphere at infinity; only its total outward flux comes to zero. This suggests the curious notion that there is an overall energy flow in the system, from $z = +\infty$ to $z = -\infty$: equal amounts moving in from positive spatial infinity and out to negative spatial infinity, so that there is no net change in the energy surrounding the charge. These peculiarities of our result are all tied to the gauge transformation properties of the theory.

The phase space estimate for the energy carried in the $p > |2k|$ modes is not wrong, precisely. However, it is of extremely limited usefulness. The energy density $\mathcal{E}$ depends upon the gauge, meaning that it is not possible to identify the amount of energy contained within a finite region of coordinate space. Similarly, it is not possible to isolate the energy carried by only a partial subset of the momentum modes. In the particular Coulomb gauge that we used, we found evidence of energy being carried away by a subset of the modes. Additionally, there appeared to be a nonzero energy flux at spatial infinity, even if there is no net deposition of energy at any finite radius. However, both these statements depend on the choice of gauge. With a gauge transformation, it is possible to reconfigure the potentials so that $\mathcal{E}$ and $\vec{S}$ are distributed differently in either real space or momentum space. In real space, this means that the nonzero $\vec{S}$ at spatial infinity is a gauge artifact.

In momentum space, things are more subtle. It is meaningful that there are modes of the field that do appear to be capable of carrying away energy (even if the precise allotment of energy to individual modes is not physically significant). The propagating modes with $|2k| < p < |2k|/(1-v^2)$ really are excited. However, the modes with $p < |2k|$ also carry negative energy, which precisely cancels the energy carried by the longer-wavelength modes! As we have already noted, excitation of the longest wavelength modes is associated with negative energies. Normally, we would expect the involvement of these modes to lead to instabilities. However, by looking only at field configurations in which the profiles are all in uniform motion, we have forced these modes to behave differently. Instead of growing exponentially in time, the unstable modes are associated with propagating solutions carrying negative energies.

Because these negative energy modes exist, some cancelation of positive and negative terms is obviously possible—and we have found that the cancelation is, in fact, exact. The very same modes which threaten the viability of the theory under other circumstances (forcing us to accept acausal radiation processes, for example) ensure that the timelike Chern–Simons theory is actually free of vacuum Cerenkov radiation. Just like in conventional Maxwell electrodynamics, a charge moving with a constant speed does not lose energy through radiation.

A detailed mode-by-mode accounting of the energy balance for the fields following the moving charge has not yet been attempted. It might be extremely interesting to see precisely how the positive- and negative-energy modes actually achieve their cancelation. However, such a calculation would obviously be very complicated, because of the theory's gauge dependence; it is not clear how much of the cancelation structure depends on the gauge chosen for the potentials. A full understanding

of these questions could reveal even more about the structure of the Lorentz-violating Chern–Simons theory specifically, and gauge theories in general.

So although the electromagnetic Chern–Simons term in the SME has been very tightly bounded, it remains of significant interest. At a purely theoretical level, the study of the theory with k_{AF} has yielded many new insights into the basic structure of quantum field theory. The theory's unstable vacuum and peculiar gauge invariance properties allow it to behave in ways that more conventional particle physics models simply cannot. If Lorentz or CPT symmetry are actually violated in some subtle way in nature (or even if the ultimate unifying theory has some other peculiar traits), what we learn now from studying the Chern–Simons theory may be important to understanding the true underlying theory. Moreover, empirically, it is still useful to work to gain the best understanding of the electromagnetic k_{AF} term, because there are other terms in the SME with similar properties, but which are much less tightly constrained.

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